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Some New Sharp Bounds on the Distance Spectral Radius of Graph *

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Abstract: The D-eigenvalues $\{\lambda_1, \lambda_2, \cdots, \lambda_n\}$ of a graph G are the eigenvalues of its distance matrix D and form the D-spectrum of G denoted by spec$D(G)$. The greatest D-eigenvalue is called the distance spectral radius of G, denoted by λ_1. In this paper we obtain some new lower and upper bounds for λ_1, and also show that all of our bounds are sharp.

1 Introduction

Let G be a connected graph with vertex set $V(G) = \{v_1, v_2, \cdots, v_n\}$. The distance matrix $D = D(G)$ of G is defined so that its (i, j)-entry, d_{ij} , is equal to $d_G(v_i, v_j)$, the distance (length of the shortest path) between the vertices v_i and v_j of G. Then the distance matrix of a connected distance graph is irreducible and symmetric. The eigenvalues of $D(G)$ are said to be the D-eigenvalues of G and form the D-spectrum of G, denoted by spec$D(G)$. Since the distance matrix is symmetric, all

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its eigenvalues $\lambda_i, i = 1, 2, \cdots, n$, are real and can be labeled so that $\lambda_1 \geq \lambda_2 \geq \cdots \geq \lambda_n$.

The ordinary spectrum of G, which is the spectrum of the adjacency matrix of G is well studied and many properties of graphs in connection with the spectrum are revealed during the past years. For details see the book [1] and the references cited therein. The greatest eigenvalue of the distance matrix of a graph G, λ_1 is called the distance spectral radius. For some recent works on distance spectrum of graphs, see [2–4]. In [3], the author gave some lower bounds for λ_1 and characterize those graphs for which these bounds are best possible. In this paper, we present some new lower and upper bounds for λ_1, and also prove that all of our bounds are sharp.

2 Main results and lemmas

Definition 2.1 *Let G be a graph with $V(G) = \{v_1, v_2, \cdots, v_n\}$ and a distance matrix $D = (d_{ij})$. Then the distance degree of v_i, denoted by D_i is given by $D_i = \sum\limits_{j=1}^{n} d_{ij}$.*

Definition 2.2 *Let G be a graph with $V(G) = \{v_1, v_2, \cdots, v_n\}$, a distance matrix $D = (d_{ij})$, and a distance degree sequence $\{D_1, D_2, \cdots, D_n\}$. Then the second distance degree of v_i, denoted by T_i is given by $T_i = \sum\limits_{j=1}^{n} d_{ij} D_j$.*

Definition 2.3 *Let G be a graph with distance degree sequence $\{D_1, D_2, \cdots, D_n\}$ and second distance degree sequence $\{T_1, T_2, \cdots, T_n\}$. Then G is pseudo k-distance regular if $\frac{T_i}{D_i} = k$ for all $1 \leq i \leq n$.*

Definition 2.4 *Let A be a matrix. We use $s_i(A)$ to denote the ith row sum of A.*

The proof of Lemma 2.1 in [5] implies the following slightly stronger version.

Lemma 2.1 *[5] Let A be a real symmetric $n \times n$ matrix, and let λ be an eigenvalue of A with an eigenvector x all of whose entries are nonnegative. Then*

$$\min_{1 \leq i \leq n} s_i(A) \leq \lambda \leq \max_{1 \leq i \leq n} s_i(A).$$

Moveover, if all entries of x are positive then either of the equalities holds if and only if the row sums of A are all equal.

Lemma 2.2 *[6] Let A be a nonnegative irreducible $n \times n$ matrix with spectral radius λ. Then λ is a simple eigenvalue of A, and if x is an eigenvector with eigenvalue λ, then all entries of x are nonzero and have the same sign.*

Corollary 2.1 *Let A be a nonnegative irreducible $n\times n$ matrix with spectral radius λ. Then*

$$\min_{1\le i\le n} s_i(A) \le \lambda \le \max_{1\le i\le n} s_i(A).$$

Equalities holds if and only if the row sums of A are all equal.

