MATCH
Communications in Mathematical and in Computer Chemistry

MATCH Commun. Math. Comput. Chem. **63** (2010) 637-656

ISSN 0340 - 6253

The Architecture and Jones Polynomials of Cycle–crossover Polyhedral Links

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(Received December 9, 2009)

Abstract

Polyhedral links, the interlinked and interlocked architectures, have been proposed for the description and analysis of knotted configurations in the backbone of DNA and proteins. Motivated by tangled polyhedral links, we utilize cycle-crossovers to cover all edges of an arbitrary convex polyhedron to produce many interlocked polyhedral frameworks. We also discuss some essential conditions for the realization of these models by DNA molecules. Meanwhile, a formula for computing Jones polynomial is obtained by using generalized Tutte polynomial and chain polynomial, which can greatly simplify the computation compared to the use of Jones skein relation.

1 Introduction

One gigantic challenge in supramolecular chemistry is to attain total control of the arrangement of molecular knots and links by the design of building blocks [1–6]. Polyhedral catenanes, the interlinked and interlocked architectures synthesized by using DNA molecules [7–18], include some DNA Platonic and Archimedean solids and DNA bipyramids and buckyballs. In addition, polyhedral links are also found in the backbone of some proteins of virus. These fancy objects hint some topological structures which give us many new targets for theoretical characterization by using mathematical methods [19].

On the theoretical side, knot theory has been applied to solving these fancy objects [20–26]. In the knot theory, it is basic and important to determine whether two knots or links are equivalent or not. Knot and link invariants are extremely useful tools. Take Jones polynomial for an example. It can distinguish many links from their mirror images and also has many chemical and physical applications. Now we recall that the Jones polynomial $V_t(L) \in \mathbb{Z}[t]$ of an oriented link L is related to the Kauffman bracket polynomial by

$$V_t(L) = (-A^3)^{-w(D)}\langle D\rangle|_{A=t^{-1/4}}, \tag{1}$$

where D is the diagram of L, $w(D)$ the writhe and $\langle D\rangle$ the Kauffman bracket polynomial of D. In knot theory, the Jones polynomials of many types of knots, such as the 2-bridge knot, prezel links, and tours links [27, 28, 29], have been calculated. Recently, Jin and Zhang established a relation between Kauffman bracket polynomial and chain polynomial for a signed chain graph G. The relation is described as

$$Q[G] = \frac{A^m}{(-A^2 - A^{-2})^{q-p+1}} Ch[R], \tag{2}$$

where m is the sum of all signs of G, and p and q are the numbers of vertices and edges of reduced graph R, respectively. According to the formula (2), Jin and Zhang gave a classification for all knots and links [30]. Meanwhile, they also calculated the Kauffman bracket polynomials of rational links by using W-polynomial [31].

In this paper, motivated by tangled polyhedral links, we fabricate a class of cycle-crossover polyhedral links by utilizing cycle-crossovers to cover all edges of an arbitrary convex polyhedron. The resulting structures have not been synthesized but might constitute interesting targets for a topology-aided molecular design. Meanwhile, we also calculate their Kauffman bracket polynomials by using chain polynomial and generalized Tutte polynomial. In biology, considering the fact that DNA double-strands could appear with two orientations, parallel or antiparallel, the cycle-crossover polyhedral links with the orientation of global parallel or antiparallel have biological meanings. And then we give out the Jones polynomials of such two cases. In mathematics, the method has the advantage of simplifying the computation of Jones polynomial. This constructed model could form the basis of future development of more complex models, and can aid synthetic chemists and biologists in testing and developing their synthetic strategies. Our result may provide further insight into the theoretical characterization of the DNA polyhedral links.

2 Construction of cycle-crossover polyhedral links

In the section, we first introduce the conception of adding tangles, and then give the definition of cycle-crossover polyhedral links.

We define a way to add n tangles $T_1, T_2, \cdots, T_n$ to obtain a new tangle called *adding tangle*. Glue NE, SE of T_i to NW, SW of T_{i+1}, respectively, where $i = 1, 2, \cdots, n$, as seen in Fig. 1.

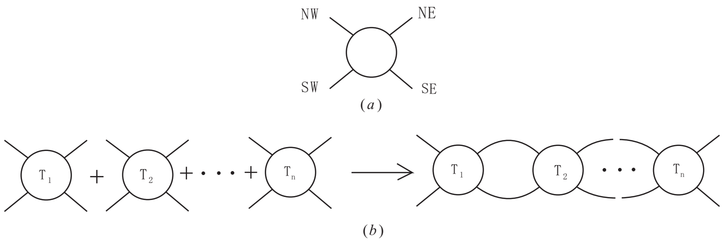


Fig. 1. (a) Four ends of a tangle. (b) Adding tangles.

