

On Laplacian Like Energy of Trees

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Abstract

Let G be a simple undirected n -vertex graph with the characteristic polynomial of its Laplacian matrix $L(G)$, $\det(\lambda I - L(G)) = \sum_{k=0}^n (-1)^k c_k \lambda^{n-k}$. Laplacian-like energy of a graph is newly proposed graph invariant, defined as the sum of square roots of Laplacian eigenvalues. For bipartite graphs, the Laplacian-like energy coincides with the recently defined incidence energy $IE(G)$ of a graph. In [D. Stevanović, *Laplacian-like energy of trees*, MATCH Commun. Math. Comput. Chem. 61 (2009), 407–417.] the author introduced a partial ordering of graphs based on Laplacian coefficients. We point out that original proof was incorrect and illustrate the error on the example using Laplacian Estrada index. Furthermore, we found the inverse of Jacobian matrix with elements representing derivatives of symmetric polynomials of order n , and provide a corrected elementary proof of the fact: Let G and H be two n -vertex graphs; if for Laplacian coefficients holds $c_k(G) \leq c_k(H)$ for $k = 1, 2, \dots, n-1$, then $LEL(G) \leq LEL(H)$. In addition, we generalize this theorem and provide a necessary condition for functions that satisfy partial ordering based on Laplacian coefficients.

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1 Introduction

Let $G = (V, E)$ be a simple undirected graph with $n = |V|$ vertices and $m = |E|$ edges. The Laplacian polynomial $P(G, \lambda)$ of G is the characteristic polynomial of its Laplacian matrix $L(G) = D(G) - A(G)$,

$$P(G, \lambda) = \det(\lambda I_n - L(G)) = \sum_{k=0}^n (-1)^k c_k \lambda^{n-k}.$$

The Laplacian matrix $L(G)$ has non-negative eigenvalues $\mu_1 \geq \mu_2 \geq \dots \geq \mu_{n-1} \geq \mu_n = 0$ [1]. From Viette's formulas,

$$c_k = \sigma_k(\mu_1, \mu_2, \dots, \mu_{n-1}) = \sum_{\substack{I \subseteq \{1, 2, \dots, n-1\}, \\ |I|=k}} \prod_{i \in I} \mu_i \quad (1)$$

is a symmetric polynomial of order $n-1$. Detailed introduction to graph Laplacians may be found in [2] and [3]. In particular, we have $c_0 = 1$, $c_n = 0$, $c_1 = 2m$, $c_{n-1} = n\tau(G)$, where $\tau(G)$ denotes the number of spanning trees of G . If G is a tree, coefficient c_{n-2} is equal to its Wiener index, which is a sum of distances between all pairs of vertices

$$c_{n-2}(T) = W(T) = \sum_{u, v \in V} d(u, v).$$

The Wiener index is considered as one of the most used topological indices with high correlation with many physical and chemical properties of molecular compounds. For recent results and applications of Wiener index see [4].

Recently, Zhou and Gutman [5] proved that the extreme values of Laplacian coefficients among all n -vertex trees are attained on one side by the path P_n , and on the other side by the star S_n . In other words,

$$c_k(S_n) \leq c_k(T) \leq c_k(P_n), \quad k = 0, 1, 2, \dots, n, \quad (2)$$

holds for all trees T of order n .

The Laplacian-like energy of graph G , LEL for short, is defined as follows:

$$LEL(G) = \sum_{k=1}^{n-1} \sqrt{\mu_k}.$$

This concept was introduced in [6] where it was shown that it has similar features as molecular graph energy [7] (for recent results on graph energy see [8]). Various properties of Laplacian coefficients and Laplacian-like energy on trees and unicyclic graphs

are investigated in [9]–[18]. It was demonstrated in [12] that LEL can be used both in graph discriminating analysis and correlating studies for modeling a variety of physical and chemical properties and biological activities.

