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On Comparing the Variable Zagreb Indices*

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Abstract

Let G be a simple graph with n vertices and m edges. The variable first and second Zagreb indices are defined to be

$$^{\lambda}M_1(G) = \sum_{u\in V}(d(u))^{2\lambda} \text{ and } {}^{\lambda}M_2(G) = \sum_{uv\in E}(d(u)d(v))^{\lambda}$$

where λ is any real number. In this paper, it is shown that ${}^{\lambda}M_1(G)/n \geq {}^{\lambda}M_2(G)/m$ for all graphs G and $\lambda \in (-\infty,\ 0)$, which implies the results in [6, 9, 13]. We also show that the relationship of numerical value between ${}^{\lambda}M_1(G)/n$ and ${}^{\lambda}M_2(G)/m$ is indefinite in the distinct trees (resp. chemical graphs and bicyclic graphs) for $\lambda \in (1, +\infty)$. With the conclusions in [9, 10], we finish discussing the direct comparison between ${}^{\lambda}M_1(G)/n$ and ${}^{\lambda}M_2(G)/m$ in trees (resp. chemical graphs) for $\lambda \in R$.

1 Introduction

The first and second Zagreb indices are among the oldest and the most famous topological indices (see [2] and references within) and they are defined as

$$M_1(G) = \sum_{u\in V}(d(u))^2 \text{ and } M_2(G) = \sum_{uv\in E} d(u)d(v)$$

where $G = (V,\ E)$ is a simple graph with n vertices and m edges, and $d(u)$ is the degree of vertex u. These indices have been generalized to the variable first and second Zagreb indices ([7]) defined as

$$^{\lambda}M_1(G) = \sum_{u\in V}(d(u))^{2\lambda} \text{ and } {}^{\lambda}M_2(G) = \sum_{uv\in E}(d(u)d(v))^{\lambda}$$

where λ is any real number. Clearly, ${}^{1}M_1(G) = M_1(G)$ and ${}^{1}M_2(G) = M_2(G)$.

A natural issue is to compare the values of the Zagreb indices on the same graph. Observe that, for general graphs, the order of magnitude of M_1 is $O(n^3)$ while the order of magnitude

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of M_2 is $O(mn^2)$. This suggests comparing M_1/n with M_2/m instead of M_1 and M_2. In [1], the AutoGraphiX system proposed the following conjecture:

Conjecture 1.1 *([1]) For all simple connected graphs G,*

$$M_1(G)/n \leq M_2(G)/m$$

and the bound is tight for complete graphs.

However, this conjecture does not hold for all general graphs ([3]), while it is proved to be true for chemical graphs ([3]), trees ([8]), unicyclic graphs ([5]), and connected bicyclic graphs except one class ([12]).

Analogously as Conjecture 1.1, many mathematicians proved that

$$^{\lambda}M_1(G)/n \leq {}^{\lambda}M_2(G)/m \tag{1}$$

is true for the following cases: all graphs and $\lambda \in [0, \frac{1}{2}]$ ([9]), all chemical graphs and $\lambda \in [0, 1]$ ([9]), all trees and $\lambda \in [0, 1]$ ([10]), all unicyclic graphs and $\lambda \in [0, 1]$ ([4]), all graphs G satisfying $\Delta(G) - \delta(G) \leq 2$ (resp. $\Delta(G) - \delta(G) \leq 3$ and $\delta(G) \neq 2$) and $\lambda \in [0, 1]$ ([6, 11]), where $\Delta(G)$ and $\delta(G)$ denote the maximum and minimum degrees of G, respectively.

On the other hand, the inequality

$$^{\lambda}M_1(G)/n \geq {}^{\lambda}M_2(G)/m \tag{2}$$

holds for the following cases: all chemical graphs and $\lambda \in (-\infty, 0]$ ([6]), all unbalanced bipartite graphs and $\lambda \in R\backslash[0, 1]$ ([9]), all unicyclic graphs and $\lambda \in (-\infty, 0]$ ([13]), all graphs G satisfying $\Delta(G) - \delta(G) \leq 2$ (resp. $\Delta(G) - \delta(G) \leq 3$ and $\delta(G) \neq 2$) and $\lambda \in (-\infty, 0]$ ([6]).