In Ref. [26], Qiu et al constructed a type of tangled polyhedral links whose building blocks are some n-twisted double lines with arbitrary twists [26]. These links are DNA polyhedra, which have been synthesized in the laboratory. For example, the construction of tangled tetrahedral link can be seen in Fig. 3(a). Here, n-twisted double line can be clarified by adding tangle.

If T_i, $i = 1, 2, \cdots, n$ is a 1-tangle, then $T = T_1 + T_2 + \cdots + T_n$ is called n-twisted double line, where n is the number of twists of T.

If T_i is a 2-tangle, then $C = T_1 + T_2 + \cdots + T_n$ is called *cycle-crossover*. Here n is called the length of C. For instance, a cycle-crossover C with length 5 is shown in Fig. 2. Motivated by the tangled polyhedral links, we extend tangled polyhedral links to cycle-crossover polyhedral links $\mathbb{L}(P)$ whose building blocks are some cycle-crossovers with arbitrary lengths. That is, $\mathbb{L}(P)$ can be constructed by using cycle-crossovers with arbitrary lengths to cover each edge of a polyhedron P. For instance, the construction of cycle-crossover tetrahedral polyhedral links as shown in Fig. 3(b).

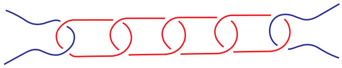


Fig. 2. Cycle-crossover C with length 5.

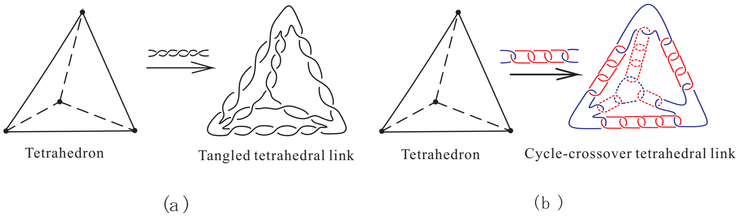


Fig. 3. Tangle and cycle-crossover are building blocks, respectively, in the construction of the tetrahedral polyhedral link.

Note that $\mathbb{L}(P)$ is alternating in our construction.

In the following, we will take two examples to illuminate the construction. For instance, every edge of a tetrahedron P is labeled with a, b, c, d, e, f, firstly. Whereafter, we use 6 cycle-crossovers $C(a), C(b) \cdots, C(f)$ with length $4, 4, 4, 2, 2, 5$, respectively, to cover each edge of P, and then the resulting link $\mathbb{L}(P)$ is the tetrahedral link, as shown in Fig. 4. Similarly, use 8 cycle-crossovers and 4 cycle-crossovers with length 5 and 2, respectively, to cover each edge of a cube, and then the resulting link is the cubic link, as shown in Fig. 5.

For the type of cycle-crossover polyhedral links, they can be realized in terms of DNA. The reason is that DNA double strands can form some types of tangles. Hence, the tangled polyhedral links and the cycle-crossover polyhedral links $\mathbb{L}(P)$ are DNA polyhedral models which can be synthesized by using some DNA molecules.

For linked molecular framework, symmetry plays an important role as a guiding principle for the design of novel molecules. Remarkably, the symmetry breaking happens during the construction of our links. For instance, the cycle-crossover cubic link and tetrahedral

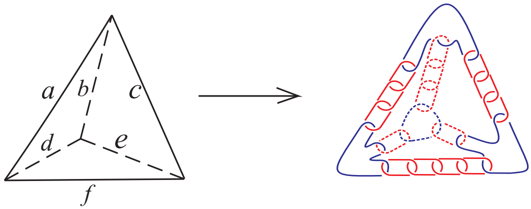


Fig. 4. Cycle-crossover tetrahedral link with the length 4, 5 and 2, respectively.

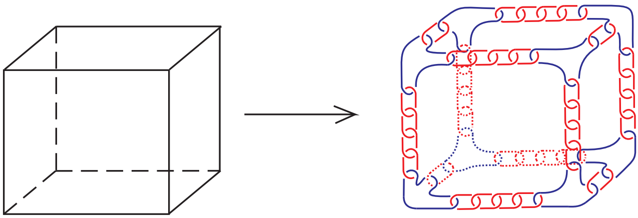


Fig. 5. Cycle-crossover cubic link with the length 5 and 2, respectively.

link reduce the symmetry group to O and T, respectively, from O_h and T_d of cube and tetrahedron.

3 Kauffman bracket polynomial of cycle-crossover polyhedral links

In the section, we first introduce the relation between plane graph and its corresponding knot or link, and then introduce the conceptions of Kauffman bracket polynomial, generalized Tutte polynomial and chain polynomial.