The signless Laplacian matrix of the graph G is defined as $Q(G) = D(G) + A(G)$. Matrix $Q(G)$ also has all real and nonnegative eigenvalues $\mu'_1 \geq \mu'_2 \geq \dots \geq \mu'_n \geq 0$, [19]. Motivated by Nikiforov's idea [20], Jooyandeh et al. [21] introduced the concept of incidence energy $IE(G)$ of a graph G , defining it as the sum of the singular values of the incidence matrix $I(G)$. It turns out that

$$IE(G) = \sum_{k=1}^n \sqrt{\mu'_k}.$$

In particular, if G is a bipartite graph, the spectra of $Q(G)$ and $L(G)$ coincide, and we have $IE(G) = LEL(G)$. This relation provides a new interpretation of the Laplacian-like energy, and offers a new insight into its possible physical or chemical meaning. Many mathematical properties of this quantity were established in [22] and [23].

Stevanović in [10] showed a connection between LEL and Laplacian coefficients.

Theorem 1.1 *Let G and H be two n -vertex graphs. If $c_k(G) \leq c_k(H)$ for $k = 1, 2, \dots, n-1$ then $LEL(G) \leq LEL(H)$. Furthermore, if a strict inequality $c_k(G) < c_k(H)$ holds for some $1 \leq k \leq n-1$, then $LEL(G) < LEL(H)$.*

We will show that this fact remains true, but the original author's proof was not correct. Here, we will provide a corrected elementary proof.

Let us move to a more general setting. Consider the open set in $\mathbb{R}^n$

$$\mathcal{X} = \{(x_1, x_2, \dots, x_n) : n > x_1 > x_2 > \dots > x_{n-1} > x_n > 0\}.$$

Let $\mathcal{C}$ denote the set of coefficients of polynomials having roots in $\mathcal{X}$,

$$\begin{aligned} \mathcal{C} &= \{(c_1, c_2, \dots, c_{n-1}, c_n) : \exists (x_1, x_2, \dots, x_n) \in \mathcal{X} \\ P(x) &= x^n - c_1 x^{n-1} + c_2 x^{n-2} - \dots + (-1)^{n-1} c_{n-1} x + (-1)^n c_n \\ &= (x - x_1)(x - x_2) \cdot \dots \cdot (x - x_n)\}. \end{aligned}$$

Let $F : \mathcal{X} \rightarrow \mathcal{C}$ be the bijection defined by Viette's formulas (1). This function represents polynomial coefficients from $\mathcal{C}$ via the roots from $\mathcal{X}$. Since every coordinate

of the vector function F is a polynomial, we conclude that F is a continuous function. Symmetric polynomials are continuously differentiable functions and we have

$$\frac{\partial c_k}{\partial x_j} = \sum_{\substack{j \in I \subseteq \{1, 2, \dots, n-1\}, \\ |I|=k}} \prod_{i \in I \setminus \{j\}} x_i. \quad (3)$$

Let $F^{-1} : \mathcal{C} \rightarrow \mathcal{X}$ be the inverse function of F . The Jacobian matrix of F is

$$J_F = \left[\frac{\partial c_i}{\partial x_j} \right]_{i,j=\overline{1,n}} = \begin{bmatrix} \frac{\partial c_1}{\partial x_1} & \frac{\partial c_1}{\partial x_2} & \dots & \frac{\partial c_1}{\partial x_{n-1}} & \frac{\partial c_1}{\partial x_n} \\ \frac{\partial c_2}{\partial x_1} & \frac{\partial c_2}{\partial x_2} & \dots & \frac{\partial c_2}{\partial x_{n-1}} & \frac{\partial c_2}{\partial x_n} \\ \frac{\partial c_3}{\partial x_1} & \frac{\partial c_3}{\partial x_2} & \dots & \frac{\partial c_3}{\partial x_{n-1}} & \frac{\partial c_3}{\partial x_n} \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ \frac{\partial c_n}{\partial x_1} & \frac{\partial c_n}{\partial x_2} & \dots & \frac{\partial c_n}{\partial x_{n-1}} & \frac{\partial c_n}{\partial x_n} \end{bmatrix}$$

