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# The Homfly Polynomial for Even Polyhedral Links

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**Abstract**

In this paper, a general approach is presented to compute the Homfly polynomials of even polyhedral links formed from a polyhedron by '$n$-branched curve and $2k$-twisted double-line covering'. We show that Homfly polynomials of the whole family of even polyhedral links can be obtained from the Tutte polynomial of the 1-skeleton of the polyhedron by special parametrizations. As applications, by using computer algebra (Maple) techniques, Homfly polynomials of even Platonic polyhedral links are calculated.

## 1 Introduction

Knots and links occurs in proteins, and knotted and linked DNA also exist in nature. In addition, chemists and molecular biologists have succeeded in synthesizing many knotted and linked molecules. See [1] and references therein. As the potential structure for synthesizing new types of topologically complex molecules, in a series of papers [2-6], Qiu etc introduced several types of polyhedral links with highly symmetry. In this paper, we restrict ourselves to one type of such polyhedral links introduced in [6], formed from a polyhedron by '$n$-branched curve and $k$-twisted double-line covering'. Now we describe the construction of such polyhedral links.

Given a polyhedron, to construct a polyhedral link, two types of basic building blocks are needed. One is an $n$-branched curve designed to replace the vertex with degree $n$ of a polyhedron. The other is an $m$-twisted double-line ($m = 0, 1, 2, \cdots$), which is proposed to replace the edge of a polyhedron. See Fig. 1 for examples. By connecting these two building blocks we obtain an alternating link called a polyhedral link[1]. A polyhedral link having $m$-twisted double-line is called a $T_m$-polyhedral link. When $m = 2k$, we call a $T_m$-polyhedral link an even polyhedral link.

Fig. 1: A 5-branched curve and a 4-twisted double-line.

In [6], the authors also determined some parameters of Platonic polyhedral links including the component number, the crossing number, the linking number, the writhe and the configuration. However, there are some errors on the calculation of the Homfly polynomials of such links. For example, the Homfly polynomial for the $T_2$-tetrahedral link in the Appendix is wrong. In this paper we shall study the calculation of the Homfly polynomials of even polyheral links.

The Homfly polynomial is a very powerful invariant of oriented links, introduced in [7] and [8] independently. It is the generalization of the Alexander polynomial [9], Conway [10] polynomial and the Jones polynomial [11]. It is closely related to many other invariants [12] [13], say, component number, the genus, the braid index of links, etc. In particular, Homfly polynomials play a significant role in the analysis of chirality problems in chemistry [14] [1].

The surface of a polyhedron is topologically homeomorphic to the sphere $S^2$. Thus the graph consisting of vertices and edges of a polyhedron, i.e the 1-skeleton, is a planar graph via the well-known stereographic projection. Conversely, a planar graph, say the 3-cycle, is not necessarily the 1-skeleton of some polyhedron. The above construction of links from polyhedra can be naturally generalized to any plane graph. Furthermore, we can use different twisted double-lines to cover the edges. To be precise, let $G$ be a plane graph with edge set $\{e_1, e_2, \cdots, e_q\}$, we use $m_i$-twisted double-line to cover the edge $e_i$ for $i = 1, 2, \cdots, q$. We call the link thus obtained the link corresponding to the plane graph $G$ and denote it by $D(G)$. When $m_i$ is even for each $i = 1, 2, \cdots, q$, we call $D(G)$ to be an even link corresponding to $G$ and denote it by $D_E(G)$. When $m_i = 2k$ for each

[1]The polyhedral link in this paper is the mirror image of the polyhedral link defined in [6].

$i = 1, 2, \cdots, q$, we denote $D(G)$ by $D_{2k}(G)$.

Let $\widehat{G}$ is the plane graph obtained from $G$ by replacing the edge $e_i$ by a path $P_i$ with length $n_i$. $D(G)$ is actually can be obtained from $\widehat{G}$ via the the well-known medial construction in knot theory [13][15]. Obviously $D_1(G)$ i.e. $m_i = 1$ for each $i = 1, 2, \cdots, q$ in $D(G)$ is the link obtained from the plane graph $G$ by the medial construction. And $D_2(G)$ were once constructed and their Homfly polynomials were once computed in [16]. In this paper, we generalize the method in [16] for computing the Hofmly polynomials of $D_2(G)$ to deal with the general even link $D_E(G)$. The orientations of $D_E(G)$ will be discussed in Section 3. We shall build a relation between the weighted Tutte polynomial of the plane graph $G$ and the Homfly polynomial of the corresponding even link $D_E(G)$. As applications, using computer algebra (Maple) techniques Homfly polynomials of some even Platonic polyhedral links are calculated.

The graph $G = (V, E)$ in this paper allows loops and multiple edges. For $e \in E$ we use $G - e$ and $G/e$ to denote the graphs obtained from $G$ by deleting and contracting (that is, deleting the edge and identifying its ends) the edge $e$, respectively. In particular, If $e$ is a loop of $G$, then $G - e = G/e$. Let $S$ be a set, we denote by $|S|$ the cardinality of the set $S$.

## 2 The Tutte and Homfly polynomials

The Tutte polynomial for graphs was constructed by Tutte in 1954 [17], building on his work seven years earlier [18]. L. Traldi generalized it to weighted graphs in 1989 [19], he introduced a dichromatic polynomial for weighted graphs. We shall call the dichromatic polynomial for weighted graphs the weighted Tutte polynomial.

A weighted graph $\widetilde{G}$ is a graph $G$ together with a weight function $w$ mapping the edge set $E$ of $G$ into some commutative ring $R$ with unity 1. If $e$ is an edge of the weighted graph $\widetilde{G}$, then $w(e)$ is the weight of $e$.

**Definition 2.1** *[19] The weighted Tutte polynomial $Q_{\widetilde{G}}(t, z)$ of a weighted graph $\widetilde{G}$ can be defined by the following recursive rules:*

1. *If $\widetilde{G}$ is an edgeless graph with $n \geq 1$ vertices, then*

$$Q_{\widetilde{G}}(t, z) = t^n. \tag{1}$$

2. *If $e$ is a loop of $\widetilde{G}$, then*

$$Q_{\widetilde{G}}(t, z) = (1 + w(e)z)Q_{\widetilde{G}-e}(t, z). \tag{2}$$

*If $e$ is not a loop of $\widetilde{G}$, then*

$$Q_{\widetilde{G}}(t,z) = Q_{\widetilde{G}-e}(t,z) + w(e)Q_{\widetilde{G}/e}(t,z). \tag{3}$$

We should point out that despite the weighted Tutte polynomial is defined to be a polynomial in $t$ and $z$, we do not require that $t$ and $z$ be indeterminates; indeed, it is often convenient to treat $t$ and $z$ as elements of $R$.

The following Lemma opens out the relation between the Tutte polynomial and the weighted Tutte polynomial, and will be used later.