A connected plane medial graph $M(G)$ is a 4-regular plane graph which can be obtained from a connected planar graph G. An example is shown in Fig. 6. A signed graph is a graph with each of its edges labeled with a sign ($+$ or $-$). According to the sign of the edge (see Fig. 7), a knot or link $L(G)$ can be easily obtained by changing each vertex of medial graph $M(G)$ into a crossing, as illustrated in Fig. 8.

There exists a one-to-one correspondence between link diagrams and signed plane

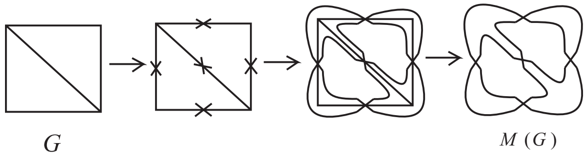


Fig. 6. A planar graph G becomes a medial graph $M(G)$.

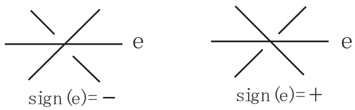


Fig. 7. Signs of edges.

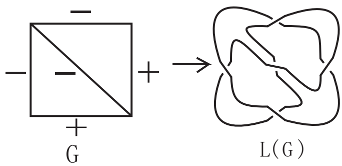


Fig. 8. A signed graph G becomes a link $L(G)$.

graphs via the medial construction [32]. Given a link diagram L, we create a corresponding planar graph in the following way. First shade the regions of the link diagram as black and white so that the unbounded region is white. Then associate a graph G to the link diagram so that the vertices correspond to the black regions and the edges correspond to the crossings shared by the black regions.

Definition 3.1. *[33, 34, 35] The Kauffman bracket polynomial $\langle L\rangle$ of a link diagram L is defined by the following three relations:*

(1) $\langle \bigcirc \rangle = 1$.

(2) $\langle \bigcirc \cup L\rangle = (-A^2 - A^{-2})\langle L\rangle$.

(3) $\langle \times \rangle = A\langle \asymp \rangle + A^{-1}\langle)(\rangle$.

Kauffman introduced the generalized Tutte polynomial for signed graphs, which is a generalization of both the Tutte polynomial for graphs and the Kauffman bracket polynomial for link diagrams.

Theorem 3.2. *[32] The generalized Tutte polynomial $Q[G]$ for a signed graph G is defined by the following properties:*

1. If e is neither an isthmus (i.e., a cut edge) nor a loop in G, then

$$Q(G) = BQ(G-e) + AQ(G/e) \qquad sign(e) = +,$$

$$Q(G) = AQ(G-e) + BQ(G/e) \qquad sign(e) = -,$$

where $G-e$, G/e are obtained from G by deleting and contracting the edge e, respectively.

2. If every edge of G is either an isthmus or a loop and G is connected, then

$$Q(G) = X^{i_+ + l_-}Y^{i_- + l_+},$$

where $X = -A^{-3}, Y = -A^3$; $i_+(i_-)$ is the number of the positive (respectively negative) isthmuses; $l_+(l_-)$ is the number of the positive (respectively negative) loops.

3. If G is the disjoint union of graphs G_1 and G_2 , then

$$Q(G) = (-A^2 - A^{-2})Q(G_1)Q(G_2).$$

The relation of Kauffman bracket polynomial and Tutte polynomial is given as follows:

Theorem 3.3. *[32] Let G be a signed plane graph, $L(G)$ be the link diagram associated with G via the medial construction. Then*

$$Q(G) = \langle L(G) \rangle.$$

To compute Kauffman bracket polynomial of $\mathbb{L}(P)$, we need find corresponding signed plane graph from $\mathbb{L}(P)$. Let R be the signed plane graph of $\mathbb{L}(P)$. Since $\mathbb{L}(P)$ is alternating, the sign of each edge of R is the same. For instance, the planar graph from cycle-crossover tetrahedral link is illustrated in Fig. 9.

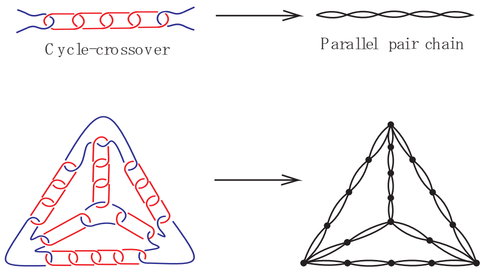


Fig. 9. The planar graph R from cycle-crossover tetrahedral link $\mathbb{L}(P)$.

Let m_d denote the length of parallel pair chain d in R. Here $m_d = e(d)/2$, where $e(d)$ is the number of edges of the chain d.

Generalizing the concept of deleting and contracting an edge to a parallel pair chain. Deleting a parallel pair chain means deleting the internal vertices in it. Contracting a chain means deleting the internal vertices in the parallel pair chain and then identifying its endvertices. Compare to Property 1 in Theorem 3.2, we can obtain a similarity theorem.

