MATCH
Communications in Mathematical and in Computer Chemistry

MATCH Commun. Math. Comput. Chem. **63** (2010) 461-468

ISSN 0340 - 6253

On Comparing Variable Zagreb Indices for Unicyclic Graphs

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(Received March 23, 2009)

Abstract

Recently, the first and second Zagreb indices are generalized into the variable Zagreb indices which are defined by ${}^{\lambda}M_1(G) = \sum_{u\in V}(d(u))^{2\lambda}$ and ${}^{\lambda}M_2(G) = \sum_{uv\in E}(d(u)d(v))^{\lambda}$, where λ is any real number. In this paper, we prove that ${}^{\lambda}M_1(G)/n \geqslant {}^{\lambda}M_2(G)/m$ for all unicyclic graphs and all $\lambda \in (-\infty, 0]$. And we also show that the relationship of numerical value between ${}^{\lambda}M_1(G)/n$ and ${}^{\lambda}M_2(G)/m$ is indefinite in the distinct unicyclic graphs for each $\lambda \in (1, +\infty)$. With the conclusion in [4], we finish discussing the relationship of ${}^{\lambda}M_1(G)/n$ and ${}^{\lambda}M_2(G)/m$ in unicyclic graphs for $\lambda \in R$.

1 Introduction

The first and second Zagreb indices are among the oldest and the most famous topological indices, which are defined as:

$$M_1(G) = \sum_{u\in V}(d(u))^2 \text{ and } M_2(G) = \sum_{uv\in E} d(u)d(v)$$

where V is the set of vertices, E is the set of edges and $d(u)$ is degree of vertex u. $|V| = n$, $|E| = m$.

Recently, the system AutoGraphiX proposed the following conjecture:

Conjecture 1.1 *For all simple connected graph G,*

$$M_1(G)/n \leqslant M_2(G)/m$$

and the bound is tight for complete graphs.

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But, in paper [5], this conjecture is proved not true, while it is proved true for chemical graphs([5]), trees([3]) and unicyclic graphs([1]). The generalization of this claim to the variable Zagreb indices has been analyzed. The variable first and second Zagreb indices are defined as:

$$^{\lambda}M_1(G) = \sum_{u \in V} (d(u))^{2\lambda} \text{ and } ^{\lambda}M_2(G) = \sum_{uv \in E} (d(u)d(v))^{\lambda}$$

where λ is any real number, with the following theorems from paper [2–4].

Theorem 1.2 *For all chemical graphs G and all* $\lambda \in [0, 1]$*, it holds that* $^{\lambda}M_1(G)/n \leqslant {}^{\lambda}M_2(G)/m$.

Theorem 1.3 *For all trees G and all* $\lambda \in [0, 1]$*, it holds that* $^{\lambda}M_1(G)/n \leqslant {}^{\lambda}M_2(G)/m$.

Theorem 1.4 *Let* $\lambda \in R\backslash[0, 1]$ *and G be any unbalanced bipartite graph. Then,* $^{\lambda}M_1(G)/n > {}^{\lambda}M_2(G)/m$.

Theorem 1.5 *For all graphs G and all* $\lambda \in [0, 1/2]$*, it holds that* $^{\lambda}M_1(G)/n \leqslant {}^{\lambda}M_2(G)/m$.

Theorem 1.6 *Let* $\lambda \in (\sqrt{2}/2, 1)$*. Then, there is a graph G such that* $^{\lambda}M_1(G)/n > {}^{\lambda}M_2(G)/m$.

Theorem 1.7 *For all unicyclic graphs G and all* $\lambda \in [0, 1]$*, it holds that* $^{\lambda}M_1(G)/n \leqslant {}^{\lambda}M_2(G)/m$.