In [10] the author has following identity

$$\frac{\partial LEL}{\partial c_k} = \sum_{i=1}^n \frac{\partial LEL}{\partial \mu_i} \cdot \frac{\partial \mu_i}{\partial c_k} = \sum_{i=1}^n \frac{\partial LEL / \partial \mu_i}{\partial c_k / \partial \mu_i}. \quad (4)$$

This is incorrect – we cannot use fact that the derivatives $\frac{\partial \mu_i}{\partial c_k}$ and $\frac{\partial c_k}{\partial \mu_i}$ are reciprocal, since c_k is a function of n variables. We can illustrate this error on the recently introduced molecular structure descriptor – Laplacian Estrada index [24]–[26], defined as

$$LEE(G) = \sum_{k=1}^n e^{\mu_k}.$$

Namely, the derivative $\frac{\partial LEE}{\partial c_k}$ is strictly positive, since $\frac{\partial LEE}{\partial c_k} = e^{\mu_i} > 0$. Therefore, according to the equation (4), this function is increasing in every coordinate c_k . The following conclusion is obviously not true – from inequalities (2) we have that $LEL(S_n) < LEL(P_n)$ and on the other side

$$LEE(P_n) = \sum_{k=1}^n e^{2+2 \cos \frac{k\pi}{n}} < e^n + 1 + (n-2)e = LEE(S_n),$$

since for $n > 5$ we have $ne^4 < e^n$.

This error can be fixed by finding the Jacobian matrix of the inverse F^{-1} . Let

$$\omega(x) = (x - x_1)(x - x_2) \dots (x - x_n)$$

be the polynomial with n distinct real roots $x_1, x_2, \dots, x_n$. The following theorem is the main contribution of this paper.

Theorem 1.2 *The Jacobian matrix of function F^{-1} equals*

$$J_{F^{-1}} = \left[(-1)^{j-1} \frac{x_i^{n-j}}{\omega'(x_i)} \right]_{i,j=1,n}$$

$$= \begin{bmatrix} \frac{x_1^{n-1}}{\omega'(x_1)} & -\frac{x_1^{n-2}}{\omega'(x_1)} & \cdots & (-1)^{n-2} \frac{x_1}{\omega'(x_1)} & (-1)^{n-1} \frac{1}{\omega'(x_1)} \\ \frac{x_2^{n-1}}{\omega'(x_2)} & -\frac{x_2^{n-2}}{\omega'(x_2)} & \cdots & (-1)^{n-2} \frac{x_2}{\omega'(x_2)} & (-1)^{n-1} \frac{1}{\omega'(x_2)} \\ \frac{x_3^{n-1}}{\omega'(x_3)} & -\frac{x_3^{n-2}}{\omega'(x_3)} & \cdots & (-1)^{n-2} \frac{x_3}{\omega'(x_3)} & (-1)^{n-1} \frac{1}{\omega'(x_3)} \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ \frac{x_n^{n-1}}{\omega'(x_n)} & -\frac{x_n^{n-2}}{\omega'(x_n)} & \cdots & (-1)^{n-2} \frac{x_n}{\omega'(x_n)} & (-1)^{n-1} \frac{1}{\omega'(x_n)} \end{bmatrix}$$

The paper is organized as follows. After a few preliminary results involving Newton's identities and complex analysis in Section 2, we prove Theorem 1.2 in Section 3. In Section 4 we provide a sufficient condition for establishing a partial ordering of graphs based on Laplacian coefficients and finally in Section 5 we give a corrected proof of Theorem 1.1.

2 Preliminary results

Before presenting the proof of Theorem 1.2, we need to prove some useful lemmas.