Theorem 3.4. *Let G be a signed planar graph whose every edges have the same sign. Let R be a planar graph from $\mathbb{L}(G)$. If d is a parallel pair chain of R, then*

$$Q[R] = \frac{(-A^{-4} - A^{4})^{m_d} - (1 - A^{4})^{m_d}}{-A^{2} - A^{-2}} Q[H] + (1 - A^{4})^{m_d} \delta Q[K], \tag{3}$$

where H and K are obtained from R by deleting and contracting the chain d respectively; $\delta = -A^{2} - A^{-2}$ if e is a loop of G, and $\delta = 1$ otherwise.

Proof: Suppose each edge is positive in R, firstly. Let $e_1, e_1^*, e_2, e_2^*, \cdots, e_{m_d}, e_{m_d}^*$ be $2m_d$ edges of parallel pair chain d, where e_i, e_i^* are two multiple edges in chain d. Then

$$\begin{aligned}
Q[R] &= BQ[R-e_1] + AQ[R/e_1] \\
&= B\{BQ[R-e_1-e_1^*] + AQ[R-e_1/e_1^*]\} + AYQ[R/e_1/e_1^*] \\
&= B^2Q[R-e_1-e_1^*] + (BA+AY)Q[R/e_1/e_1^*] \\
&= B^2\{BQ[R-e_1-e_1^*-e_2] + AQ[R-e_1-e_1^*/e_2]\} + (BA+AY)Q[R/e_1/e_1^*] \\
&= B^2\{BXQ[R-e_1-e_1^*-e_2/e_2^*] + AYQ[R-e_1-e_1^*/e_2/e_2^*]\} \\
&\quad +(BA+AY)Q[R/e_1/e_1^*] \\
&= B^2\{(BX+AY)Q[R-e_1-e_1^*-e_2/e_2^*]\} + (BA+AY)Q[R/e_1/e_1^*] \\
&= \cdots \\
&= B^2(XB+AY)^{m_d-1}Q[H] + (BA+AY)Q[R/e_1/e_1^*]. \qquad (4)
\end{aligned}$$

Let $p = XB + AY$, $q = BA + AY$.

(i) If edge e is neither isthmus nor loop of G, then $Q[R/e_1/e_1^*/\cdots/e_{m_d}/e_{m_d}^*] = Q[K]$. Therefore, from (4), we have

$$\begin{aligned}
Q[R] &= B^2p^{m_d-1}Q[H] + qQ[R/e_1/e_1^*] \\
&= B^2p^{m_d-1}Q[H] + q\{B^2(p^{m_d-2}Q[H] + qQ[R/e_1/e_1^*/e_2/e_2^*]\} \\
&= \cdots \\
&= B^2(p^{m_d-1} + p^{m_d-2}q + p^{m_d-3}q^2 + \cdots + q^{m_d-1}) + q^{m_d}Q[K] \\
&= B^2\frac{p^{m_d}-q^{m_d}}{p-q}Q[H] + (1-A^4)^{m_d}Q[K] \\
&= \frac{(-A^{-4}-A^4)^{m_d} - (1-A^4)^{m_d}}{-A^2-A^{-2}}Q[H] + (1-A^4)^{m_d}Q[K].
\end{aligned}$$

(ii) If edge e is a loop of G, then

$$Q[R/e_1/e_1^*/\cdots/e_{m_d-1}/e_{m_d-1}^*] = Y^2Q[K].$$

So, from (4), we have

$$\begin{aligned}
Q[R] &= B^2p^{m_d-1}Q[H] + qQ[R/e_1/e_1^*]\\
&= B^2p^{m_d-1}Q[H] + q\{B^2(p^{m_d-2}Q[H] + qQ[R/e_1/e_1^*/e_2/e_2^*]\}\\
&= \cdots\\
&= B^2(p^{m_d-1} + p^{m_d-2}q + p^{m_d-3}q^2 + \cdots + pq^{m_d-2}) + q^{m_d-1}Y^2Q[K]\\
&= B^2\frac{p^{m_d} - q^{m_d}}{p-q}Q[H] + (1-A^4)^{m_d}\delta Q[K]\\
&= \frac{(-A^{-4}-A^4)^{m_d} - (1-A^4)^{m_d}}{-A^2-A^{-2}}Q[H] + (1-A^4)^{m_d}\delta Q[K],
\end{aligned}$$

where $\delta = -A^2 - A^{-2}$.