It is known to all that the variable Zagreb indices are often used in the study of unicyclic molecules. In this paper, we show that the relationship of numerical value between $^{\lambda}M_1(G)/n$ and $^{\lambda}M_2(G)/m$ for $\lambda \in R\backslash[0, 1]$. With the conclusion in [4], we finish discussing the relationship of the variable first and second Zagreb indices in unicyclic graphs for $\lambda \in R$.

2 Comparing Variable Zagreb Indices for Unicyclic Graphs for $\lambda \leqslant 0$

Theorem 2.1 *Let G be a connected unicyclic graph with n vertices and m edges. Then*

$$^{\lambda}M_1(G)/n \geqslant {}^{\lambda}M_2(G)/m, \lambda \in (-\infty, 0]$$

Moreover, if $\lambda \in (-\infty, 0)$*, then* $^{\lambda}M_1(G)/n = {}^{\lambda}M_2(G)/m$ *holds if and only if G is a cycle.*

Proof. If G is a cycle, it is easy to see that $^{\lambda}M_1(G)/n = {}^{\lambda}M_2(G)/m, \lambda \in (-\infty, 0]$. So we may assume that G is not a cycle in the following proof.

Since G is a connected unicyclic graph, we have $n = m$. Moreover, by the definition of $^{\lambda}M_1(G)$ and $^{\lambda}M_2(G)$, it is obvious that $^{0}M_1(G) = \sum_{u \in V} (d(u))^{2 \cdot 0} = n = m = {}^{0}M_2(G) = \sum_{uv \in E} (d(u)d(v))^{0}$. So we only need to prove $^{\lambda}M_1(G) > {}^{\lambda}M_2(G), \lambda \in (-\infty, 0)$. We prove this conclusion by induction on n.

If $n = 4$, since G is not a cycle, G is a connected unicyclic graph that has a triangle and a pendant vertex. Then we have

$$^{\lambda}M_1(G) = 2^{2\lambda} + 2^{2\lambda} + 3^{2\lambda} + 1^{2\lambda} = 4^{\lambda} + 4^{\lambda} + 9^{\lambda} + 1,$$

$$^{\lambda}M_2(G) = (2\times 2)^{\lambda} + (2\times 3)^{\lambda} + (3\times 2)^{\lambda} + (3\times 1)^{\lambda} = 4^{\lambda} + 6^{\lambda} + 6^{\lambda} + 3^{\lambda}.$$

Then

$$\begin{aligned} ^{\lambda}M_1(G) - {}^{\lambda}M_2(G) &= 4^{\lambda} + 4^{\lambda} + 9^{\lambda} + 1 - 4^{\lambda} - 6^{\lambda} - 6^{\lambda} - 3^{\lambda} \\ &> 1 + 9^{\lambda} - 4^{\lambda} - 3^{\lambda}. \end{aligned}$$

Suppose $f_1(\lambda) = 1 + 9^{\lambda} - 4^{\lambda} - 3^{\lambda}, \lambda \in (-\infty, 0)$. Then

$$\begin{aligned} f_1'(\lambda) &= 9^{\lambda}\ln 9 - 4^{\lambda}\ln 4 - 3^{\lambda}\ln 3 \\ &= 2\cdot 9^{\lambda}\ln 3 - 4^{\lambda}\ln 4 - 3^{\lambda}\ln 3 \\ &= (9^{\lambda} - 3^{\lambda})\ln 3 + (9^{\lambda}\ln 3 - 4^{\lambda}\ln 4) \\ &< 0. \end{aligned}$$

Note that $f_1(\lambda)$ is decreasing on $(-\infty, 0)$ in λ. We have $f_1(\lambda) > f_1(0) = 1+9^0-4^0-3^0$. Therefore, $^{\lambda}M_1(G) - {}^{\lambda}M_2(G) > 0, \lambda \in (-\infty, 0)$.

