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Some Inequalities for Szeged-Like Topological Indices of Graphs

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Abstract

The PI, vertex PI and edge Szeged index are three of most important Szeged-like topological indices introduced very recently. The aim of this paper is to present some sharp inequalities between PI, Vertex PI, Szeged and edge Szeged indices of graphs.

1 Introduction

Throughout this paper we only consider finite connected graph. Let G be a graph with vertex and edge sets $V(G)$ and $E(G)$, respectively. As usual, the distance between the vertices u and v of G is denoted by $d_G(u, v)$ ($d(u, v)$ for short) and it is defined as the number of edges in a shortest path connecting the vertices u and v.

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Suppose *Graph* denotes the class of all graphs. A map *Top* from *Graphs* into real numbers is called a topological index, if $G \cong H$ implies that $Top(G) = Top(H)$. The Wiener index was the first reported topological index based on graph distances, see [28]. This index is defined as the sum of all distances between vertices of the graph under consideration, see for detail [7, 8].

Let $e = uv$ be an edge of the graph G. The number of vertices of G whose distance to the vertex u is smaller than the distance to the vertex v is denoted by $n_u(e)$. Analogously, $n_v(e)$ is the number of vertices of G whose distance to the vertex v is smaller than the distance to the vertex u. The edge variants of $n_u(e)$ and $n_v(e)$ are denoted by $m_u(v)$ and $m_v(e)$, respectively. We now define four topological indices of the PI, vertex PI, Szeged and edge Szeged indices of the graph G as follows:

$$\begin{aligned}
PI(G) &= \sum_{e=uv} [m_u(e) + m_v(e)] \quad ([20,\ 19,\ 3]), \\
PI_v(G) &= \sum_{e=uv} [n_u(e) + n_v(e)] \quad ([22,\ 23,\ 1,\ 29]), \\
Sz(G) &= \sum_{e=uv} [n_u(e) \cdot n_v(e)] \quad ([12]), \\
Sz_e(G) &= \sum_{e=uv} [m_u(e) \cdot m_v(e)] \quad ([13,\ 21,\ 27]).
\end{aligned}$$

These topological indices attracted recently much attention [2, 4, 5, 9, 10, 11, 14, 15, 16, 17, 18, 24, 25].

Define $N_G(u)$ to be the set of all vertices adjacent to u. The diameter $diam(G)$ is the greatest distance between two vertices of G. The complete graph on vertices is denoted by K_n. Other notations are standard and taken mainly from [6, 26].

2 Main Results

A graph G is called $k-$regular if $deg_G(v) = k$, for all $v \in V(G)$; a regular graph is one that is $k-$regular for some k. A $k-$regular graph G is said to be strongly regular with parameters (v, k, r, s) if $|v(G)| = v$, any two adjacent vertices of G have exactly r common neighbors and any two non-adjacent vertices of G have exactly s common neighbors. In this case, we denote G by $Srg(v, k, r, s)$. It is well-known that if G is strongly regular with parameters (v, k, r, s) then $r > 0$. Moreover, strongly regular graphs have diameter 2.

Theorem 1. Let $G = Srg(v, k, r, s)$ be a connected graph. Then $PI_v(G) = kv(k-r)$

and $Sz(G) = m(k-r)^2$.

Proof. Let G be a strongly regular graph with parameters (v, k, r, s). It is easy to see that $s = 0$ if and only if G is a graph in which its components are complete graphs with the same vertices. Since G is connected, $G \cong K_v$, as desired. Suppose $s \geq 1$ and $e = uv$ is an edge of G. Define A_e to be the set of all vertices which are equidistant from u and v. Choose a vertex x outside $N_G(u) \cup N_G(v)$. Since strongly regular graphs have diameter 2, $d(u, x) = d(v, x) = 2$. Thus $n_u(e) = n_v(e) = k - r$. Therefore, $PI_v(G) = kv(k-r)$ and $Sz(G) = m(k-r)^2$. $\square$

A chordless cycle C of a graph H is a graph cycle of length at least four such that the graph cycle is an induced subgraph. A chordal graph is a simple graph possessing no chordless cycles.

Theorem 2. Suppose G is a graph.

(a) If for every $e = uv \in E(G)$, $Min\{n_u(e), n_v(e)\} = 1$ then G is complete or a chordal graph of diameter 2.

(b) If $Min\{m_u(e), m_v(e)\} = 1$ then G is a cycle of length ≤ 4.