(iii) If edge e is a isthmus of G, then we have

$$\begin{aligned}
Q[R] &= pQ[G/e_1/e_1^*] = p^2Q[G/e_1/e_1^*/e_2/e_2^*] = \cdots = p^{m_d}G[K]\\
&= \frac{p^{m_d} - q^{m_d}}{-A^2-A^{-2}}(-A^2-A^{-2})Q[K] + q^{m_d}Q[K]\\
&= \frac{p^{m_d} - q^{m_d}}{-A^2-A^{-2}}Q[H] + q^{m_d}Q[K]\\
&= \frac{(-A^{-4}-A^4)^{m_d} - (1-A^4)^{m_d}}{-A^2-A^{-2}}Q[H] + (1-A^4)^{m_d}Q[K].
\end{aligned}$$

□

Theorem 3.4 gives a recursion formula to calculate the Kauffman bracket polynomial of R. In the following, we will simplify the formula by chain polynomial. First, we introduce the definition of chain polynomial, which is proposed by Read and Whitehead for the purpose of studying the chromatic polynomial for homeomorphism class of graphs.

Definition 3.5. *[36] The chain polynomial $Ch[G]$ of a graph G is defined as*

$$Ch[G] = \sum_{(Y,U)} F[Y]w^{\nu(U)},$$

where Y is a spanning subgraph of G, $U = Y - E(G)$; $F[Y]$ is the flow polynomial, and $\nu(U)$ the number of chain in U.

Proposition 3.6. *[36] If graph G consists of two graphs G_1 and G_2 which have at most one vertex in common, then $Ch[G] = Ch[G_1]Ch[G_2]$.*

The following proposition is obvious.

Proposition 3.7. *[36] Let a be an edge of G, and let $G - a$ and G/a be the graph obtained from G by deleting and contracting the edge a respectively. Then if a is a loop, $Ch[G] = (a - w)Ch[G - a]$, and otherwise, $Ch[G] = (a - 1)Ch[G - a] + Ch[G/a]$.*

The following theorem gives a relation of Kauffman bracket polynomial for a cycle-crossover polyhedral link $\mathbb{L}(P)$ and chain polynomial of the polyhedron P. Applying Theorem 3.4 to each parallel pair chain of R, we have

Theorem 3.8. *Let P be a convex polyhedron. Let R be a planar graph from $\mathbb{L}(P)$. In $Ch[P]$, if we replace w by $-A^{-4}-1-A^4$, and replace d by $(\dfrac{A^{-4}+A^4}{A^4-1})^{m_d}$ for every parallel pair chain d, then we have*

$$Q[R] = \frac{(1-A^4)^m}{(-A^2-A^{-2})^{q-p+1}} Ch[P], \tag{5}$$

where m denotes the sum of the length of cycle-crossover in R, and p the number of vertices of P, q the number of edges of P .

Proof: Suppose that $i \in E(R)$ is a parallel pair chain of R, by (3), we have

$$Q[R] = (1-A^4)^{m_i} \left\{ \frac{(\frac{A^{-4}+A^4}{A^4-1})^{m_i} - 1}{-A^2-A^{-2}} Q[H] + \delta Q[K] \right\}.$$

Let $\alpha = \dfrac{A^{-4}+A^4}{A^4-1}$. Then

$$Q[R] = (1-A^4)^{m_i} \left\{ \frac{\alpha^{m_i} - 1}{-A^2-A^{-2}} Q[H] + \delta Q[K] \right\}.$$

Applying (3) successively with each cycle-crossover in turn, we obtain

$$Q[R] = (1-A^4)^{\sum_{i=1}^{e(P)} m_i} \sum_{D \subset E(P)} \left\{ \prod_{d \in D} \left(\frac{\alpha^{m_i} - 1}{-A^2-A^{-2}} \right) \right\} \delta^L Q[R_D],$$

where R_D is the graph from R by deleting the parallel pair chains in D and contracting those in $S = E(P) - D$, and L is the number of times that a loop has been contracted.

$$\begin{aligned} Q[R] &= (1-A^4)^{\sum_{i=1}^{e(P)} m_i} \sum_{D \subset E(P)} \left\{ \prod_{d \in D} (\alpha^{m_i} - 1) \right\} (-A^2-A^{-2})^{-|D|+k\langle S\rangle - 1} \delta^{|S|-p+k\langle S\rangle} \\ &= \frac{(1-A^4)^{\sum_{i=1}^{e(P)} m_i}}{(-A^2-A^{-2})^{q-p+1}} \sum_{D \subset E(P)} \left\{ \prod_{d \in D} (\alpha^{m_i} - 1) \right\} ((-A^2-A^{-2})\delta)^{|S|-p+k\langle S\rangle}, \end{aligned}$$

where $k\langle S\rangle$ denotes the number of connected components of induced subgraph of S, and $|S|$ the size of S.