In this paper, we show that ${}^{\lambda}M_1(G)/n \geq {}^{\lambda}M_2(G)/m$ for all graphs G and $\lambda \in (-\infty, 0)$, which implies the results in [6, 9, 13]. Moreover, the relationship of numerical value between ${}^{\lambda}M_1(G)/n$ and ${}^{\lambda}M_2(G)/m$ is proved to be indefinite in the distinct trees (resp. chemical graphs and bicyclic graphs) for each $\lambda \in (1, +\infty)$. With the conclusions in [9, 10], we finish discussing the direct comparison between ${}^{\lambda}M_1(G)/n$ and ${}^{\lambda}M_2(G)/m$ in trees (resp. chemical graphs) for $\lambda \in R$.

2 Main results

To begin with, we introduce some lemmas which are useful in this paper. We start with the special case of the rearrangement inequality (proof is given for the sake of the completeness of the results).

Lemma 2.1 *Let $a,\ b,\ c,\ d$ be positive real numbers with $a \geq b$ and $c \geq d$. Then*

$$ac + bd \geq ad + bc\ .$$

Moreover, the equality above holds if and only if $a = b$ or $c = d$.

Proof. Suppose $a = b + \varepsilon_1$ and $c = d + \varepsilon_2$, where $\varepsilon_1,\ \varepsilon_2 \geq 0$. Thus

$$ac + bd = (b + \varepsilon_1)(d + \varepsilon_2) + bd = (b + \varepsilon_1)d + b(d + \varepsilon_2) + \varepsilon_1\varepsilon_2\ .$$

Hence

$$ac + bd = ad + bc + \varepsilon_1\varepsilon_2 \geq ad + bc$$

and the equality holds if and only if $\varepsilon_1 = 0$ or $\varepsilon_2 = 0$, that is, $a = b$ or $c = d$. □

From Lemma 1 in [9], it follows that (again proof is given for the sake of the completeness of the results)

Lemma 2.2 *Let $a,\ b$ be positive integers and $\lambda \in (-\infty,\ 0)$. Let*

$$f(a,\ b) = a^{\lambda} \cdot b^{\lambda} \cdot \left(\frac{1}{a} + \frac{1}{b}\right) - a^{2\lambda-1} - b^{2\lambda-1}\ .$$

Then $f(a,\ b) \leq 0$, and the equality holds if and only if $a = b$.

Proof. On the one hand, if $a = b$, it is obvious that

$$f(a,\ a) = a^{2\lambda} \cdot \frac{2}{a} - 2a^{2\lambda-1} = 0\ .$$

Now it will suffice to show that $f(a,\ b) < 0$ if $a \neq b$. Note that the expression of $f(a,\ b)$ above is symmetric in a and b. Hence, we may assume that $a > b$. Denote $x = \frac{a}{b} > 1$. Then we have

$$\frac{a \cdot b \cdot f(a,\ b)}{b^{2\lambda+1}} = x^{\lambda} + x^{\lambda+1} - x^{2\lambda} - x = x \cdot \left(1 - x^{\lambda-1}\right) \cdot \left(x^{\lambda} - 1\right)\ .$$

Therefore, $f(a,\ b)$ has the same sign as $x \cdot \left(1 - x^{\lambda-1}\right) \cdot \left(x^{\lambda} - 1\right)$. Note that

$$x > 1 > 0,\ 1 - x^{\lambda-1} > 0,\ and\ x^{\lambda} - 1 < 0\ for\ each\ \lambda < 0\ .$$

Hence $x \cdot \left(1 - x^{\lambda-1}\right) \cdot \left(x^{\lambda} - 1\right) < 0$ for $\lambda < 0$, and this completes the proof. □

Let G be a simple graph. We denote the number of vertices of degree i in G by n_i and the number of edges that connect vertices of degree i and j by m_{ij}, where we do not distinguish m_{ij}

and m_{ji}. Let N denote the set of the degrees of vertices in G. Let $\{i,\ j\},\ \{k,\ l\} \in N^2,\ \lambda < 0$, and suppose

$$\mu = \sum_{k\le l\in N} m_{kl} \cdot \sum_{k\le l\in N} m_{kl}\left(\frac{1}{k}+\frac{1}{l}\right),\ \ and$$

$$g^{\lambda}_{\{i,\ j\},\ \{k,\ l\}} = i^{\lambda}\cdot j^{\lambda}\cdot\left(\frac{1}{k}+\frac{1}{l}\right)+k^{\lambda}\cdot l^{\lambda}\cdot\left(\frac{1}{i}+\frac{1}{j}\right)-i^{2\lambda-1}-j^{2\lambda-1}-k^{2\lambda-1}-l^{2\lambda-1}\ .$$