Suppose that it holds for all connected unicyclic graphs with vertices less than n. Since G is not a cycle, there exists a pendant vertex v and its unique neighbor vertex u. Denote by $N_G(u)$ the set of the neighbor vertices of u. Let $N_G(u) = \{v, v_1, v_2, \cdots, v_k\}, (k \geqslant 1)$ and $N_G[u] = N_G(u)\cup\{u\}$, where $v_i \in V(G), (1 \leqslant i \leqslant k)$. Let $V(G) = N_G[u] \cup \{x_1, x_2, \cdots, x_{n-k-2}\}$.

Case 1 When $k = 1$.

Then $N_G(u) = \{v, v_1\}$. Denote $N_G(v_1) = \{u, u_1, \cdots, u_p\}, (p \geqslant 1)$, where $u_i \in V(G), (1 \leqslant i \leqslant p)$.

Subcase 1.1 When $p = 1$.

Let $G' = G - v$. Then G' is a connected unicyclic graph with $n-1$ vertices. Since u is a pendant vertex in G', G' is not a cycle. By the induction hypothesis, we have $^{\lambda}M_1(G') > {}^{\lambda}M_2(G'), \lambda \in (-\infty, 0)$. Now we compare $^{\lambda}M_1(G)$ and $^{\lambda}M_2(G)$.

$$^{\lambda}M_1(G) = {}^{\lambda}M_1(G') + 2^{2\lambda} - 1 + 1 = {}^{\lambda}M_1(G') + 4^{\lambda},$$

$$^{\lambda}M_2(G) = {}^{\lambda}M_2(G') + 4^{\lambda} - 2^{\lambda} + 2^{\lambda} = {}^{\lambda}M_2(G') + 4^{\lambda}.$$

Then

$$^{\lambda}M_1(G) - {}^{\lambda}M_2(G) = {}^{\lambda}M_1(G') + 4^{\lambda} - {}^{\lambda}M_2(G') - 4^{\lambda} > 0.$$

Subcase 1.2 When $p \geqslant 2$.

Suppose $V(G) = \{v\} \cup N_G[v_1] \cup \{y_1, y_2, \cdots, y_{n-k-3}\}$, where $y_i \in V(G), (1 \leqslant i \leqslant n-k-3)$. Let $G'' = G - v - u$. Then G'' is a connected unicyclic graph with $n-2$ vertices. If G'' is a cycle, ${}^{\lambda}M_1(G'') = {}^{\lambda}M_2(G'')$. If G'' is not a cycle, by the induction hypothesis, ${}^{\lambda}M_1(G'') > {}^{\lambda}M_2(G'')$. Now we compare ${}^{\lambda}M_1(G)$ and ${}^{\lambda}M_2(G)$.

$$ {}^{\lambda}M_1(G) = {}^{\lambda}M_1(G'') + (p+1)^{2\lambda} - p^{2\lambda} + 2^{2\lambda} + 1, $$

$$ {}^{\lambda}M_2(G) = {}^{\lambda}M_2(G'') + [2(p+1)]^{\lambda} + 2^{\lambda} - [p^{\lambda} - (p+1)^{\lambda}] \sum_{i=1}^{p} (d_G(u_i))^{\lambda}. $$

Then

$$ \begin{aligned} {}^{\lambda}M_1(G) - {}^{\lambda}M_2(G) &= {}^{\lambda}M_1(G'') - {}^{\lambda}M_2(G'') + (p+1)^{2\lambda} - p^{2\lambda} + 2^{2\lambda} + 1 - [2(p+1)]^{\lambda} - 2^{\lambda} \\ &\quad + [p^{\lambda} - (p+1)^{\lambda}] \sum_{i=1}^{p} (d_G(u_i))^{\lambda} \\ &> (p+1)^{2\lambda} - p^{2\lambda} + 2^{2\lambda} + 1 - [2(p+1)]^{\lambda} - 2^{\lambda}. \end{aligned} $$