Since

$$\prod_{d\in D}(\alpha^{m_i}-1)=\sum_{U\subset D}(-1)^{|D|-|U|}\alpha^{\nu(U)},$$

where $\nu(U)=\sum\limits_{d\in D} m_i$. So we have

$$Q[R]=\frac{(1-A^4)^{\sum\limits_{i=1}^{e(P)} m_i}}{(-A^2-A^{-2})^{q-p+1}}\sum_{D\subset E(P)}\left\{\sum_{U\subset D}(-1)^{|D|-|U|}\alpha^{\nu(U)}\right\}((-A^2-A^{-2})\delta)^{|S|-p+k\langle S\rangle}.$$

Let $Y=E(P)-U$, then the coefficient of α^{m_i} is

$$\begin{aligned}
&\frac{(1-A^4)^{\sum\limits_{i=1}^{e(P)} m_i}}{(-A^2-A^{-2})^{q-p+1}}\sum_{S\subset Y}(-1)^{|Y|-|S|}((-A^2-A^{-2})\delta)^{|S|-p+k\langle S\rangle}\\
=&\frac{(1-A^4)^{\sum\limits_{i=1}^{e(P)} m_i}}{(-A^2-A^{-2})^{q-p+1}}F(Y,(-A^2-A^{-2})\delta).
\end{aligned}$$

Finally, we get

$$\begin{aligned}
Q[R] &= \frac{(1-A^4)^{\sum\limits_{i=1}^{e(P)} m_i}}{(-A^2-A^{-2})^{q-p+1}}\sum_{S\subset Y}F(Y,(-A^2-A^{-2})\delta)\alpha^{\nu(U)}\\
&= \frac{(1-A^4)^{\sum\limits_{i=1}^{e(P)} m_i}}{(-A^2-A^{-2})^{q-p+1}}Ch[P]\\
&= \frac{(1-A^4)^{m}}{(-A^2-A^{-2})^{q-p+1}}Ch[P].
\end{aligned}$$

□

Theorem 3.8 implies that it is easier to compute by (5) than by Kauffman bracket skein formula. For instance, for cycle-crossover cubic link (see Fig. 8), the computable complex of using Kauffman bracket skein formula is 2^{96}. However, the computable complex of using formula (4) at most is 2^{12}. This indicates that our result can more simplify the process of the calculation.

4 Jones polynomials of cycle-crossover polyhedral links

To distinguish two knots or links, Vaughan Jones introduced Jones polynomial in 1984. It is an invariant of an oriented knot or link which assigns to each oriented knot or link a laurent polynomial in the variable $t^{\frac{1}{2}}$ with integer coefficients.

Theorem 4.1. *[38] The Jones polynomial of an oriented link L, denoted variously by $V_L(z)$, is defined by the following axioms:*

1. if L is the trivial knot then $V_L(z) = 1$.

2. $t^{-1}V_{L_+} - tV_{L_-} = (t^{\frac{1}{2}} - t^{-\frac{1}{2}})V_{L_0}$, where L_+, L_- and L_0 are three oriented link diagrams that are identical except in one small region where they differ by the crossing changes or smoothing shown in the fig. 11.

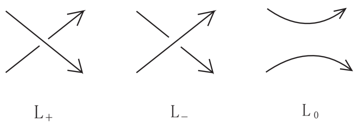


Fig. 11. Three oriented diagrams that are almost identical.

Jones polynomial can also be calculated by writhe and Kauffman bracket polynomial.

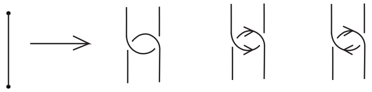
Theorem 4.2. *[39] The Jones polynomial of an oriented link L is expressed as*

$$V_t(L) = X_L(A)|_{t^{-1/4}},$$

where $X_L(A) = (-A^3)^{-w(L)}\langle L\rangle$, $w(L)$ and $\langle L\rangle$ denote the writhe and Kauffman bracket polynomial of L, respectively.

According to Theorem 4.2, if the writhe of $\mathbb{L}(P)$ can be calculated, then its Jones polynomial can be easily obtained.

In the following, we will characterize the realizable qualities (require some essential conditions for their synthesis by DNA molecule) of $\mathbb{L}(P)$. Firstly, we introduce a conception of hopf operation.



(a) (b) (c)

Fig. 10. (a) Hopf operation. (b) Parallel orientation. (c) Antiparallel orientation.

Given an edge e, define a hopf operation to be a change that replaces the edge e with an opened hopf link obtained by hopf link minus two arcs, as shown in Fig. 10(a). If we apply hopf operations to every edge of a graph G, then a corresponding link $L(G)$ can be obtained. There is a one-to-one correspondence between G and $L(G)$ via hopf operations.