Theorem 2.1 *Let G be a connected graph with distance degree sequence $\{D_1, D_2, \cdots, D_n\}$, second distance degree sequence $\{T_1, T_2, \cdots, T_n\}$, and distance spectral radius λ_1. Then*

$$\min\{m_i : \quad 1 \le i \le n\} \le \lambda_1 \le \max\{m_i : \quad 1 \le i \le n\}. \tag{1}$$

where $m_i = \frac{T_i}{D_i}$. Moveover, any equality holds if and only if G is pseudo distance regular.

Proof. Let $M = diag(D_1, \cdots, D_n)$. Then (i, j)-entry of $M^{-1}DM$ is $\frac{d_{ij}D_j}{D_i}$, and

$$s_i(M^{-1}DM) = \frac{T_i}{D_i} = m_i \qquad (1 \le i \le n).$$

It is not difficult to see that $M^{-1}DM$ is a nonnegative irreducible $n \times n$ matrix with spectral radius λ_1. Now we use Corollary 2.1 by taking $A = M^{-1}DM$, the desired result holds.

Now we assume that G is pseudo distance regular, then $m_i = \frac{T_i}{D_i} = k$ for all i, and hence $\min\{m_i : \quad 1 \le i \le n\} = \max\{m_i : \quad 1 \le i \le n\} = k$. Thus both of the equalities hold.

Conversely, if one of the equalities holds, by Corollary 2.1, the row sums of $M^{-1}DM$ are all equal. That is, $m_i = \frac{T_i}{D_i}$ $(1 \le i \le n)$ are all equal, which may implies that G is a pseudo distance regular graph. $\square$

Theorem 2.2 *Let G be a connected graph with second distance degree sequence $\{T_1, T_2, \cdots, T_n\}$, and distance spectral radius λ_1. Then*

$$\min\{\sqrt{T_i} : \quad 1 \leq i \leq n\} \leq \lambda_1 \leq \max\{\sqrt{T_i} : \quad 1 \leq i \leq n\}. \tag{2}$$

Moveover, any equality holds if and only if G has same value of T_i for all i.

Proof. Let $D = (d_{ij})$ be the distance matrix of G and $\{D_1, D_2, \cdots, D_n\}$ be the distance degree sequence of G. Since $(D^2)_{ij} = \sum\limits_{k=1}^{n} d_{ik}d_{kj}$, we have

$$\begin{aligned} s_i(D^2) &= \sum_{j=1}^{n}\sum_{k=1}^{n} d_{ik}d_{kj} \\ &= \sum_{k=1}^{n} d_{ik}\sum_{j=1}^{n} d_{kj} \\ &= \sum_{k=1}^{n} d_{ik}D_k \\ &= T_i \end{aligned}$$

Let x be an eigenvector corresponding to λ_1, all of whose entries are positive, that is, $Dx = \lambda_1 x$, then $D^2x = \lambda_1^2 x$. By Lemma 2.1,

$$\min\{T_i : \quad 1 \leq i \leq n\} \leq \lambda_1^2 = \lambda(D^2) \leq \max\{T_i : \quad 1 \leq i \leq n\}.$$

Thus $\min\{\sqrt{T_i} : \quad 1 \leq i \leq n\} \leq \lambda_1 \leq \max\{\sqrt{T_i} : \quad 1 \leq i \leq n\}$.

Now we assume that G has same value of T_i for all i, then $\min\{\sqrt{T_i} : \quad 1 \leq i \leq n\} = \max\{\sqrt{T_i} : \quad 1 \leq i \leq n\}$, both of the equalities hold.