Suppose $g_1(x) = (x+1)^{2\lambda} - x^{2\lambda} + 2^{2\lambda} + 1 - [2(x+1)]^{\lambda} - 2^{\lambda}, (x \geqslant 2)$. Then

$$ \begin{aligned} g_1'(x) &= 2\lambda(x+1)^{2\lambda-1} - 2\lambda x^{2\lambda-1} - 2\lambda[2(x+1)]^{\lambda-1} \\ &= 2\lambda\{(x+1)^{2\lambda-1} - x^{2\lambda-1} - [2(x+1)]^{\lambda-1}\} \\ &> 0. \end{aligned} $$

Note that $g_1(x)$ is increasing in $x \geqslant 2$. We have $g_1(x) \geqslant g_1(2) = 1 + 9^{\lambda} - 2^{\lambda} - 6^{\lambda}$. Suppose $f_2(\lambda) = 1 + 9^{\lambda} - 2^{\lambda} - 6^{\lambda}, \lambda \in (-\infty, 0)$. Then

$$ \begin{aligned} f_2'(\lambda) &= 9^{\lambda} \ln 9 - 2^{\lambda} \ln 2 - 6^{\lambda} \ln 6 \\ &= 2 \cdot 9^{\lambda} \ln 3 - 2^{\lambda} \ln 2 - 6^{\lambda} \ln 2 - 6^{\lambda} \ln 3 \\ &= (9^{\lambda} - 6^{\lambda}) \ln 3 + (9^{\lambda} \ln 3 - \frac{2^{\lambda} + 6^{\lambda}}{2} \ln 4) \\ &< (9^{\lambda} - 6^{\lambda}) \ln 3 + \frac{(9^{\lambda} - 2^{\lambda}) + (9^{\lambda} - 6^{\lambda})}{2} \ln 4. \\ &< 0. \end{aligned} $$

Note that $f_2(\lambda)$ is decreasing on $(-\infty, 0)$ in λ. We have $f_2(\lambda) > f_2(0) = 1 + 9^0 - 2^0 - 6^0 = 0$. It can be seen that $g_1(x) \geqslant g_1(2) = 1 + 9^{\lambda} - 2^{\lambda} - 6^{\lambda} = f_2(\lambda) > f_2(0) = 0$. Then we know that ${}^{\lambda}M_1(G) - {}^{\lambda}M_2(G) > 0, \lambda \in (-\infty, 0)$.

Therefore, ${}^{\lambda}M_1(G) - {}^{\lambda}M_2(G) > 0, \lambda \in (-\infty, 0)$ when $k = 1$.

Case 2 When $k \geqslant 2$.

Since $G' = G - v$, G' is a connected unicyclic graph with $n-1$ vertices. If G' is a cycle, ${}^{\lambda}M_1(G') = {}^{\lambda}M_2(G')$. If G' is not a cycle, by the induction hypothesis, ${}^{\lambda}M_1(G') > {}^{\lambda}M_2(G')$. Now we compare ${}^{\lambda}M_1(G)$ and ${}^{\lambda}M_2(G)$.

$$ {}^{\lambda}M_1(G) = {}^{\lambda}M_1(G') + (k+1)^{2\lambda} - k^{2\lambda} + 1, $$

$$^{\lambda}M_2(G) = {}^{\lambda}M_2(G') - [k^{\lambda} - (k+1)^{\lambda}]\sum_{i=1}^{k}(d_G(v_i))^{\lambda} + (k+1)^{\lambda}.$$

Then

$$\begin{aligned} {}^{\lambda}M_1(G) - {}^{\lambda}M_2(G) &= {}^{\lambda}M_1(G') - {}^{\lambda}M_2(G') + (k+1)^{2\lambda} - k^{2\lambda} + 1 - (k+1)^{\lambda} \\ &\quad +[k^{\lambda} - (k+1)^{\lambda}]\sum_{i=1}^{k}(d_G(v_i))^{\lambda} \\ &> (k+1)^{2\lambda} - k^{2\lambda} + 1 - (k+1)^{\lambda}. \end{aligned}$$