**Lemma 2.2** *Let $G^w$ be a connected weighted graph the weight of each of whose edges is $w$. Let $T_G(x,y)$ be the Tutte polynomial of the unweighted graph $G$. Then*

$$Q_{G^w}(t,z) = w^{|V|-1}t\ T_G(1+\frac{t}{w}, 1+wz). \tag{4}$$

**Proof.** Given a graph $G=(V,E)$, its rank is equal to $|V|-k(G)$, where $k(G)$ is the number of connected components of $G$. The Tutte polynomial is defined as [20]

$$T_G(x,y) = \sum_{A\subseteq E}(x-1)^{r(E)-r(A)}(y-1)^{|A|-r(A)}, \tag{5}$$

where $r(A)$ is the rank of the spanning subgraph $(V,A)$. Note that the weighted Tutte polynomial $Q_{\widetilde{G}}(t,z)$ can also be written as [19]

$$Q_{\widetilde{G}}(t,z) = \sum_{A\subseteq E}\left(\prod_{a\in A} w(a)\right) t^{k(A)}z^{|A|-r(A)}. \tag{6}$$

When the weight of each edge of $\widetilde{G}$ is $w$, we have

$$\begin{aligned} Q_{G^w}(t,z) &= \sum_{A\subseteq E} w^{|A|}t^{k(A)}z^{|A|-r(A)} \\ &= \sum_{A\subset E} w^{r(A)}t^{|V|-r(A)}(wz)^{|A|-r(A)} \\ &= w^{r(E)}t\sum_{A\subseteq E}(\frac{t}{w})^{r(E)-r(A)}(wz)^{|A|-r(A)} \\ &= w^{|V|-1}t\ T_G(1+\frac{t}{w}, 1+wz). \end{aligned}$$

This completes the proof of Lemma 2.2.□

**Definition 2.3** *[12] The Homfly polynomial of an oriented link $L$, denoted by $P_L(v,z)$, can be defined by the three following axioms.*

1. *$P_L(v,z)$ is invariant under ambient isotopy of $L$.*

2. *If $L$ is the trivial knot then*

$$P_L(v,z) = 1. \tag{7}$$

3. *Skein relation:*

$$v^{-1}P_{L_+}(v,z) - vP_{L_-}(v,z) - zP_{L_0}(v,z) = 0, \tag{8}$$

*where $L_+, L_-$ and $L_0$ are link diagrams which are identical except near one crossing where they are as in Fig. 2 and are called a skein triple.*

Fig. 2: $L_+$ (positive), $L_-$ (negative) and $L_0$

The Homfly polynomial possesses the following basic properties, see [7], [12], [21]:

(1) If $L$ is the connected sum of links $L_1$ and $L_2$, then

$$P_L(v,z) = P_{L_1}(v,z)P_{L_2}(v,z).$$

(2) If $L$ is the disjoint union of links $L_1$ and $L_2$, then

$$P_L(v,z) = \frac{v^{-1}-v}{z}P_{L_1}(v,z)P_{L_2}(v,z).$$

(3) If $L^r$ is the reverse of $L$, then $P_{L^r}(v,z) = P_L(v,z)$.

(4) If $L^m$ is the mirror image of $L$, then $P_{L^m}(v,z) = P_L(-v^{-1},z)$.

# 3 Orientations of even links

To compute the Homfly polynomials of links, we need to assign orientations to them firstly.

Let $G$ be a plane graph, $D_E(G)$ be an even link constructed from $G$. We orient each $n$-branched curve counterclockwise as shown in Fig. 3 (left). The orientations naturally

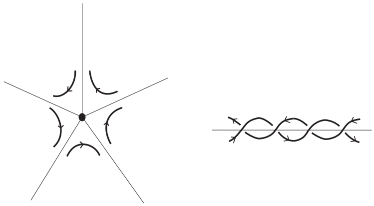


Fig. 3: An oriented 5-branched curve and an oriented 4-twisted double-line.

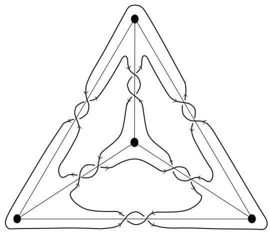


Fig. 4: Orientations of $D_2(K_4)$.

induce orientations of even twisted double-lines, see Fig. 3 (right). Thus we obtain an orientation of the whole even link. As an example, $D_2(K_4)$ is oriented as shown in Fig. 4

Under the orientation given above, $D_E(G)$ is a positive link, i.e. each crossing is positive. Let $f(G)$ be the number of faces (including the unbounded face) of the plane graph $G$, then $D_E(G)$ has $f(G)$ components, which were discussed in [6].

In the next section we shall establish a relation between the Homfly polynomial of $D_E(G)$ and the weighted Tutte polynomial of $\widetilde{G}$.

# 4 A general result

Let $G$ be a plane graph with edge set $\{e_1, e_2, \cdots, e_q\}$. Let $D_E(G)$ be the link diagram obtained from $G$ by replacing each edge $e_i$ by the $2n_i$-twisted double-line for $i = 1, 2, \cdots, q$.

The following lemma is the key to prove the main theorem.

**Lemma 4.1** *Let $e$ be an edge of the plane graph $G$. Let $D_E(G)$ be the even link corresponding to $G$. If the edge $e$ is replaced by the $2n$-twisted double-line, then*

*(1) when $e$ is a loop,*

$$P_{D_E(G)} = (\frac{z}{v^{-1} - v} + (\frac{v^{-1} - v}{z} - \frac{z}{v^{-1} - v})v^{2n})P_{D_E(G-e)}; \tag{9}$$

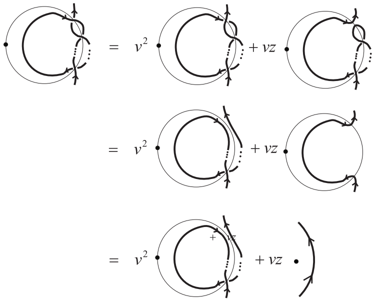
*(2) otherwise,*

$$P_{D_E(G)} = \frac{z}{v^{-1} - v}(1 - v^{2n})P_{D_E(G-e)} + v^{2n}P_{D_E(G/e)}, \tag{10}$$

*where $D_E(G-e)$ and $D_E(G/e)$ are the oriented even links obtained from $G-e$ and $G/e$ with original replacements unchanged, respectively.*

**Proof.** By the definition and properties of Homfly polynomial in Section 2, we obtain the following equations which we describe pictorially for simplicity.