Conversely, if one of the equalities holds, that is, $\lambda_1^2 = \min\{T_i : \quad 1 \leq i \leq n\}$ or $\lambda_1^2 = \max\{T_i : \quad 1 \leq i \leq n\}$. By Corollary 2.1, $s_i(D^2) = T_i^2$ $(1 \leq i \leq n)$ all are equal. So G has same value of T_i for all i. $\square$

Theorem 2.3 *Let G be a connected graph of order n, and λ_1 be the distance spectral radius, then*

$$\lambda_1 \leq \max\{\sqrt{m_i m_j} : \quad 1 \leq i, j \leq n\}, \tag{3}$$

where $m_i = \frac{T_i}{D_i}$. Moreover, the equality holds if and only if G is a pseudo distance regular graph.

Proof. Let $M = diag(D_1, \cdots, D_n)$ and $x = (x_1, x_2, \cdots, x_n)^T$ be an eigenvector of $M^{-1}DM$ corresponding to the eigenvalue λ_1. Also let one entry, say x_i, be equal to 1 and the other entries be less than or equal to 1, that is, $x_i = 1$ and $0 \leq x_k \leq 1$ for any k. Let $x_j = \max\{x_k : \ k \neq i\}$.

Now the (i, j)-entry of $M^{-1}DM$ is $\dfrac{d_{ij}D_j}{D_i}$, and

$$M^{-1}DMx = \lambda_1 x \qquad (4)$$

From the ith equation of (4),

$$\begin{aligned}
\lambda_1 x_i &= \sum_{k=1}^{n} \frac{d_{ik}D_k x_k}{D_i} \\
&= \frac{1}{D_i}\sum_{k=1}^{n} d_{ik}D_k x_k \\
&\leq \frac{d_{ii}D_i x_i}{D_i} + \frac{x_j}{D_i}\sum_{k=1,k\neq i}^{n} d_{ik}D_k \\
&= \frac{x_j}{D_i}\sum_{k=1,k\neq i}^{n} d_{ik}D_k \\
&= \frac{x_j}{D_i}\sum_{k=1}^{n} d_{ik}D_k \\
&= \frac{T_i}{D_i}x_j = m_i x_j \qquad (5)
\end{aligned}$$

From the jth equation of (4),

$$\begin{aligned}
\lambda_1 x_j &= \sum_{k=1}^{n} \frac{d_{jk}D_k x_k}{D_j} \\
&= \frac{1}{D_j}\sum_{k=1}^{n} d_{jk}D_k x_k \\
&\leq \frac{1}{D_j}\sum_{k=1}^{n} d_{jk}D_k \\
&= \frac{T_j}{D_j} \\
&= m_j \qquad (6)
\end{aligned}$$

Combing (5), (6) and $x_i = 1$, we get $\lambda_1^2 \leq m_i m_j$.

Therefore, $\lambda_1 \leq \sqrt{m_i m_j} \leq \max\{\sqrt{m_i m_j} : \ 1 \leq i, j \leq n\}$.

Now we assume that G is pseudo distance regular, so $\frac{T_i}{D_i} = k$ or $T_i = kD_i$ for all i. Then

$$D(D_1, D_2, \cdots, D_n)^T = k(D_1, D_2, \cdots, D_n)^T$$

showing that $(D_1, D_2, \cdots, D_n)^T$ is an eigenvector corresponding to k. Note that $\lambda_1 \leq \sqrt{k^2}$, we have $\lambda_1 = k$. Thus the equality holds.

Conversely, if λ_1 attains the upper bound then all equalities in the above argument must hold. In particular, from (6) that $x_k = 1$, for $1 \leq k \leq n$, that is, $x = (1, 1, \cdots, 1)^T$. Hence $M^{-1}DM(1, 1, \cdots, 1)^T = \lambda_1(1, 1, \cdots, 1)^T$, this then implies that

$$\frac{T_k}{D_k} = \frac{\sum\limits_{j=1}^{n} d_{kj} D_j}{D_j} = \lambda_1$$

or in other words G is pseudo distance regular. $\square$

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