Under biological viewpoint, two strands of DNA polyhedron have parallel or antiparallel orientation [37]. Accordingly, for all oriented $\mathbb{L}(P)$, there are only two kinds of oriented $\mathbb{L}(P)$, global parallel or antiparallel, have biological meanings. The right figure in Fig. 10(a) is a local of $\mathbb{L}(P)$. If all locals of $\mathbb{L}(P)$ are not parallel or antiparallel (see Fig. 10(b) and (c)), then we call it local parallel or antiparallel. If every local of $\mathbb{L}(P)$ are parallel or antiparallel , then we call it global parallel or antiparallel. Thus, this brings on an issue which $\mathbb{L}(P)$ can have global parallel or antiparallel orientation.

Theorem 4.3. *(1) If G is an arbitrary graph, then $L(G)$ has global antiparallel orientation.*

(2) If G is a bipartite graph, then $L(G)$ has global parallel orientation.

Proof: Let G^* be a graph by replacing every crossing of $L(G)$ with a vertex. Let $F_1(G^*), F_2(G^*), \cdots, F_{f(G)}(G^*)$ be $f(G)$ faces in G^*, which homologous to $f(G)$ faces in G.

(1) Let $L(G)$ be a cycle-crossover link obtained by applying hopf operation to every edge of G. Assign a clockwise or anticlockwise orientation to boundary of $F_i(G^*)$, then G^* can product antiparallel orientation. Hence $L(G)$ has global antiparallel orientation.

(2) Since G is bipartite, the length of every face of G denoted by $l(F_i)$ is even, where $i = 1, 2, \cdots, f(G)$. Assign an orientation to each edge of G^*, such that those edges in $F_i(G^*)$ have $l(F_i)/2$ clockwise edges and $l(F_i)/2$ anticlockwise edges. And these orientations are alternating. Then $L(P)$ has global parallel orientation. $\square$

If a $\mathbb{L}(P)$ has m parallel (resp. antiparallel) orientation and n antiparallel (resp. antiparallel) orientation, how we modify the $\mathbb{L}(P)$ such that it has global parallel or antiparallel orientation.

Corollary 4.4. *If a $\mathbb{L}(P)$ has n locals, there exist $n-1$ locals with parallel (resp. antiparallel) orientation and a local with antiparallel (resp. parallel) orientation, then $\mathbb{L}'(P)$ with global parallel (resp. antiparallel) orientation can be obtained from $\mathbb{L}(P)$ by increasing odd cycles, as shown in Fig. 12.*

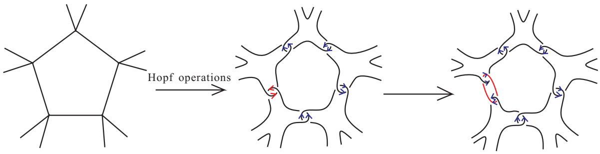


Fig. 12. An antiparallel orientation becomes a parallel orientation after adding a cycle.

Accordingly, we need only calculate Jones polynomials of the two significative cases.

For a cycle-crossover polyhedral link $\mathbb{L}(P)$ with global parallel orientation, if it is right-handed, then $w(L) = 2\sum_{i=1}^{e(P)} m_i$. Other $w(L) = -2\sum_{i=1}^{e(P)} m_i$. Now we suppose the links are all left-handed.

(1) If the orientation of $\mathbb{L}(P)$ is global parallel, then

$$w(L) = -2\sum_{i=1}^{e(P)} m_i.$$

So

$$\begin{aligned} X_L(A) &= \frac{(-A^3)^{2\sum_{i=1}^{e(P)} m_i}(1-A^4)^{\sum_{i=1}^{e(P)} m_i}}{(-A^{-2}-A^2)^{q-p+1}} Ch[P] \\ &= \frac{(A^6-A^{10})^{\sum_{i=1}^{e(P)} m_i}}{(-A^{-2}-A^2)^{q-p+1}} Ch[P]. \end{aligned}$$

Therefore

$$V_t(L) = \frac{(A^6-A^{10})^{\sum_{i=1}^{e(P)} m_i}}{(-A^{-2}-A^2)^{q-p+1}} Ch[P]\Big|_{A=t^{-1/4}}.$$

(2) If the orientation of $\mathbb{L}(P)$ is global antiparallel, then

$$w(L) = 2\sum_{i=1}^{e(P)} m_i.$$

So

$$\begin{aligned} X_L(A) &= \frac{(-A^3)^{-2\sum_{i=1}^{e(P)} m_i}(1-A^4)^{\sum_{i=1}^{e(P)} m_i}}{(-A^{-2}-A^2)^{q-p+1}} Ch[P] \\ &= \frac{(A^{-6}-A^{-2})^{\sum_{i=1}^{e(P)} m_i}}{(-A^{-2}-A^2)^{q-p+1}} Ch[P]. \end{aligned}$$

Therefore

$$V_t(L) = \frac{(A^{-6}-A^{-2})^{\sum_{i=1}^{e(P)} m_i}}{(-A^{-2}-A^2)^{q-p+1}} Ch[P]\Big|_{t^{-1/4}}.$$

5 An example

In order to obtain the Kauffman bracket polynomials of cycle-crossover tetrahedral link, it suffices to calculate the chain polynomial of the tetrahedron in Fig. 7. Firstly, we label the edges a, b, c, d, e, f. Then the chain polynomial of the tetrahedron can be obtained by Proposition 3.7.

$$\begin{aligned} Ch[P] &= abcdef - w(abc + aef + bdf + cde + ad + be + cf) \\ &\quad + w(w+1)(a+b+c+d+f) - w(w+1)(w+2). \end{aligned}$$

Now the Kauffman bracket polynomial can be obtained by Theorem 3.8.