Lemma 2.3 *Let G be a simple graph with n vertices and m edges. Then*

$${}^{\lambda}M_2(G)/m - {}^{\lambda}M_1(G)/n = \frac{1}{\mu}\cdot \sum_{\substack{i\le j,\ k\le l,\\ \{i,\ j\},\ \{k,\ l\}\subseteq N^2}} \left(g^{\lambda}_{\{i,\ j\},\ \{k,\ l\}}\cdot m_{ij}\cdot m_{kl}\right)\ .$$

Proof. On the one hand, we have

$$\frac{{}^{\lambda}M_2(G)}{m} = \frac{\sum_{uv\in E}[d(u)d(v)]^{\lambda}}{m} = \frac{\sum_{i\le j\in N}\left(m_{ij}\cdot i^{\lambda}\cdot j^{\lambda}\right)}{\sum_{i\le j\in N} m_{ij}},\ and$$

$$\frac{{}^{\lambda}M_1(G)}{n} = \frac{\sum_{v\in V}[d(v)]^{2\lambda}}{\sum_{i\in N} n_i} = \frac{\sum_{i\in N}\left(n_i\cdot i^{2\lambda}\right)}{\sum_{i\in N}\left[\left(m_{ii}+\sum_{j\in N} m_{ij}\right)\cdot\frac{1}{i}\right]}$$

$$= \frac{\sum_{i\in N}\left[\left(m_{ii}+\sum_{j\in N} m_{ij}\right) i^{2\lambda-1}\right]}{\sum_{i\le j\in N} m_{ij}\left(\frac{1}{i}+\frac{1}{j}\right)} = \frac{\sum_{i\le j\in N} m_{ij}\left(i^{2\lambda-1}+j^{2\lambda-1}\right)}{\sum_{i\le j\in N} m_{ij}\left(\frac{1}{i}+\frac{1}{j}\right)}\ .$$

Therefore,

$${}^{\lambda}M_2(G)/m - {}^{\lambda}M_1(G)/n = \frac{\sum_{i\le j\in N}\left(m_{ij}\cdot i^{\lambda}\cdot j^{\lambda}\right)}{\sum_{k\le l\in N} m_{kl}} - \frac{\sum_{i\le j\in N} m_{ij}\left(i^{2\lambda-1}+j^{2\lambda-1}\right)}{\sum_{k\le l\in N} m_{kl}\left(\frac{1}{k}+\frac{1}{l}\right)}$$

$$= \frac{1}{\mu}\cdot\left\{\left[\sum_{i\le j\in N} m_{ij}\cdot i^{\lambda}\cdot j^{\lambda}\right]\left[\sum_{k\le l\in N} m_{kl}\left(\frac{1}{k}+\frac{1}{l}\right)\right]-\left[\sum_{k\le l\in N} m_{kl}\right]\left[\sum_{i\le j\in N} m_{ij}\left(i^{2\lambda-1}+j^{2\lambda-1}\right)\right]\right\}$$

$$= \frac{1}{\mu}\cdot\sum_{\substack{i\le j,\ k\le l,\\ i,\ j,\ k,\ l\in N}}\left\{\left[i^{\lambda}j^{\lambda}\left(\frac{1}{k}+\frac{1}{l}\right)-i^{2\lambda-1}-j^{2\lambda-1}\right]m_{ij}m_{kl}\right\}\ .$$

Collecting in the same summand the case where roles of $(i,\ j)$ and $(k,\ l)$ are reversed, it follows that ${}^{\lambda}M_2(G)/m - {}^{\lambda}M_1(G)/n$

$$= \frac{1}{\mu}\cdot\sum_{\substack{i\le j,\ k\le l,\\ \{i,\ j\},\ \{k,\ l\}\subseteq N^2}}\left\{\left[i^{\lambda}j^{\lambda}\left(\frac{1}{k}+\frac{1}{l}\right)+k^{\lambda}l^{\lambda}\left(\frac{1}{i}+\frac{1}{j}\right)-i^{2\lambda-1}-j^{2\lambda-1}-k^{2\lambda-1}-l^{2\lambda-1}\right]m_{ij}m_{kl}\right\}\ .$$

Hence ${}^{\lambda}M_2(G)/m - {}^{\lambda}M_1(G)/n = \frac{1}{\mu}\cdot\sum_{\substack{i\le j,\ k\le l,\\ \{i,\ j\},\ \{k,\ l\}\subseteq N^2}}\left(g^{\lambda}_{\{i,\ j\},\ \{k,\ l\}}\cdot m_{ij}\cdot m_{kl}\right)$. □

A graph G is called k-regular if $d(v) = k$ for all $v \in V(G)$.