1. If $e$ is a loop, then




So we have

$$P_{D_E(G)} = v^2 P_{D_E(G):e(2n-2)} + vzP_{D_E(G-e)}, \tag{11}$$

$$P_{D_E(G):e(2n-2)} = v^2 P_{D_E(G):e(2n-4)} + vzP_{D_E(G-e)}, \tag{12}$$

$$P_{D_E(G):e(2n-4)} = v^2 P_{D_E(G):e(2n-6)} + vzP_{D_E(G-e)}, \tag{13}$$

$$\cdots$$

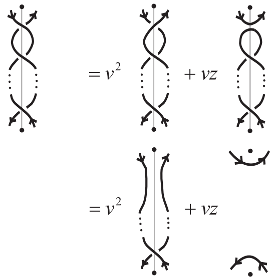
$$P_{D_E(G):e(2)} = v^2(\frac{v^{-1} - v}{z})P_{D_E(G-e)} + vzP_{D_E(G-e)}, \tag{14}$$

where $D_E(G) : e(2k)$ is the same to $D_E(G)$ except that the edge $e$ is replaced by a $2k$-twisted double-line.

It deserves pointing out (14) has another factor $\frac{v^{-1}-v}{z}$ produced by a trivial knot. Multiplying (12) by $v^2$, (13) by $v^4$ and so on, then adding all the equations, we obtain

$$\begin{aligned}
P_{D_E(G)} &= vz(1 + v^2 + \cdots + v^{2n-2})P_{D_E(G-e)} + v^{2n}(\frac{v^{-1} - v}{z})P_{D_E(G-e)} \\
&= (\frac{z}{v^{-1} - v}(1 - v^{2n}) + \frac{v^{-1} - v}{z}v^{2n})P_{D_E(G-e)} \\
&= (\frac{z}{v^{-1} - v} + (\frac{v^{-1} - v}{z} - \frac{z}{v^{-1} - v})v^{2n})P_{D_E(G-e)}.
\end{aligned}$$

2. If $e$ is not a loop, then




Thus we have

$$P_{D_E(G)} = v^2 P_{D_E(G):e(2n-2)} + vzP_{D_E(G-e)}, \tag{15}$$

$$P_{D_E(G):e(2n-2)} = v^2 P_{D_E(G):e(2n-4)} + vzP_{D_E(G-e)}, \tag{16}$$

$$P_{D_E(G):e(2n-4)} = v^2 P_{D_E(G):e(2n-6)} + vzP_{D_E(G-e)}, \tag{17}$$

$$\cdots$$

$$P_{D_E(G):e(2)} = v^2 P_{D_E(G/e)} + vzP_{D_E(G-e)}. \tag{18}$$

Multiplying (16) by $v^2$, (17) by $v^4$ and so on, then adding all the equations, we obtain

$$\begin{aligned} P_{D_E(G)} &= vz(1 + v^2 + \cdots + v^{2n-2})P_{D_E(G-e)} + v^{2n} P_{D_E(G/e)} \\ &= \frac{z}{v^{-1} - v}(1 - v^{2n})P_{D_E(G-e)} + v^{2n} P_{D_E(G/e)}. \end{aligned}$$

This completes the proof of Lemma 4.1. □

Using Lemma 4.1, we can prove the following main theorem.

**Theorem 4.2** *Let $G$ be a plane graph. Let $D_E(G)$ be the oriented even link described above. Let $\widetilde{G}$ be the weighted graph with $w(e) = \frac{v^{-1}-v}{z}\frac{v^{2n}}{1-v^{2n}}$ if $e$ is replaced by the $2n$-twisted double-line. Then*

$$P_{D_E(G)}(v, z) = (\frac{z}{v^{-1} - v})\left(\prod_{e \in E(G)} \nu(e)\right) Q_{\widetilde{G}}(\frac{v^{-1} - v}{z}, \frac{v^{-1} - v}{z}), \tag{19}$$

*where $\nu(e)$ is $\frac{z}{v^{-1}-v}(1 - v^{2n})$ if the edge $e$ is replaced by the $2n$-twisted double-line.*

**Proof.** By induction on the number of edges of $G$. If $G$ ia an edgeless graph with $p$ vertices, $D_E(G)$ is the unlink with $p$ components. Thus $P_{D_E(G)}(v,z) = (\frac{v^{-1}-v}{z})^{p-1}$ by the second property of the Homfly polynomial. The right hand of (19) equals to

$$(\frac{z}{v^{-1}-v})(\frac{v^{-1}-v}{z})^p = (\frac{v^{-1}-v}{z})^{p-1}.$$

Theorem 4.2 holds. If $E(G) \neq \emptyset$, suppose that $e$ is an edge and it is replaced in $D_E(G)$ by the $2n$-twisted double-line.

1. when $e$ is a loop.

   By induction hypothesis and (2), we have

$$\begin{aligned}
& P_{D_E(G-e)}(v,z) \\
= & (\frac{z}{v^{-1}-v})\left(\prod_{e\in E(G-e)} v(e)\right) Q_{\widetilde{G}-e}(\frac{v^{-1}-v}{z},\frac{v^{-1}-v}{z}) \\
= & (\frac{z}{v^{-1}-v})\left(\prod_{e\in E(G-e)} v(e)\right) \frac{1}{1+\frac{v^{-1}-v}{z}w(e)} Q_{\widetilde{G}}(\frac{v^{-1}-v}{z},\frac{v^{-1}-v}{z}) \\
= & (\frac{z}{v^{-1}-v})\left(\prod_{e\in E(G-e)} v(e)\right) \frac{1}{1+(\frac{v^{-1}-v}{z})^2\frac{v^{2n}}{1-v^{2n}}} \times \\
& Q_{\widetilde{G}}(\frac{v^{-1}-v}{z},\frac{v^{-1}-v}{z}) \\
= & (\frac{z}{v^{-1}-v})\frac{1}{1+(\frac{v^{-1}-v}{z})^2\frac{v^{2n}}{1-v^{2n}}}\frac{1}{\frac{z}{v^{-1}-v}(1-v^{2n})}\left(\prod_{e\in E(G)} v(e)\right) \times \\
& Q_{\widetilde{G}}(\frac{v^{-1}-v}{z},\frac{v^{-1}-v}{z}) \\
= & \frac{1}{\frac{z}{v^{-1}-v}+(\frac{v^{-1}-v}{z}-\frac{z}{v^{-1}-v})v^{2n}}(\frac{z}{v^{-1}-v})\left(\prod_{e\in E(G)} v(e)\right) \times \\
& Q_{\widetilde{G}}(\frac{v^{-1}-v}{z},\frac{v^{-1}-v}{z}).
\end{aligned}$$

   By Lemma 4.1 (1), we have

$$\begin{aligned}
P_{D_E(G)} &= (\frac{z}{v^{-1}-v}+(\frac{v^{-1}-v}{z}-\frac{z}{v^{-1}-v})v^{2n})P_{D_E(G-e)} \\
&= (\frac{z}{v^{-1}-v})\left(\prod_{e\in E(G)} \nu(e)\right) Q_{\widetilde{G}}(\frac{v^{-1}-v}{z},\frac{v^{-1}-v}{z}).
\end{aligned}$$

   Hence, Theorem 4.2 holds.

2. when $e$ is not a loop.