Proof. (a) Suppose G is not chordal. Then there is a chordless cycle $C : u_1, u_2, \ldots, u_n, u_1$. Consider the edge $e = u_2u_3 \in C$. Since $d(u_1, u_2) < d(u_1, u_3$ and $d(u_3, u_4) < d(u_3, u_1)$, $Min\{n_u(e), n_v(e)\} > 1$ lead to a contradiction. So G is chordal. If $diam(G) \geq 3$ then there are vertices u and v such that $d(u, v) = 3$. Consider the path $P : u, w, z, v$ in G. Since $d(u, w) < d(u, v)$ and $d(z, v) < d(z, u)$, $Min\{n_u(wz), n_v(wz)\} > 1$ lead again to a contradiction. So $diam(G) = 2$ and the proof is complete. The proof of part (b) is the same as (a). $\square$

Theorem 3. Let G be connected graph with exactly n and m vertices and edges, respectively. Then the following statements are holds:

(a) $PI_v(G) \leq 2Sz(G)$ with equality if and only if G is a complete graph.

(b) $PI_v(G) \geq \frac{4}{n}Sz(G)$ with equality if and only if n is even and G is the complete bipartite graph $K_{\frac{n}{2},\frac{n}{2}}$.

(c) $PI_v(G)^2 > 4Sz(G)$, $PI_v(G)^2 > (2m+2Sz(G)$ and $P_v(G)^2 > PI_v(G)+2Sz(G)$.

(d) If for each edge $e = uv \in E(G)$, $Min\{n_u(e), n_v(e)\} = 1$ then $PI_v(G) > Sz(G)$.

(e) If for each edge $e = uv \in E(G)$, $Min\{n_u(e), n_v(e)\} > 1$ then $PI_v(G) < Sz(G)$.

Proof. (a) Since for every edge $e = uv$, $n_u(e) + n_v(e) \leq 2n_u(e)n_v(e)$, $PI_v(G) \leq$

$2Sz(G)$. The equality is satisfied if and only if for each edge $e = uv$, $n_u(e) = n_v(e) = 1$ if and only if G is complete. To prove (b), we notice that $(n_u(e) + n_v(e))^2 \geq 4n_u(e)n_v(e)$. Thus,

$$\begin{aligned} n\,PI_v(G) &= \sum_{e=uv} n[n_u(e) + n_v(e)] \\ &\geq \sum_{e=uv} [n_u(e) + n_v(e)]^2 \\ &\geq \sum_{e=uv} 4n_u(e)n_v(e) = 4Sz(G)\ . \end{aligned}$$

Obviously, the equality is satisfied if and only if $n_u(e) = n_v(e) = \frac{n}{2}$, for each edge $e = uv$, if and only if G is a complete bipartite graph $K_{\frac{n}{2},\frac{n}{2}}$. On the other hand, $PI_v(g)^2 \geq \sum_{e=uv}[n_u(e) + n_v(e)]^2 \geq 4Sz(G)$, leads to the proof of (c). Other parts of (c) are obtained in a similar way. To prove (d), it is enough to notice that if $Min\{n_u(e), n_v(e)\} = 1$ then $n_u(e) + n_v(e) > n_u(e)n_v(e)$. Hence $PI_v(G) > Sz(G)$. The proof of (e) is similar to (d) and it is omitted. □

Theorem 4. Let G be $n-$vertex connected graph. Then the following statements are holds:

(a) If $Min_{e=uv\in E(G)}\{m_u(e), m_v(e)\} \geq 1$ then for each edge $e = uv \in E(G)$, $Min\{m_u(e), m_v(e)\} \geq 1$ then $PI(G) \leq 2Sz_e(G)$ with equality if and only if G is a cycle graph of length ≤ 4.

(b) $PI(G) \geq \frac{4}{m-1}Sz_e(G)$ with equality if and only if $m-1$ is even and G is a tree with an odd number of vertices or a cycle of odd length.

(c) $PI(G)^2 > 4Sz_e(G)$ and if $Min_{e=uv\in E(G)}\{m_u(e), m_v(e)\} \geq 1$ then $PI(G)^2 > (2m + 2Sz(G)$.

(d) If for each edge $e = uv \in E(G)$, $Min\{m_u(e), m_v(e)\} = 1$ then $PI(G) > Sz_e(G)$.

(e) If for each edge $e = uv \in E(G)$, $Min\{m_u(e), m_v(e)\} > 1$ then $PI(G) < Sz_e(G)$.

(f) If for each edge $e = uv \in E(G)$, $Min\{m_u(e), m_v(e)\} = 0$ then G is isomorphic to K_2.

Proof. The proof is similar to those of Theorem 3 and is omitted. □

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