Lemma 2.1 *For every $2 \leq i \leq n$ and $1 \leq j \leq n$ holds:*

$$\frac{\partial c_i}{\partial x_j} = c_{i-1} - x_j \cdot \frac{\partial c_{i-1}}{\partial x_j}. \quad (5)$$

Proof. Notice that from the equation (3), $\partial c_i / \partial x_j$ is a polynomial of degree $i - 1$ in variables $x_1, x_2, \dots, x_{j-1}, x_{j+1}, \dots, x_n$. On the right side, we have the sum of product of all $(i - 1)$ -tuples that does not contain x_j . We can derive this set if we consider all $(i - 1)$ -tuples and subtract those that contain x_j – which gives the relation (5). $\square$

Lemma 2.2 *Let $\omega(x)$ be a polynomial of real coefficients of degree n having distinct real roots $x_1, x_2, \dots, x_n$. For any nonnegative integer $0 \leq k \leq n - 2$ holds:*

$$\sum_{i=1}^n \frac{x_i^k}{\omega'(x_i)} = 0.$$

Proof. Consider the rational function $f(z)$ of complex variable z , defined as follows:

$$f(z) = \frac{z^k}{\omega(z)}.$$

By comparing degrees it holds $|f(z)| \leq \frac{C}{|z|^2}$ for some constant C and large enough $|z|$. So, for the integral over the circumference S_R of radius R and center at the origin we have

$$|I| = \left| \int_{S_R} f(z) dz \right| \leq \int_{S_R} |f(z)| dz \leq \int_{S_R} \frac{C}{|z|^2} dz = \frac{2\pi C}{R},$$

for large enough R . On the other hand, using the Cauchy Residue theorem [27] we have:

$$I = \int_{S_R} f(z) dz = 2\pi i \sum_{i=1}^n \text{Res}(x_i, f) = 2\pi i \sum_{i=1}^n \frac{x_i^k}{\omega(x_i)}.$$

Therefore, for large enough radius R , we have inequality:

$$\left| \sum_{i=1}^n \frac{x_i^k}{\omega(x_i)} \right| \leq \frac{C}{R},$$

and taking the limit as $R \rightarrow \infty$ we obtain that the sum equals zero. $\square$

Lemma 2.3 *Let $\omega(x)$ be a polynomial of real coefficients of degree n having distinct real roots $x_1, x_2, \dots, x_n$. It holds:*

$$\sum_{i=1}^n \frac{x_i^{n-1}}{\omega'(x_i)} = 1.$$

Proof. Consider the function

$$f(z) = \frac{z^{n-1}}{\omega(z)} - \frac{1}{z} = \frac{z^n - \omega(z)}{z\omega(z)}.$$

Similarly, we get that $|f(z)| \leq \frac{C}{|z|^2}$. Using the additive property of integrals and

$$\int_{S_R} \frac{dz}{z} = \int_0^{2\pi} \frac{iRe^{i\phi} d\phi}{Re^{i\phi}} = 2\pi i,$$

the relation follows. $\square$

For $i \geq 1$ denote by $s_i(x_1, x_2, \dots, x_n)$ the i -th power sum

$$s_i = \sum_{k=1}^n \frac{x_k^i}{\omega'(x_k)}.$$

The Newton identities [28] (also known as Newton-Girard formulae) give relations between power sums and elementary symmetric polynomials.

$$k \cdot c_k = \sum_{i=1}^k (-1)^{i-1} c_{k-i} \cdot s_i. \quad (6)$$

For $k > n$, by definition is $c_k = 0$. We already proved that $s_0 = s_1 = \dots = s_{n-2} = 0$ and $s_{n-1} = 1$. For $k = 2, \dots, n+1$, holds:

$$\begin{aligned} (n+k-1)c_{n+k-1} &= c_{n+k-2}s_1 - c_{n+k-3}s_2 + \dots + (-1)^{n-1}c_{k+1}s_{n-2} \\ &\quad + (-1)^{n-2}c_k s_{n-1} + (-1)^{n-1}c_{k-1}s_n + \dots + (-1)^{n+k-2}c_0 s_{n+k-1}, \end{aligned}$$

or equivalently

$$c_k \cdot s_{n-1} - c_{k-1} \cdot s_n + c_{k-2} \cdot s_{n+1} - \dots + (-1)^{k-1} \cdot c_1 \cdot s_{n+k-2} + (-1)^k \cdot s_{n+k-1} = 0. \quad (7)$$