   By induction hypothesis, we have

$$P_{D_E(G-e)}(v,z) = (\frac{z}{v^{-1}-v})\left(\prod_{e\in E(G-e)} v(e)\right) Q_{\widetilde{G}-e}(\frac{v^{-1}-v}{z}, \frac{v^{-1}-v}{z}).$$

   and

$$P_{D_E(G/e)}(x,y,z) = (\frac{z}{v^{-1}-v})\left(\prod_{e\in E(G/e)} v(e)\right) Q_{\widetilde{G}/e}(\frac{v^{-1}-v}{z}, \frac{v^{-1}-v}{z}).$$

   By Lemma 4.1 (2) and Equation (3), we have

$$\begin{aligned}
& P_{D_E(G)}(v,z) \\
= & \frac{z}{v^{-1}-v}(1-v^{2n})P_{D_E(G-e)} + v^{2n}P_{D_E(G/e)} \\
= & (\frac{z}{v^{-1}-v})^2(1-v^{2n})\left(\prod_{e\in E(G-e)} v(e)\right) Q_{\widetilde{G}-e}(\frac{v^{-1}-v}{z}, \frac{v^{-1}-v}{z}) + \\
& v^{2n}(\frac{z}{v^{-1}-v})\left(\prod_{e\in E(G/e)} v(e)\right) Q_{\widetilde{G}/e}(\frac{v^{-1}-v}{z}, \frac{v^{-1}-v}{z}) \\
= & \frac{z}{v^{-1}-v}\left(\prod_{e\in E(G)} v(e)\right) Q_{\widetilde{G}-e}(\frac{v^{-1}-v}{z}, \frac{v^{-1}-v}{z}) + (\frac{z}{v^{-1}-v}) \times \\
& \left(\prod_{e\in E(G)} v(e)\right) \frac{v^{-1}-v}{z}\frac{v^{2n}}{1-v^{2n}} Q_{\widetilde{G}/e}(\frac{v^{-1}-v}{z}, \frac{v^{-1}-v}{z}) \\
= & \frac{z}{v^{-1}-v}\left(\prod_{e\in E(G)} v(e)\right) \times \\
& \left(Q_{\widetilde{G}-e}(\frac{v^{-1}-v}{z}, \frac{v^{-1}-v}{z}) + w(e)Q_{\widetilde{G}/e}(\frac{v^{-1}-v}{z}, \frac{v^{-1}-v}{z})\right) \\
= & \frac{z}{v^{-1}-v}\left(\prod_{e\in E(G)} v(e)\right) Q_{\widetilde{G}}(\frac{v^{-1}-v}{z}, \frac{v^{-1}-v}{z}).
\end{aligned}$$

   Hence, Theorem 4.2 also holds. This complete the proof of Theorem 4.2.

□

The following corollary is a direct consequence of Theorem 4.2.

**Corollary 4.3** *Let $P$ be a polyhedron with $q$ edges. Let $D_{2n}(P)$ be the $T_{2n}$-polyhedral link constructed from $P$. Assigning an orientation to $D_{2n}(P)$ as described in Section 3. Let*

*$P^w$ be the weighted 1-skeleton of $P$ each of whose edges is assigned the weight $\frac{v^{-1}-v}{z}\frac{v^{2n}}{1-v^{2n}}$. Then*

$$P_{D_{2n}(P)}(v,z) = (\frac{z}{v^{-1}-v})^{q+1}(1-v^{2n})^q Q_{P^w}(\frac{v^{-1}-v}{z}, \frac{v^{-1}-v}{z}).$$

**Theorem 4.4** *Let $P$ be a polyhedron with $p$ vertices and $q$ edges. Let $D_{2n}(P)$ be the $T_{2n}$-polyhedral link constructed from $P$. Assigning an orientation to $D_{2n}(P)$ as described in Section 3. Let $T_P(x,y)$ be the Tutte polynomial of the 1-skeleton of $P$. Then*

$$\begin{aligned} P_{D_{2n}(P)}(v,z) &= (\frac{z}{v^{-1}-v}(1-v^{2n}))^{q-p+1} v^{2n(p-1)} \times \\ & \quad T_P(v^{-2n}, 1+(\frac{v^{-1}-v}{z})^2 \frac{v^{2n}}{1-v^{2n}}). \end{aligned} \tag{20}$$

**Proof.** By Corollary 4.3 and Lemma 2.2, we obtain

$$\begin{aligned} & P_{D_{2n}(P)}(v,z) \\ = & (\frac{z}{v^{-1}-v})^{q+1}(1-v^{2n})^q Q_{P^w}(\frac{v^{-1}-v}{z}, \frac{v^{-1}-v}{z}) \\ = & (\frac{z}{v^{-1}-v})^{q+1}(1-v^{2n})^q (\frac{v^{-1}-v}{z}\frac{v^{2n}}{1-v^{2n}})^{p-1}\frac{v^{-1}-v}{z} \times \\ & T_P(v^{-2n}, 1+(\frac{v^{-1}-v}{z})^2 \frac{v^{2n}}{1-v^{2n}}) \\ = & (\frac{z}{v^{-1}-v}(1-v^{2n}))^{q-p+1} v^{2n(p-1)} T_P(v^{-2n}, 1+(\frac{v^{-1}-v}{z})^2 \frac{v^{2n}}{1-v^{2n}}). \end{aligned}$$

This completes the proof of Theorem 4.4. □

**Remark 4.5** *In the case of $n = 1$, Theorem 4.4 reduces to Proposition 1 in [16].*

# 5 Tutte polynomials for 1-skeletons of Platonic polyhedra

The Tutte polynomial is a very powerful invariant of the graph up to isomorphism. It contains a great deal of information of the graph. See [20] for a survey. However, computing the Tutte polynomials is, in general, very difficult. In fact, one has proved that the problem of evaluating the Tutte polynomial at any point $(a, b)$ is #P-hard except that when $(a, b)$ is on one special hyperbola and 8 special points [22]. The Tutte polynomial of some (relatively simple) well-known graph families has been computed, including complete graphs [23], ladders (prisms) [24] [25], wheels (pyramids) [24] [26], and some other families [27] [28] [29].

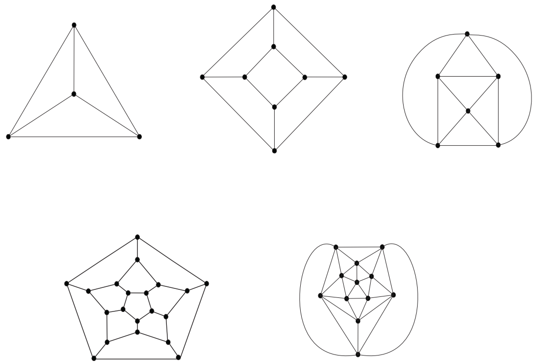


Fig. 5: 1-skeletons of five regular polyhedra.