Let $\alpha = \dfrac{A^{-4}+A^4}{A^4-1}$, $w = -A^{-4} - 1 - A^4$. Then

$$\begin{aligned} \langle \mathbb{L}(P) \rangle &= \frac{(1-A^4)^{m_a+m_b+m_c+m+m_e+m_f}}{(-A^2-A^{-2})^3}\{\alpha^{m_a+m_b+m_c+m_d+m_e+m_f} - w(\alpha^{m_a+m_b+m_c} \\ &\quad + \alpha^{m_a+m_e+m_f} + \alpha^{m_b+m_d+m_f} + \alpha^{m_c+m_d+m_e} + \alpha^{m_a+m_d} + \alpha^{m_b+m_e} + \alpha^{m_c+m_f}) \\ &\quad + w(w+1)(\alpha^{m_a} + \alpha^{m_b} + \alpha^{m_c} + \alpha^{m_d} + \alpha^{m_e} + \alpha^{m_f}) - w(w+1)(w+2)\}. \\ &= \frac{(1-A^4)^{21}}{(-A^2-A^{-2})^3}\{\alpha^{21} - w(\alpha^{12} + 2\alpha^{11} + \alpha^9 + \alpha^8 + 2\alpha^6) + w(w+1)(3\alpha^4 + \alpha^5 \\ &\quad + 2\alpha^2) - w(w+1)(w+2)\} \end{aligned}$$

$$
\begin{aligned}
= \ & A^{-204} - 3A^{-200} + 27A^{-196} - 73A^{-192} + 351A^{-188} - 861A^{-184} + 2933A^{-180} \\
& -6567A^{-176} + 17747A^{-172} - 36463A^{-168} + 83006A^{-164} - 157128A^{-160} \\
& +312233A^{-156} - 545777A^{-152} + 967428A^{-148} - 1561783A^{-144} + 2499751A^{-140} \\
& -3718271A^{-136} + 5399195A^{-132} - 7359660A^{-128} + 9679490A^{-124} \\
& -11988919A^{-120} + 14191162A^{-116} - 15805331A^{-112} + 16688244A^{-108} \\
& -16529323A^{-104} + 15409821A^{-100} - 13411804A^{-96} + 10899369A^{-92} \\
& -8205147A^{-88} + 5704980A^{-84} - 3629849A^{-80} + 2100470A^{-76} - 1092946A^{-72} \\
& +506670A^{-68} - 206343A^{-64} + 72893A^{-60} - 21920A^{-56} + 5497A^{-52} \\
& -1110A^{-48} + 171A^{-44} - 18A^{-40} + A^{-36}.
\end{aligned}
$$

According to (1), we have

$$
\begin{aligned}
V_t(L) \ = \ & t^{-51} - 3t^{-50} + 27t^{-49} - 73t^{-48} + 351t^{-47} - 861t^{-46} + 2933t^{-45} - 6567t^{-44} \\
+ \ & 17747t^{-43} - 36463t^{-42} + 83006t^{-41} - 157128t^{-40} + 312233t^{-39} - 545777t^{-38} \\
+ \ & 967428t^{-37} - 1561783t^{-36} + 2499751t^{-35} - 3718271t^{-34} + 5399195t^{-33} \\
- \ & 7359660t^{-32} + 9679490t^{-31} - 11988919t^{-30} + 14191162t^{-29} - 15805331t^{-28} \\
+ \ & 16688244t^{-27} - 16529323t^{-26} + 15409821t^{-25} - 13411804t^{-24} + 10899369t^{-23} \\
- \ & 8205147t^{-22} + 5704980t^{-21} - 3629849t^{-20} + 2100470t^{-19} - 1092946t^{-18} \\
+ \ & 506670t^{-17} - 206343t^{-16} + 72893t^{-15} - 21920t^{-14} + 5497t^{-13} - 1110t^{-12} \\
+ \ & 171t^{-11} - 18t^{-10} + t^{-9} \ .
\end{aligned}
$$

Acknowledgements

This work was supported by grants from The National Natural Science Foundation of China (No.10831001 and No.20973085).

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