Suppose $g_2(x) = (x+1)^{2\lambda} - x^{2\lambda} + 1 - (x+1)^{\lambda}, (x \geqslant 2)$. Then

$$\begin{aligned} g_2'(x) &= 2\lambda(x+1)^{2\lambda-1} - 2\lambda x^{2\lambda-1} - \lambda(x+1)^{\lambda-1} \\ &= \lambda[2(x+1)^{2\lambda-1} - 2x^{2\lambda-1} - (x+1)^{\lambda-1}] \\ &> 0. \end{aligned}$$

Note that $g_2(x)$ is increasing in $x \geqslant 2$. We have $g_2(x) \geqslant g_2(2) = 1 + 9^{\lambda} - 4^{\lambda} - 3^{\lambda} = f_1(\lambda)$. From the foregoing proof, it has been know that $f_1(\lambda) > 0$. It can be seen that $g_2(x) \geqslant g_2(2) = 1 + 9^{\lambda} - 4^{\lambda} - 3^{\lambda} = f_1(\lambda) > 0$. Therefore, we have ${}^{\lambda}M_1(G) - {}^{\lambda}M_2(G) > 0, \lambda \in (-\infty, 0)$ when $k \geqslant 2$.

This completes the proof of theorem. □

3 Comparing Variable Zagreb Indices for Unicyclic Graphs for $\lambda > 1$

Now we discuss the changing situation of the Zagreb indices when $\lambda > 1$.

Case 1

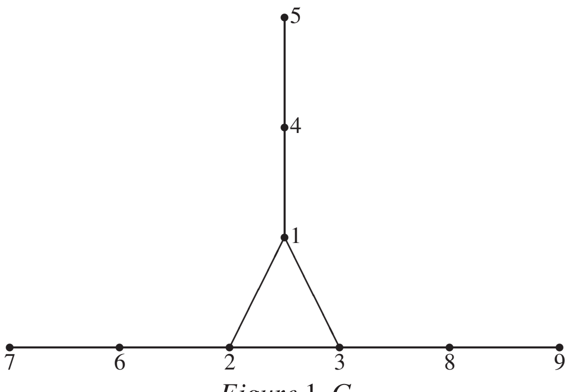

Figure 1. G_1

Since $G = G_1$(see Fig.1),

$$^{\lambda}M_2(G_1) - {}^{\lambda}M_1(G_1) = 3(6^{\lambda} + 2^{\lambda} - 4^{\lambda} - 1).$$

Suppose $h_1(\lambda) = 3(6^{\lambda} + 2^{\lambda} - 4^{\lambda} - 1), (\lambda > 1)$. Then

$$h_1'(\lambda) = 3(6^{\lambda} \ln 6 + 2^{\lambda} \ln 2 - 4^{\lambda} \ln 4) > 0.$$

Note that $h_1(\lambda)$ is increasing in $\lambda > 1$. We have $h_1(\lambda) > h(1) = 6 > 0$. Therefore $\forall \lambda_1 > 1, {}^{\lambda_1}M_2(G_1) - {}^{\lambda_1}M_1(G_1) > 0$.

Case 2

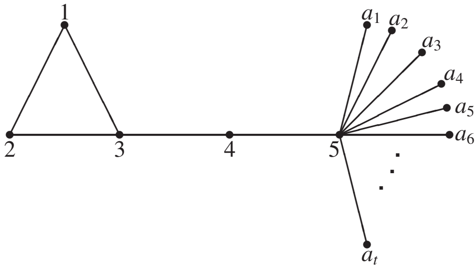

Figure 2. G_2

Since $G = G_2$(see Fig.2),

$$^{\lambda}M_2(G_2) - {}^{\lambda}M_1(G_2) = 6^{\lambda} + 6^{\lambda} + 6^{\lambda} + [2(t+1)]^{\lambda} + t(t+1)^{\lambda} - 4^{\lambda} - 4^{\lambda} - 9^{\lambda} - (t+1)^{2\lambda} - t.$$