There are five Platonic polyhedra together [30] and their 1-skeletons are shown in Fig. 5. Note that the 1-skeletons of hexahedron and octahedron are dual graphs, and the 1-skeletons of dodecahedron and icosahedron are dual graphs. Let $G$ be a connected plane graph with dual $G^*$, then [20]

$$T_{G^*}(x,y) = T_G(y,x). \tag{21}$$

Note that the tetrahedron is the wheel graph with three spokes, the hexahedron is the prisms with four rungs. Using the formulae given in [24] and [25], we can obtain the Tutte polynomials of 1-skeletons of tetrahedron and hexahedron. The Tutte polynomial of 1-skeleton of dodecahedron can be found in [31]. We point out that the Maple 12 has a function called TuttePolynomial in the GraphTheory package, which can also be used to calculate the Tutte polynomial of 1-skeletons of five Platonic polyhedra. Let

$$T_G(x,y) = \sum_{i,j} c_{i,j} x^i y^j. \tag{22}$$

By (21), here we only list non-zero coefficients of the Tutte polynomial of 1-skeletons of tetrahedron, hexahedron and dodecahedron in Tables 1-3, respectively.

| $c_{i,j}$ | j=0 | j=1 | j=2 | j=3 |
|---|---|---|---|---|
| i=0 | | 2 | 3 | 1 |
| i=1 | 2 | 4 | | |
| i=2 | 3 | | | |
| i=3 | 1 | | | |

Table 1: $c_{i,j}$'s for tetrahedron.

| $c_{i,j}$ | j=0 | j=1 | j=2 | j=3 | j=4 | j=5 |
|---|---|---|---|---|---|---|
| i=0 | | 11 | 25 | 20 | 7 | 1 |
| i=1 | 11 | 46 | 39 | 8 | | |
| i=2 | 32 | 52 | 12 | | | |
| i=3 | 40 | 24 | | | | |
| i=4 | 29 | 6 | | | | |
| i=5 | 15 | | | | | |
| i=6 | 5 | | | | | |
| i=7 | 1 | | | | | |

Table 2: $c_{i,j}$'s for hexahedron.

| $c_{i,j}$ | j=0 | j=1 | j=2 | j=3 | j=4 | j=5 | j=6 | j=7 |
|---|---|---|---|---|---|---|---|---|
| i=0 | | 4 412 | 17 562 | 30 686 | 31 540 | 21 548 | 10 439 | 3 693 |
| i=1 | 4 412 | 38 864 | 95 646 | 115 448 | 82 550 | 38 322 | 12 046 | 2 542 |
| i=2 | 25 714 | 128 918 | 218 682 | 185 071 | 90 860 | 27 825 | 5 390 | 610 |
| i=3 | 72 110 | 245 880 | 295 915 | 174 870 | 57 735 | 11 230 | 1 240 | 60 |
| i=4 | 131 380 | 320 990 | 275 910 | 112 365 | 24 140 | 2 775 | 140 | |
| i=5 | 176 968 | 316 256 | 193 791 | 53 350 | 7 175 | 468 | 12 | |
| i=6 | 189 934 | 250 692 | 108 884 | 19 810 | 1 620 | 60 | | |
| i=7 | 170 690 | 167 140 | 50 850 | 5 870 | 270 | | | |
| i=8 | 132 920 | 96 400 | 19 980 | 1 350 | 30 | | | |
| i=9 | 91 740 | 48 710 | 6 510 | 220 | | | | |
| i=10 | 56 852 | 21 530 | 1 674 | 20 | | | | |
| i=11 | 31 792 | 8 198 | 306 | | | | | |
| i=12 | 16 016 | 2 610 | 30 | | | | | |
| i=13 | 7 216 | 660 | | | | | | |
| i=14 | 2 871 | 120 | | | | | | |
| i=15 | 989 | 12 | | | | | | |
| i=16 | 286 | | | | | | | |
| i=17 | 66 | | | | | | | |
| i=18 | 11 | | | | | | | |
| i=19 | 1 | | | | | | | |

| $c_{i,j}$ | j=8 | j=9 | j=10 | j=11 |
|---|---|---|---|---|
| i=0 | 950 | 170 | 19 | 1 |
| i=1 | 330 | 20 | | |
| i=2 | 30 | | | |

Table 3: $c_{i,j}$'s for dodecahedron, see [31].

# 6 Computational results for Homfly polynomials of even Platonic polyhedral links

In this section, we use Corollary 4.3 and Theorem 4.4 to compute the Homfly polynomials of even Platonic polyhedral links.

**1.** $T_{2n}$-tetrahedral link $A_n$.

The edges of 1-skeleton $T$ of the tetrahedron are labeled as shown in Fig. 6 and each edge is assigned the weight $w$. Now we calculate $Q_{T^w}(t,t)$ according to (6). The contribution of its each edge subset is listed in Table 4.

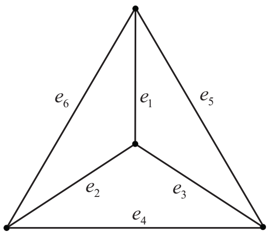



Fig. 6: Labels of edges of $K_4$.

| Edge subset $F$ | Contribution |
|---|---|
| $\emptyset$ | $t^4$ |
| 1-edge subsets | $6wt^3$ |
| 2-edge subsets | $15w^2t^2$ |
| $\{e_1,e_2,e_6\},\{e_1,e_3,e_5\},\{e_2,e_3,e_4\},\{e_4,e_5,e_6\}$ | $4w^3t^3$ |
| other 3-edge subsets | $16w^3t$ |
| 4-edge subsets | $15w^4t^2$ |
| 5-edge subsets | $6w^5t^3$ |
| 6-edge subsets | $w^6t^4$ |

Table 4: The contribution of each edge subset of $T$.

Hence, we have

$$Q_T(t,t) = t^4(1+w^6) + t^3(6w+4w^3+6w^5) + t^2(15w^2+15w^4) + 16tw^3.$$

By Corollary 4.3, we obtain

$$\begin{aligned} P_{A(n)}(v,z) &= (\frac{z}{v^{-1}-v})^3\{(1-v^{2n})^6 + 6(1-v^{2n})^5v^{2n} + \\ &\quad 15(1-v^{2n})^4v^{4n} + 16(1-v^{2n})^3v^{6n}\} + \end{aligned}$$

$$(\frac{z}{v^{-1}-v})\{4(1-v^{2n})^3v^{6n}+15(1-v^{2n})^2v^{8n}\}+$$
$$(\frac{v^{-1}-v}{z})\{6(1-v^{2n})v^{10n}\}+$$
$$(\frac{v^{-1}-v}{z})^3\{v^{12n}\}.$$

However, the expression we obtained is not a polynomial. By applying computer algebra (MAPLE) techniques, we can transform the expression into a polynomial for each $n = 1, 2, \cdots$. Some computational results are listed in Table 5.