3 The proof of Theorem 1.2

The theorem states that

$$J_F \cdot J_{F^{-1}} = J_{F^{-1}} \cdot J_F = I_n = [\delta_{i,j}]_{i,j=\overline{1,n}},$$

where I_n is identity matrix of order n . Therefore, we have to prove the following equalities for every $1 \leq i, j \leq n$,

$$\sum_{k=1}^n (-1)^{k-1} \frac{x_i^{n-k}}{\omega'(x_i)} \cdot \frac{\partial c_k}{\partial x_j} = \delta_{i,j} \quad (8)$$

and

$$\sum_{k=1}^n \frac{\partial c_i}{\partial x_k} \cdot \frac{x_k^{n-j}}{\omega'(x_k)} = (-1)^{j-1} \cdot \delta_{i,j}. \quad (9)$$

Define the polynomials $\omega_i(x)$, $i = 1, 2, \dots, n$ as

$$\omega_i(x) = (x - x_1)(x - x_2) \cdot \dots \cdot (x - x_{i-1})(x - x_{i+1}) \cdot \dots \cdot (x - x_n).$$

By the derivation of the product, we get

$$\omega'(x) = \sum_{i=1}^n \omega_i(x).$$

The proof of (8) follows from the identity (case $i = j$):

$$\begin{aligned} \omega'(x_i) &= \omega_i(x_i) \\ &= (x_i - x_1)(x_i - x_2) \cdot \dots \cdot (x_i - x_{i-1})(x_i - x_{i+1}) \cdot \dots \cdot (x_i - x_n) \\ &= \frac{\partial c_1}{\partial x_i} \cdot x_i^{n-1} - \frac{\partial c_2}{\partial x_i} \cdot x_i^{n-2} + \frac{\partial c_3}{\partial x_i} \cdot x_i^{n-3} - \dots + (-1)^{n-2} \frac{\partial c_{n-1}}{\partial x_i} \cdot x_i + (-1)^{n-1} \frac{\partial c_n}{\partial x_i}. \end{aligned}$$

For $i \neq j$, similarly we get:

$$0 = \omega_j(x_i) = \sum_{k=1}^n (-1)^{k-1} x_i^{n-k} \cdot \frac{\partial c_k}{\partial x_j}.$$

Since the first row in J_F equals $(1, 1, 1, \dots, 1)$, from Lemma 2.2 and Lemma 2.3 follows that first row in the matrix product $J_F \cdot J_{F^{-1}}$ is $(1, 0, 0, \dots, 0)$. For the rows $i = 2, 3, \dots, n$ we proceed by mathematical induction using the recurrent formula (5).

$$\begin{aligned} \sum_{k=1}^n \frac{\partial c_i}{\partial x_k} \cdot \frac{x_k^{n-j}}{\omega'(x_k)} &= \sum_{k=1}^n \left(c_{i-1} - x_k \cdot \frac{\partial c_{i-1}}{\partial x_k} \right) \cdot \frac{x_k^{n-j}}{\omega'(x_k)} \\ &= c_{i-1} \cdot \sum_{k=1}^n \frac{x_k^{n-j}}{\omega'(x_k)} - \sum_{k=1}^n \frac{\partial c_{i-1}}{\partial x_k} \cdot \frac{x_k^{n-j+1}}{\omega'(x_k)} \end{aligned}$$

For $j > 1$, we have that the sum equals $-(-1)^{j-1} \delta_{i-1,j-1} = (-1)^j \delta_{i,j}$. The case $j = 1$, we have to consider independently.