Suppose $h_2(t) = 6^{\lambda} + 6^{\lambda} + 6^{\lambda} + [2(t+1)]^{\lambda} + t(t+1)^{\lambda} - 4^{\lambda} - 4^{\lambda} - 9^{\lambda} - (t+1)^{2\lambda} - t, (t \geqslant 1, \lambda > 1)$. Then

$$\begin{aligned} h_2'(t) &= 2\lambda[2(t+1)]^{\lambda-1} + (t+1)^{\lambda} + \lambda t(t+1)^{\lambda-1} - 2\lambda(t+1)^{2\lambda-1} - 1 \\ &= 2^{\lambda}\lambda(t+1)^{\lambda-1} + (t+1)^{\lambda} + \lambda t(t+1)^{\lambda-1} - 2\lambda(t+1)^{2\lambda-1} - 1. \end{aligned}$$

Moreover,

$$\begin{aligned} h_2''(t) &= 2^{\lambda}\lambda(\lambda-1)(t+1)^{\lambda-2} + \lambda(t+1)^{\lambda-1} + \lambda(t+1)^{\lambda-1} \\ &\quad + \lambda(\lambda-1)t(t+1)^{\lambda-2} - 2\lambda(2\lambda-1)(t+1)^{2\lambda-2} \\ &= [2^{\lambda}\lambda(\lambda-1) + 2\lambda t + 2\lambda + \lambda(\lambda-1)t - 2\lambda(2\lambda-1)(t+1)^{\lambda}](t+1)^{\lambda-2}. \end{aligned}$$

Suppose $l(t) = 2^{\lambda}\lambda(\lambda-1) + 2\lambda t + 2\lambda + \lambda(\lambda-1)t - 2\lambda(2\lambda-1)(t+1)^{\lambda}, (t \geqslant 1, \lambda > 1)$. Then

$$l'(t) = 2\lambda + \lambda(\lambda-1) - 2\lambda(2\lambda-1)\lambda(t+1)^{\lambda-1}$$

$$
\begin{aligned}
&< \lambda[2+\lambda-1-2\lambda(2\lambda-1)] \\
&= \lambda(1+4\lambda)(1-\lambda) \\
&< 0.
\end{aligned}
$$

Note that $l(t)$ is decreasing in $t \geqslant 1$, we have

$$
\begin{aligned}
l(t) \leqslant l(1) &= 2^{\lambda}\lambda^2 - 2^{\lambda}\lambda + 2\lambda + 2\lambda + \lambda^2 - \lambda - 2\lambda(2\lambda-1)2^{\lambda} \\
&= \lambda 2^{\lambda} + \lambda^2 + 3\lambda - 3\lambda^2 2^{\lambda}.
\end{aligned}
$$

Suppose $r(\lambda) = \lambda 2^{\lambda} + \lambda^2 + 3\lambda - 3\lambda^2 2^{\lambda}, (\lambda > 1)$. Then

$$
\begin{aligned}
r'(\lambda) &= 2^{\lambda} + \lambda 2^{\lambda}\ln 2 + 2\lambda + 3 - 6\lambda 2^{\lambda} - 3\lambda^2 2^{\lambda}\ln 2 \\
&= (2^{\lambda} + 2\lambda + 3 - 6\lambda 2^{\lambda}) + (1-3\lambda)\lambda 2^{\lambda}\ln 2 \\
&= [(2^{\lambda} - \lambda 2^{\lambda}) + (2\lambda - 2\lambda 2^{\lambda}) + (3 - 3\lambda 2^{\lambda})] + (1-3\lambda)\lambda 2^{\lambda}\ln 2 \\
&< 0.
\end{aligned}
$$