Theorem 2.4 *Let G be a simple graph with n vertices and m edges. Then*

$${}^{\lambda}M_1(G)/n \ge {}^{\lambda}M_2(G)/m\ for\ \lambda\in(-\infty,\ 0)\ .$$

Moreover, the equality holds if and only if G is a regular graph.

Proof. If $\lambda \in (-\infty,\ 0)$, then by Lemma 2.3,

$$^{\lambda}M_2(G)/m - {}^{\lambda}M_1(G)/n = \frac{1}{\mu} \cdot \sum_{\substack{i \le j,\ k \le l, \\ \{i,\ j\},\ \{k,\ l\} \subseteq N^2}} \left(g^{\lambda}_{\{i,\ j\},\ \{k,\ l\}} \cdot m_{ij} \cdot m_{kl} \right) .$$

Hence, we need to show that $g^{\lambda}_{\{i,\ j\},\ \{k,\ l\}} \le 0$ for each $\{i,\ j\},\ \{k,\ l\} \subseteq N^2$.

Without loss of generality, suppose $j = max\{i,\ j,\ k,\ l\}$ and $k \le l$. Thus

$$\begin{aligned} \frac{\partial g^{\lambda}_{\{i,\ j\},\ \{k,\ l\}}}{\partial j} &= \frac{\lambda \cdot i^{\lambda} \cdot j^{\lambda-1}}{k} + \frac{\lambda \cdot i^{\lambda} \cdot j^{\lambda-1}}{l} - \frac{k^{\lambda} \cdot l^{\lambda}}{j^2} - (2\lambda - 1) \cdot j^{2\lambda-2} \\ &= (1 - 2\lambda) \cdot \frac{j^{2\lambda}}{j^2} + \lambda \cdot \frac{i^{\lambda} \cdot j^{\lambda}}{k \cdot j} + \lambda \cdot \frac{i^{\lambda} \cdot j^{\lambda}}{l \cdot j} - \frac{k^{\lambda} \cdot l^{\lambda}}{j^2} . \end{aligned}$$

Note that $\lambda < 0$, and then

$$(-\lambda) \cdot \frac{j^{2\lambda}}{j^2} \le (-\lambda) \cdot \frac{i^{\lambda} \cdot j^{\lambda}}{k \cdot j},\ (-\lambda) \cdot \frac{j^{2\lambda}}{j^2} \le (-\lambda) \cdot \frac{i^{\lambda} \cdot j^{\lambda}}{l \cdot j},\ \frac{j^{2\lambda}}{j^2} \le \frac{k^{\lambda} \cdot l^{\lambda}}{j^2} .$$

It follows that

$$\frac{\partial g^{\lambda}_{\{i,\ j\},\ \{k,\ l\}}}{\partial j} \le 0 .$$

Hence it is sufficient to prove the claim when $j = max\{i,\ k,\ l\}$.

Case 1. $j = i$. In this case, we have

$$i^{\lambda} \cdot j^{\lambda} \le k^{\lambda} \cdot l^{\lambda} \ and\ \frac{1}{k} + \frac{1}{l} \ge \frac{1}{i} + \frac{1}{j} .$$

Then by Lemma 2.1,

$$g^{\lambda}_{\{i,\ j\},\ \{k,\ l\}} \le i^{\lambda} \cdot j^{\lambda} \cdot \left(\frac{1}{i} + \frac{1}{j}\right) + k^{\lambda} \cdot l^{\lambda} \cdot \left(\frac{1}{k} + \frac{1}{l}\right) - i^{2\lambda-1} - j^{2\lambda-1} - k^{2\lambda-1} - l^{2\lambda-1}$$

$$= \left[i^{\lambda} \cdot j^{\lambda} \cdot \left(\frac{1}{i} + \frac{1}{j}\right) - i^{2\lambda-1} - j^{2\lambda-1} \right] + \left[k^{\lambda} \cdot l^{\lambda} \cdot \left(\frac{1}{k} + \frac{1}{l}\right) - k^{2\lambda-1} - l^{2\lambda-1} \right] .$$

Combining this with Lemma 2.2, we conclude that $g^{\lambda}_{\{i,\ j\},\ \{k,\ l\}} \le 0$.