$$\begin{aligned} \sum_{k=1}^n \frac{\partial c_i}{\partial x_k} \cdot \frac{x_k^{n-1}}{\omega'(x_k)} &= c_{i-1} s_{n-1} - \sum_{k=1}^n \frac{\partial c_{i-1}}{\partial x_k} \cdot \frac{x_k^n}{\omega'(x_k)} \\ &= c_{i-1} s_{n-1} - c_{i-2} s_n + \sum_{k=1}^n \frac{\partial c_{i-2}}{\partial x_k} \cdot \frac{x_k^{n+1}}{\omega'(x_k)}. \end{aligned}$$

Applying this substitutions i times and using the identity (7) we get that the first element in the i -th row equals zero. This finishes the inductive proof of (9).

4 Partial ordering of graphs based on Laplacian coefficients

First, we will give some general method for deriving the sign of special summation that we need. Let f be any smooth function and let

$$P(x) = c_0 x^{n-1} + c_1 x^{n-2} + \dots + c_{n-2} x + c_{n-1}$$

be the interpolating polynomial for f at the points $x_1, x_2, \dots, x_n$ of degree $n - 1$. Using the Lagrange interpolating theorem, we have:

$$P(x) = \sum_{i=1}^n f(x_i) \cdot \frac{(x - x_1)(x - x_2) \cdot \dots \cdot (x - x_{i-1})(x - x_{i+1}) \cdot \dots \cdot (x - x_n)}{\omega'(x_i)}.$$

The sum

$$S = \sum_{i=1}^n \frac{f(x_i)}{\omega'(x_i)} = \sum_{i=1}^n \frac{P(x_i)}{\omega'(x_i)}$$

equals to the leading coefficient of P . Using Taylor expansion of $P(x)$, we have

$$S = \sum_{i=1}^n \frac{f(x_i)}{\omega'(x_i)} = c_1 = \frac{1}{(n-1)!} P^{(n-1)}(x).$$

Since $f(x) - P(x)$ has n real roots, we conclude by Rolle's theorem that $f^{(n-1)}(x) - P^{(n-1)}(x)$ has at least one real root. Therefore, for some ξ in the interval containing all x_i 's we have

$$S = \frac{1}{(n-1)!} f^{(n-1)}(\xi).$$

If the function $f^{(n-1)}(x)$ preserves the sign, we can uniquely determine the sign of sum S .

5 The proof of Theorem 1.1

Consider the open set in $\mathbb{R}^{n-1}$

$$\mathcal{M} = \{(\mu_1, \mu_2, \dots, \mu_{n-1}) : n > \mu_1 > \mu_2 > \dots > \mu_{n-1} > 0\}.$$

Let $\mathcal{C}$ denote the set of coefficients of polynomials having roots in $\mathcal{M}$,

$$\begin{aligned} \mathcal{C} &= \{(c_1, c_2, \dots, c_{n-1}) : (\exists (\mu_1, \mu_2, \dots, \mu_{n-1}) \in \mathcal{M}) \\ P(x) &= x^{n-1} - c_1 x^{n-1} + c_2 x^{n-2} + \dots + (-1)^{n-1} c_{n-1} \\ &= (x - \mu_1)(x - \mu_2) \dots (x - \mu_{n-1})\}. \end{aligned}$$

The Laplacian-like energy function $LEL : \mathcal{M} \rightarrow \mathbb{R}$, defined by

$$LEL(\mu_1, \mu_2, \dots, \mu_{n-1}) = \sqrt{\mu_1} + \sqrt{\mu_2} + \dots + \sqrt{\mu_{n-1}} + 0,$$

may then be represented as an implicit function of coefficients from $\mathcal{C}$. By the chain rule, for arbitrary k , $1 \leq k \leq n$, we have

$$\frac{\partial LEL}{\partial c_k} = \sum_{i=1}^n \frac{\partial LEL}{\partial \mu_i} \cdot \frac{\partial \mu_i}{\partial c_k} = \sum_{i=1}^n \frac{1}{2\sqrt{\mu_i}} \cdot \frac{\partial \mu_i}{\partial c_k}.$$