| link | Homfly polynomial |
|---|---|
| $A(1)$ | $(6v^9+6v^7+3v^5+v^3)z^3+(-11v^{11}+7v^9+4v^7)z+$ $(6v^{13}-12v^{11}+6v^9)z^{-1}+(-v^{15}+3v^{13}-3v^{11}+v^9)z^{-3}$ |
| $A(2)$ | $(18v^{19}+24v^{17}+24v^{15}+6v^{21}+21v^{13}+15v^{11}+10v^9+6v^7+3v^5$ $+v^3)z^3+(-11v^{23}+4v^{15}-11v^{21}+7v^{19}+7v^{17}+4v^{13})z+$ $(6v^{25}-6v^{23}-6v^{21}+6v^{19})z^{-1}+(-v^{27}+3v^{25}-3v^{23}+v^{21})z^{-3}$ |
| $A(3)$ | $(54v^{25}+36v^{29}+54v^{23}+6v^{33}+51v^{21}+48v^{27}+45v^{19}+18v^{31}+$ $36v^{17}+28v^{15}+21v^{13}+15v^{11}+10v^9+6v^7+3v^5+v^3)z^3+$ $(-11v^{33}+7v^{27}-11v^{31}+7v^{25}+4v^{23}+7v^{29}+4v^{21}-11v^{35}+$ $4v^{19})z+(6v^{37}+6v^{29}-6v^{35}-6v^{31})z^{-1}+(-v^{39}+v^{33}+3v^{37}-$ $3v^{35})z^{-3}$ |
| $A(4)$ | $(6v^{45}+90v^{35}+96v^{33}+96v^{31}+78v^{37}+93v^{29}+36v^{41}+87v^{27}+$ $18v^{43}+78v^{25}+60v^{39}+66v^{23}+55v^{21}+45v^{19}+36v^{17}+28v^{15}+$ $21v^{13}+15v^{11}+10v^9+6v^7+3v^5+v^3)z^3+(-11v^{47}+7v^{35}$ $-11v^{45}+7v^{39}-11v^{41}-11v^{43}+7v^{37}+4v^{31}+4v^{29}+7v^{33}+$ $4v^{27}+4v^{25})z+(6v^{49}-6v^{41}-6v^{47}+6v^{39})z^{-1}+(-v^{51}+v^{45}$ $+3v^{49}-3v^{47})z^{-3}$ |

Table 5: Homfly polynomials of $A(n)$ for $n = 1, 2, 3, 4$.

**2.** $T_{2n}$-hexahedral link $B_n$ and $T_{2n}$-octahedral link $C_n$.

According to Table 2 and Theorem 4.4, we can obtain the formulae of Homfly polynomials of $B_n$ and $C_n$. However, they are too lengthy and we omit them here. Using Maple, we can obtain some computational results listed in Table 6.

**3.** $T_{2n}$-dodecahedral link $D_n$ and $T_{2n}$-icosahedral link $E_n$.

Similarly, we can obtain the formulae of Homfly polynomials of $D_n$ and $E_n$. We compute $P_{D(1)}(v, z)$ and $P_{E(1)}(v, z)$ by using Maple and list them in Tables 7 and 8.

# 7 Concluding remarks

Although Theorem 4.2 is not very difficult in mathematics, it converted the computation of the Homfly polynomial of all even polyhedral links to that of the Tutte polynomial

| link | Homfly polynomial |
|---|---|
| $B(1)$ | $(64v^{13}+104v^{17}+35v^{11}+96v^{15}+64v^{19}+15v^9+5v^7+v^5)z^5+(6v^{19}+72v^{17}+6v^{11}+18v^{13}-154v^{21}+52v^{15})z^3+(137v^{23}-211v^{21}+23v^{19}+39v^{17}+12v^{15})z+(-58v^{25}-150v^{21}+166v^{23}+8v^{17}+34v^{19})z^{-1}+(12v^{27}-48v^{25}+72v^{23}-48v^{21}+12v^{19})z^{-3}+(-v^{29}+10v^{23}+5v^{27}-5v^{21}+v^{19}-10v^{25})z^{-5}$ |
| $B(2)$ | $(1160v^{37}+64v^{43}+1456v^{35}+744v^{39}+1584v^{33}+911v^{25}+1155v^{27}+685v^{23}+1544v^{31}+489v^{21}+320v^{41}+330v^{19}+1384v^{29}+210v^{17}+126v^{15}+70v^{13}+35v^{11}+15v^9+5v^7+v^5)z^5+(-462v^{43}-136v^{39}+174v^{29}+228v^{31}+6v^{19}+106v^{27}-456v^{41}+60v^{25}+268v^{33}+36v^{23}-154v^{45}+90v^{37}+18v^{21}+222v^{35})z^3+(137v^{45}+137v^{47}+12v^{31}-211v^{43}+39v^{35}-211v^{41}+12v^{29}+39v^{33}+23v^{37}+23v^{39})z+(-58v^{49}+58v^{47}-8v^{37}+8v^{35}-108v^{43}+108v^{45}+42v^{39}-42v^{41})z^{-1}+(12v^{51}+12v^{41}-36v^{49}+24v^{47}+24v^{45}-36v^{43})z^{-3}+(-v^{53}+10v^{47}+5v^{51}-5v^{45}+v^{43}-10v^{49})z^{-5}$ |
| $C(1)$ | $(28v^{11}+76v^{13}+7v^9+v^7+139v^{15}+133v^{17})z^7+(183v^{15}+55v^{13}+177v^{17}+8v^{11}-423v^{19})z^5+(-984v^{19}+264v^{17}+12v^{13}+136v^{15}+572v^{21})z^3+(-441v^{23}+48v^{15}+1275v^{21}+297v^{17}-1179v^{19})z+(6v^{15}+190v^{17}-820v^{19}-850v^{23}+214v^{25}+1260v^{21})z^{-1}+(660v^{21}+330v^{25}-330v^{19}-660v^{23}-66v^{27}+66v^{17})z^{-3}+(-72v^{19}+180v^{25}-72v^{27}+12v^{29}-240v^{23}+12v^{17}+180v^{21})z^{-5}+(-v^{31}+21v^{21}-35v^{23}-21v^{27}+7v^{29}-7v^{19}+v^{17}+35v^{25})z^{-7}$ |
| $C(2)$ | $(7420v^{29}+8190v^{31}+5908v^{27}+4228v^{25}+5628v^{35}+133v^{41}+2932v^{37}+1660v^{21}+2764v^{23}+916v^{19}+931v^{39}+462v^{17}+7650v^{33}+210v^{15}+84v^{13}+28v^{11}+7v^9+v^7)z^7+(2262v^{33}+1473v^{27}+773v^{25}-423v^{43}+40v^{19}+2770v^{31}+355v^{23}+2282v^{29}-2115v^{41}+135v^{21}-3345v^{37}-162v^{35}-4053v^{39}+8v^{17})z^5+(-192v^{35}+928v^{33}+420v^{29}-2688v^{37}+572v^{45}+172v^{27}+732v^{41}+672v^{31}+36v^{25}+1716v^{43}-2380v^{39}+12v^{23})z^3+(1275v^{41}-1179v^{37}-1179v^{39}+1275v^{43}+297v^{35}+48v^{31}+48v^{29}+297v^{33}-441v^{47}-441v^{45})z+(196v^{35}-196v^{37}-624v^{39}+636v^{43}-636v^{45}+214v^{49}-6v^{33}+624v^{41}-214v^{47}+6v^{31})z^{-1}+(-66v^{51}+66v^{41}-330v^{45}+66v^{37}-198v^{39}+330v^{43}-66v^{47}+198v^{49})z^{-3}+(108v^{43}+12v^{39}-60v^{41}-60v^{45}-60v^{47}+108v^{49}+12v^{53}-60v^{51})z^{-5}+(-v^{55}+21v^{45}-35v^{47}-21v^{51}+v^{41}-7v^{43}+35v^{49}+7v^{53})z^{-7}$ |