Case 2. $j = l$. Without loss of generality, suppose $i \ge k$. Note that

$$\frac{\partial g^{\lambda}_{\{i,\ j\},\ \{k,\ j\}}}{\partial k} = (1 - 2\lambda) \cdot \frac{k^{2\lambda}}{k^2} + \lambda \cdot \frac{j^{\lambda} \cdot k^{\lambda}}{j \cdot k} + \lambda \cdot \frac{j^{\lambda} \cdot k^{\lambda}}{i \cdot k} - \frac{i^{\lambda} \cdot j^{\lambda}}{k^2} .$$

Since

$$\frac{k^{2\lambda}}{k^2} \ge \frac{i^{\lambda} \cdot j^{\lambda}}{k^2},\ (-\lambda) \cdot \frac{k^{2\lambda}}{k^2} \ge (-\lambda) \cdot \frac{j^{\lambda} \cdot k^{\lambda}}{j \cdot k},\ (-\lambda) \cdot \frac{k^{2\lambda}}{k^2} \ge (-\lambda) \cdot \frac{j^{\lambda} \cdot k^{\lambda}}{i \cdot k}$$

it follows that

$$\frac{\partial g^{\lambda}_{\{i,\ j\},\ \{k,\ l\}}}{\partial k} \ge 0$$

Hence it will suffice to prove the claim when $k = i$. By Lemma 2.2,

$$g^{\lambda}_{\{i,\ j\},\ \{i,\ j\}} = 2 \cdot \left[i^{\lambda} \cdot j^{\lambda} \cdot \left(\frac{1}{i} + \frac{1}{j} \right) - i^{2\lambda-1} - j^{2\lambda-1} \right] \leq 0$$

Therefore, we obtain that ${}^{\lambda}M_1(G)/n \geq {}^{\lambda}M_2(G)/m\ for\ \lambda \in (-\infty,\ 0)$.

Moreover, from the foregoing proof and combining Lemmas 2.1 and 2.2, the equality above holds if and only if $g^{\lambda}_{\{i,\ j\},\ \{k,\ l\}} = 0$ for all $m_{ij} \cdot m_{kl} > 0$, which means $i = j = k = l$ for each $\{i,\ j\},\ \{k,\ l\} \subseteq N^2$, that is, G is a regular graph. □

Remark 1 *If $\lambda \in (-\infty,\ 0)$, by Theorem 2.4, for all chemical graphs ([6]), unbalanced bipartite graphs ([9]), trees, unicyclic graphs ([13]), bicyclic graphs, the inequality ${}^{\lambda}M_1(G)/n \geq {}^{\lambda}M_2(G)/m$ holds.*

Finally, we discuss the relationship between ${}^{\lambda}M_1(G)/n$ and ${}^{\lambda}M_2(G)/m$ for trees (resp. chemical graphs and bicyclic graphs) for each $\lambda \in (1, +\infty)$.

The star graph S_n is a tree on n vertices with one vertex having degree $n - 1$ and the other vertices having degree 1. A complete bipartite graph is a simple bipartite graph with bipartition $(X,\ Y)$ in which each vertex of X is joined to each vertex of Y; if $|X| = n_1$ and $|Y| = n_2$, such a graph is denoted by $K_{n_1,\ n_2}$.

Example 1 *Let $G_1 = S_n$ $(n > 1)$. It is obvious that G_1 is a tree, and*

$$ {}^{\lambda}M_2(G_1)/m - {}^{\lambda}M_1(G_1)/n = \frac{n \cdot (n-1)^{\lambda} - (n-1)^{2\lambda} - (n-1)}{n} < 0\ for\ \lambda > 1\ .$$

Example 2 *Let $G_2(t)$ be the graph shown as in Fig. 1. Clearly, $G_2(t)$ is a tree of order $3t + 2$ with t vertices having degree 4 and the other vertices having degree 1. By directly computing, we have*

$$\begin{aligned} {}^{\lambda}M_2(G_2(t))/m - {}^{\lambda}M_1(G_2(t))/n &= \frac{(2t+2) \cdot 4^{\lambda} + (t-1) \cdot 16^{\lambda}}{3t+1} - \frac{2t + 2 + t \cdot 16^{\lambda}}{3t+2} \\ &= \frac{(2t+2)\left[(3t+2)\left(4^{\lambda} - 1\right) - 16^{\lambda} + 1\right]}{(3t+1)(3t+2)}\ . \end{aligned}$$

Therefore, $\forall \lambda > 1$, we can find a positive integer T to cause $(3t + 2)\left(4^{\lambda} - 1\right) - 16^{\lambda} + 1 > 0$ when $t \geq T$, which implies ${}^{\lambda}M_2(G_2(t))/m - {}^{\lambda}M_1(G_2(t))/n > 0$.