Therefore,

$$\frac{\partial LEL}{\partial c_k} = \frac{1}{2} \sum_{i=1}^n \frac{1}{\sqrt{\mu_i}} \cdot (-1)^{k-1} \cdot \frac{\mu_i^{n-1-k}}{\omega'(\mu_i)} = \frac{(-1)^{k-1}}{2} \sum_{i=1}^n \frac{\mu_i^{n-1-k}}{\omega'(\mu_i) \sqrt{\mu_i}}.$$

Let $f(x) = x^{n-k-\frac{3}{2}}$ and according to the previous consideration we have to examine the sign of $(n-2)$ -th derivative of f .

$$f^{(n-2)}(x) = \left(x^{n-k-\frac{3}{2}}\right)^{(n-1)} = \left(n-k-\frac{3}{2}\right) \left(n-k-\frac{5}{2}\right) \cdots \left(\frac{3}{2}-k\right) x^{1/2-k}.$$

This means that the sign of $f^{(n-2)}(x)$ is equals to $(-1)^{k-1}$, and finally

$$\frac{\partial LEL}{\partial c_k} > 0,$$

and the Laplacian-like energy function is strictly increasing on $\mathcal{C}$ in each coordinate.

So far we have dealt with the case of distinct eigenvalues only. The remaining step is to consider the closure of $\mathcal{M}$

$$\overline{\mathcal{M}} = \{(\mu_1, \mu_2, \dots, \mu_{n-1}) : n \geq \mu_1 \geq \mu_2 \geq \dots \geq \mu_{n-1} \geq 0\}.$$

Its image under equations (1) is the set $\overline{\mathcal{C}}$ of coefficients of polynomials having roots in $\overline{\mathcal{M}}$. The previous proof on $\overline{\mathcal{M}}$ can not be applied since F is not injection - but, it can be done simply using the continuity.

Suppose that we have two points x_1 and x_2 in $\overline{\mathcal{M}} \setminus \mathcal{M}$ such that $c_k(x_1) < c_k(x_2)$ for some $k \in \{1, 2, \dots, n-1\}$, $c_j(x_1) = c_j(x_2)$ for $j \neq k$ and $LEL(x_1) > LEL(x_2)$. Functions c_k and LEL are continuous, so we can find disjoint balls $B_1(x_1, \varepsilon) = \{x \mid \|x - x_1\| < \varepsilon\}$ and $B_2(x_2, \varepsilon) = \{x \mid \|x - x_2\| < \varepsilon\}$ such that $c_k(x) < c_k(y)$ and $LEL(x) > LEL(y)$ for all $x \in B_1(x_1, \varepsilon)$ and $y \in B_2(x_2, \varepsilon)$. The set $\overline{\mathcal{M}} \setminus \mathcal{M}$ has empty interior, so there exists points $x' \in B_1(x_1, \varepsilon) \cap \mathcal{M}$ and $y' \in B_1(x_1, \varepsilon) \cap \mathcal{M}$ such that $c_k(x') < c_k(y')$ and $LEL(x') > LEL(y')$. Surfaces $c_j(x) = C_j$ (for $j \neq k$ and some constants C_j) can be arbitrarily close to the surfaces passing through points x_1 and x_2 by continuity. Therefore, it follows that such surfaces have common points with both $B_1(x_1, \varepsilon)$ and $B_2(x_2, \varepsilon)$ - so we can choose x' and y' such that $c_j(x') = c_j(y')$ for every $j \neq k$.

This is a contradiction with the first part of the proof, since $x', y' \in \mathcal{M}$. Finally, the Laplacian-like energy function LEL , which is strictly increasing on $\mathcal{C}$ in each coordinate, must be strictly increasing on $\overline{\mathcal{C}}$ as well.

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