Note that $r(\lambda)$ is decreasing in $\lambda > 1$, we have $r(\lambda) < r(1) = 2+1+3-6 = 0$. It can be seen that $l(t) \leqslant l(1) = \lambda 2^{\lambda} + \lambda^2 + 3\lambda - 3\lambda^2 2^{\lambda} = r(\lambda) < r(1) = 0$. Therefore, $h_2''(t) = (t+1)^{\lambda-2}\cdot l(t) < 0$, then $h_2(t)$ is a concave function, and $h_2'(t)$ is decreasing in $t \geqslant 1$. Hence we have

$$
\begin{aligned}
h_2'(t) \leqslant h_2'(1) &= 2^{\lambda}\lambda 2^{\lambda-1} + 2^{\lambda} + \lambda 2^{\lambda-1} - 2\lambda 2^{2\lambda-1} - 1 \\
&= 2^{\lambda} + \lambda 2^{\lambda-1} - \lambda 2^{2\lambda-1} - 1.
\end{aligned}
$$

Suppose $s(\lambda) = 2^{\lambda} + \lambda 2^{\lambda-1} - \lambda 2^{2\lambda-1} - 1, (\lambda > 1)$. Then

$$
\begin{aligned}
s'(\lambda) &= 2^{\lambda}\ln 2 + 2^{\lambda-1} + \lambda 2^{\lambda-1}\ln 2 - 2^{2\lambda-1} - \lambda 2^{2\lambda}\ln 2 \\
&= (2^{\lambda-1} - 2^{2\lambda-1}) + (2^{\lambda} + \lambda 2^{\lambda-1} - \lambda 2^{2\lambda})\ln 2 \\
&= (2^{\lambda-1} - 2^{2\lambda-1}) + [(2^{\lambda} - 2^{\lambda}\lambda 2^{\lambda-1}) + (\lambda 2^{\lambda-1} - 2^{\lambda}\lambda 2^{\lambda-1})]\ln 2 \\
&< 0.
\end{aligned}
$$

Note that $s(\lambda)$ is decreasing in $\lambda > 1$, we have $s(\lambda) < s(1) = 2+1-2-1 = 0$. Therefore, $h_2'(t) \leqslant h_2'(1) = 2^{\lambda} + \lambda 2^{\lambda-1} - \lambda 2^{2\lambda-1} - 1 = s(\lambda) < s(1) = 0$. Then $h_2(t)$ is strictly decreasing in $t \geqslant 1$. By the discussion above, $h_2(t)$ is a strictly decreasing concave function. Therefore, $\forall \lambda_2 > 1$, we can find a positive integer T to cause ${}^{\lambda_2}M_2(G_2) - {}^{\lambda_2}M_1(G_2) < 0$ when $t \geqslant T$.

Combining Case 1 and Case 2, $\forall \lambda > 1$, we can find a suitable graph G_1^* to cause ${}^{\lambda}M_2(G_1^*) - {}^{\lambda}M_1(G_1^*) > 0$, or we can find a suitable graph G_2^* to cause ${}^{\lambda}M_2(G_2^*) - {}^{\lambda}M_1(G_2^*) < 0$. Therefore, when $\lambda > 1$, the relationship of numerical value between ${}^{\lambda}M_1(G)/n$ and ${}^{\lambda}M_2(G)/m$ is indefinite to the distinct unicyclic graphs.

4 Conclusion

With the foregoing discussion and the conclusion in [4], the relationship of ${}^{\lambda}M_1(G)/n$ and ${}^{\lambda}M_2(G)/m$ in unicyclic graphs for $\lambda \in R$ can be seen that:

(i) ${}^{\lambda}M_1(G)/n \geqslant {}^{\lambda}M_2(G)/m, \lambda \in (-\infty, 0)$,

(ii) ${}^{\lambda}M_1(G)/n \leqslant {}^{\lambda}M_2(G)/m, \lambda \in [0, 1]$,

(iii) the relationship of numerical value between ${}^{\lambda}M_1(G)/n$ and ${}^{\lambda}M_2(G)/m$ is indefinite in the distinct unicyclic graphs when $\lambda \in (1, +\infty)$.

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