Table 6: Homfly polynomials of $B(n)$ and $C(n)$ for $n=1,2$.

| link | Homfly polynomial |
|---|---|
| $D(1)$ | $(80076v^{29} + 147180v^{31} + 394820v^{35} + 859040v^{43} + 390180v^{47} + 18656v^{25} + 121020v^{49} + 867700v^{41} + 250680v^{33} + 7876v^{23} + 571000v^{37} + 748020v^{39} + 2991v^{21} + 40296v^{27} + 1001v^{19} + 683100v^{45} + 286v^{17} + 66v^{15} + 11v^{13} + v^{11})z^{11} + (78140v^{33} + 360330v^{39} - 463310v^{51} + 16128v^{29} + 3640v^{45} - 469710v^{47} + 336090v^{43} + 6140v^{27} + 37452v^{31} + 108v^{21} + 426180v^{41} + 2010v^{25} + 247140v^{37} + 540v^{23} - 727890v^{49} + 12v^{19} + 147000v^{35})z^9 + (782108v^{53} + 11604v^{33} + 125456v^{43} + 246v^{27} + 4008v^{31} - 519973v^{47} - 655171v^{49} - 118690v^{45} + 1152v^{29} + 114563v^{39} + 28830v^{35} + 62684v^{37} + 326v^{51} + 162827v^{41} + 30v^{25})z^7 + (-780286v^{55} + 3180v^{35} + 10230v^{37} + 52285v^{41} + 870v^{33} - 428705v^{49} + 238305v^{51} - 298300v^{47} + 160v^{31} + 1161940v^{53} + 60605v^{43} + 25680v^{39} + 20v^{29} - 45984v^{45})z^5 + (3515v^{39} - 225v^{45} + 11805v^{41} + 144645v^{51} + 1160685v^{53} - 1500000v^{55} + 517360v^{57} + 1020v^{37} + 30v^{33} - 110955v^{47} - 252915v^{49} + 24885v^{43} + 150v^{35})z^3 + (6655v^{43} - 118510v^{49} + 6055v^{45} - 1565805v^{55} + 1023045v^{57} - 241605v^{59} + 1515v^{41} - 8740v^{51} + 240v^{39} - 22505v^{47} + 60v^{37} + 919595v^{53})z + (40v^{47} + 556810v^{53} - 448972v^{59} + 81622v^{61} + 68v^{41} + 2400v^{45} + 12v^{39} + 1000v^{43} - 64260v^{51} - 1082380v^{55} + 990270v^{57} - 36610v^{49})z^{-1} + (-20023v^{63} + 60v^{43} + 430v^{45} - 5541v^{49} - 43869v^{51} + 1212v^{47} - 511665v^{55} + 582753v^{57} - 380019v^{59} + 134259v^{61} + 242403v^{53})z^{-3} + (270v^{47} + 260v^{49} + 71540v^{53} - 161980v^{55} - 15400v^{51} - 181960v^{59} + 93950v^{61} - 27490v^{63} + 3500v^{65} + 30v^{45} + 217280v^{57})z^{-5} + (-3015v^{51} + 13260v^{53} + 49770v^{57} - 50610v^{59} - 14760v^{63} + 3715v^{65} - 415v^{67} + 235v^{49} + 20v^{47} - 32340v^{55} + 34140v^{61})z^{-7} + (-3600v^{55} + 6300v^{61} - 300v^{67} + 30v^{69} + 30v^{49} + 1350v^{53} - 300v^{51} - 7560v^{59} + 6300v^{57} + 1350v^{65} - 3600v^{63})z^{-9} + (-v^{71} - 11v^{51} + 55v^{53} - 165v^{55} + 330v^{57} - 462v^{59} + 462v^{61} - 330v^{63} + 165v^{65} + v^{49} + 11v^{69} - 55v^{67})z^{-11}$ |

Table 7: Homfly polynomials of $D(n)$ and $E(n)$ for $n = 1$.