From Examples 1 and 2, when $\lambda > 1$, we can find a suitable tree G_1^* such that ${}^{\lambda}M_2(G_1^*)/m - {}^{\lambda}M_1(G_1^*)/n < 0$, and a suitable tree G_2^* such that ${}^{\lambda}M_2(G_2^*)/m - {}^{\lambda}M_1(G_2^*)/n > 0$. Consequently, when $\lambda \in (1, +\infty)$, the relationship of numerical value between ${}^{\lambda}M_1(G)/n$ and ${}^{\lambda}M_2(G)/m$ is indefinite for distinct trees.

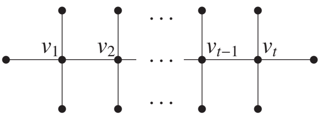

Figure 1. $G_2(t)$

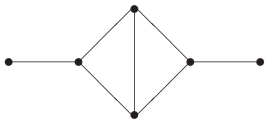

Figure 2. G_3

Example 3 *Let G_3 be the graph shown as in Fig. 2. Obviously, G_3 is a chemical graph (also a bicyclic graph), and*

$$^{\lambda}M_2(G_3)/m - {}^{\lambda}M_1(G_3)/n = \frac{2\cdot 3^{\lambda}+5\cdot 9^{\lambda}}{7} - \frac{2+4\cdot 9^{\lambda}}{6} > 0 \; for \; \lambda > 1 \, .$$

Example 4 *Let $G_4 = K_{2,\,3}$. Then G_4 is a chemical graph (also a bicyclic graph),*

$$^{\lambda}M_2(G_4)/m - {}^{\lambda}M_1(G_4)/n = \frac{5\cdot 6^{\lambda} - 3\cdot 4^{\lambda} - 2\cdot 9^{\lambda}}{5} < 0 \; for \; \lambda > 1 \, .$$

From Examples 3 and 4, when $\lambda > 1$, there is a suitable chemical graph (resp. bicyclic graph) G_3^* such that $^{\lambda}M_2(G_3^*)/m - {}^{\lambda}M_1(G_3^*)/n > 0$, and a suitable chemical graph (resp. bicyclic graph) G_4^* such that $^{\lambda}M_2(G_4^*)/m - {}^{\lambda}M_1(G_4^*)/n < 0$. Therefore, when $\lambda \in (1, +\infty)$, the relationship between $^{\lambda}M_1(G)/n$ and $^{\lambda}M_2(G)/m$ is indefinite for distinct chemical graphs (resp. bicyclic graphs).

With the foregoing discussions and the conclusions in [9, 10], we conclude that the relationships between $^{\lambda}M_1(G)/n$ and $^{\lambda}M_2(G)/m$ in trees (resp. chemical graphs, unicyclic graphs [13]) for $\lambda \in R$ are listed as follows.

(i) $^{\lambda}M_1(G)/n \geq {}^{\lambda}M_2(G)/m$ for $\lambda \in (-\infty,\ 0)$;

(ii) $^{\lambda}M_1(G)/n \leq {}^{\lambda}M_2(G)/m$ for $\lambda \in [0,\ 1]$ ([9, 10]) ;

(iii) The relationship of numerical value between $^{\lambda}M_1(G)/n$ and $^{\lambda}M_2(G)/m$ is indefinite when $\lambda \in (1, +\infty)$.

Remark 2 *For the relationship between $^{\lambda}M_1(G)/n$ and $^{\lambda}M_2(G)/m$ in bicyclic graphs, the conclusions (i) and (iii) are also true.*

Moreover, it is known that when $\lambda \in [0,\ \frac{1}{2}]$, $^{\lambda}M_1(G)/n \leq {}^{\lambda}M_2(G)/m$ for all graphs (including bicyclic graphs) ([9]); when $\lambda = 1$, the inequality $M_1/n \leq M_2/m$ holds for connected bicyclic graphs except one class ([12]).

Consequently, the relationship between $^{\lambda}M_1(G)/n$ and $^{\lambda}M_2(G)/m$ in bicyclic graphs remains to be determined for $\lambda \in (\frac{1}{2},\ 1)$.

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