| link | Homfly polynomial |
| --- | --- |
| $E(1)$ | $(295920v^{33} + 1285740v^{37} + 1310v^{25} + 102228v^{31} + v^{19} + 19v^{21} + 29267v^{29} + 1111968v^{41}$<br>$+190v^{23} + 6905v^{27} + 1651392v^{39} + 699060v^{35})z^{19} + (20v^{23} + 23224v^{29} + 114296v^{31}+$<br>$3552v^{27} + 440298v^{33} + 370v^{25} + 1317580v^{35} + 3389040v^{39} - 7171160v^{43} - 959380v^{41}+$<br>$2842160v^{37})z^{17} + (22594964v^{45} + 64395v^{31} + 730v^{27} + 30v^{25} - 25856130v^{43} + 354045v^{33}$<br>$+4031490v^{37} + 8520v^{29} + 1445226v^{35} + 5140515v^{39} - 7783785v^{41})z^{15} + (190645v^{33}+$<br>$1740v^{29} + 1129945v^{35} - 59800115v^{43} + 7437100v^{39} - 46655060v^{47} + 4431220v^{37}$<br>$-16928795v^{41} + 22630v^{31} + 60v^{27} + 110170630v^{45})z^{13} + (-22812393v^{41}+$<br>$659965v^{35} + 3862895v^{37} + 300202017v^{45} + 5215v^{31} + 73005v^{33} + 200v^{29}+$<br>$9819945v^{39} - 111856893v^{43} + 71159652v^{49} - 251113608v^{47})z^{11} + (290775v^{35}$<br>$-176551770v^{43} + 12v^{29} + 10866813v^{39} + 20225v^{33} + 2656095v^{37} - 711794115v^{47}$<br>$+768v^{31} - 85371335v^{51} - 20496318v^{41} + 397584615v^{49} + 582794235v^{45})z^{9}$<br>$+(1427516v^{37} + 873918290v^{45} + 3990v^{33} - 10365022v^{41} + 96180v^{35} + 60v^{31}-$<br>$233952316v^{43} + 1132146488v^{49} + 9669032v^{39} + 83530946v^{53} - 480700768v^{51}-$<br>$1375774396v^{47})z^{7} + (-1990744125v^{47} + 23420v^{35} - 68090965v^{55} + 592450v^{37}+$<br>$6823830v^{39} + 1661175v^{41} - 257615085v^{43} + 510v^{33} + 464967175v^{53} + 1047885195v^{45}$<br>$-1348993875v^{51} + 2143490295v^{49})z^{5} + (1021291320v^{45} + 9227400v^{41} + 46805540v^{57}$<br>$+3990v^{35} - 234240060v^{43} + 1267292280v^{53} + 2978118150v^{49} + 186240v^{37} - 369009060v^{55}$<br>$-2252801280v^{47} + 3798100v^{39} + 30v^{33} - 2470672650v^{51})z^{3} + (-2038731660v^{47}-$<br>$3267441090v^{51} + 1656300v^{39} - 27292965v^{59} - 964041720v^{55} + 10386465v^{41}-$<br>$175298085v^{43} + 815226300v^{45} + 3183095790v^{49} + 243575385v^{57} + 420v^{35} + 42900v^{37}$<br>$+2218821960v^{53})z + (20v^{35} + 2763798880v^{53} - 1491955920v^{47} + 2681942292v^{49}+$<br>$602064450v^{57} + 534349350v^{45} - 134559176v^{59} + 13518806v^{61} - 3275514972v^{51}-$<br>$107726480v^{43} + 559584v^{39} + 7543466v^{41} + 6820v^{37} - 1594027120v^{55})z^{-1}+$<br>$(-1847390490v^{55} + 666v^{37} - 54210981v^{43} - 2551620786v^{51} - 310385433v^{59}-$<br>$886597767v^{47} + 1802797326v^{49} + 2571450552v^{53} - 5673571v^{63} + 287341615v^{45}$<br>$+4048563v^{41} + 927838351v^{57} + 143732v^{39} + 62258223v^{61})z^{-3} + (970469280v^{49}$<br>$+30v^{37} - 427388610v^{47} - 1574002650v^{55} + 2004600v^{65} + 1830429810v^{53}+$<br>$126253050v^{45} + 27210v^{39} - 1562192280v^{51} + 131972790v^{61} + 986213580v^{57}-$<br>$22242180v^{43} - 24027630v^{63} - 439192740v^{59} + 1675740v^{41})z^{-5} + (417127425v^{49}$<br>$-166130250v^{47} - 1007592300v^{55} - 7387575v^{43} + 44980650v^{45} + 756595125v^{57}+$<br>$3600v^{39} - 752991525v^{51} + 543075v^{41} + 1006047900v^{53} - 589875v^{67} - 420731025v^{59}$<br>$-45963450v^{63} + 168423450v^{61} + 7664775v^{65})z^{-7} + (300v^{39} + 142194v^{69} - 1963416v^{43}$<br>$-283771488v^{51} + 425978982v^{53} - 1990416v^{67} + 12935454v^{65} + 142226994v^{61}-$<br>$51458316v^{47} + 426407982v^{57} + 141735594v^{49} - 487108908v^{55} + 137994v^{41}-$<br>$51731316v^{63} + 12830454v^{45} - 284372088v^{59})z^{-9} + (27213v^{41} - 2876085v^{67}-$<br>$12447435v^{47} + 37355409v^{49} - 82201119v^{51} + 176213895v^{57} + 137024745v^{53}-$<br>$137065929v^{59} - 27393v^{71} + 410883v^{69} - 37385985v^{63} - 409635v^{43} + 12v^{39}-$<br>$176196735v^{55} + 2870805v^{45} + 12462555v^{65} + 82244799v^{61})z^{-11} + (-17734080v^{51}$<br>$-64960v^{43} + 7389200v^{65} - 64960v^{71} + 4060v^{73} - 2273600v^{47} - 46446400v^{55}+$<br>$4060v^{41} + 7389200v^{49} - 46446400v^{59} + 32512480v^{61} + 52252200v^{57} + 487200v^{69}$<br>$+487200v^{45} - 2273600v^{67} + 32512480v^{53} - 17734080v^{63})z^{-13} + (-295800v^{47}+$<br>$5383560v^{53} - 7395v^{43} - 435v^{75} + 59160v^{45} - 1035300v^{67} + 295800v^{69}-$<br>$10574850v^{59} + 1035300v^{49} + 7395v^{73} + 435v^{41} - 2691780v^{51} + 10574850v^{57}$<br>$-5383560v^{63} + 2691780v^{65} - 59160v^{71} - 8459880v^{55} + 8459880v^{61})z^{-15}+$<br>$(4590v^{45} + 91800v^{49} + 556920v^{65} + 1312740v^{61} + 556920v^{53} - 24480v^{47}-$<br>$954720v^{55} - 540v^{43} + 30v^{77} + 4590v^{73} + 91800v^{69} - 24480v^{71} + 1312740v^{57}$<br>$-257040v^{51} - 540v^{75} + 30v^{41} - 257040v^{67} - 954720v^{63} - 1458600v^{59})z^{-17}+$<br>$(-v^{79} + v^{41} - 3876v^{71} - 11628v^{51} - 969v^{47} + 171v^{45} + 27132v^{53} + 19v^{77}+$<br>$969v^{73} - 75582v^{63} - 50388v^{55} + 3876v^{49} - 19v^{43} + 75582v^{57} - 171v^{75} + 50388v^{65}$<br>$-27132v^{67} + 11628v^{69} - 92378v^{59} + 92378v^{61})z^{-19}$ |

Table 8: Homfly polynomials of $D(n)$ and $E(n)$ for $n = 1$.

of the polyhedron in a unified way. Software packages which can use compute the Homfly polynomial usually can only deal with knots and links with small crossing numbers. By using Theorem 4.2, once we obtain the Tutte polynomial of one polyhedron the Homfly polynomials of all even polyhedral links (which can have very large number of crossings.) can be obtained almost immediately. In addition, the Tutte polynomial of many polyhedra can be obtained by Maple.

Secondly, Theorem 4.2 can be naturally generalized to the polyhedral links obtained from a polyhedron by taking some edges to be replaced by positive even-twisted double-lines and other edges to be replaced by negative even-twisted double-lines with different numbers of twists. Note that such links may be non-alternating. Actually Theorem 4.2 can be generalized further as L. Traldi did in [19].

Thirdly, naturally one will ask whether there exists a similar approach to deal with odd polyhedral links. We suspect the existence of this similar approach. We point out that as for the Jones polynomial, we can compute the its main part, i.e. the Kauffman brackets of (odd or even) polyhedral links via the chain polynomial of the polyhedron [32].

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