MATCH
Communications in Mathematical and in Computer Chemistry

MATCH Commun. Math. Comput. Chem. **62** (2009) 715-724

ISSN 0340 - 6253

Variable Zagreb Indices of K_{r+1}-free Graphs

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(Received January 8, 2008)

Abstract

Recently, in the paper [1], it has been established that for every K_{r+1}-free graph G with $n = xr$ vertices, it holds $M_1(G) \le M_1(T_{r+1}(n))$ and $M_2(G) \le M_2(T_{r+1}(n))$, where $T_{r+1}(n)$ is Turan's graph, M_1 is the first Zagreb index and M_2 is the second Zagreb index. In Turán's theorem [2], it is stated that $m(G) \le m(T_{r+1}(n))$, where m is the number of edges of graph. This can be reformulated as ${}^{1/2}M_1(G) \le {}^{1/2}M_1(T_{r+1}(n))$, where ${}^{1/2}M_1$ is the first variable Zagreb index for $\lambda = 1/2$. In this paper, we analyze generalizations of these results, i.e. we study for which λ it holds ${}^{\lambda}M_1(G) \le {}^{\lambda}M_1(T_{r+1}(n))$ and for which λ it holds ${}^{\lambda}M_2(G) \le {}^{\lambda}M_2(T_{r+1}(n))$.

1. Introduction

The first and second Zagreb indices are among the oldest and the most famous topological indices (see [3-6] and references within) and they are defined as:

$$M_1 = \sum_{i \in V} d_i^2 \text{ and } M_2 = \sum_{(i,j) \in E} d_i d_j ,$$

where V is the set of vertices, E is set of edges and d_i is degree of vertex i.

These indices have been generalized to variable first and second Zagreb indices [7] defined as

$$M_1 = \sum_{i \in V} d_i^{2\lambda} \text{ and } M_2 = \sum_{(i,j) \in E} \left(d_i d_j\right)^{\lambda} .$$

Some recent studies of mathematical properties of variable Zagreb indices can be found in [8-10].

Turán's graph $T_{r+1}(n)$ is $r-$partite complete graph in which the sizes of any two classes of the partition can differ for at most 1. We say that Turán's graph is completely balanced if all classes have the same size. In this case n is multiple of r, i.e. there is an integer x such that $n = rx$. Turán's theorem reads as:

Theorem A. If G is K_{r+1}-free graph with n vertices, then $m(G) \le \frac{1}{2}\left(1-\frac{1}{r}\right)n^2$ and the equality holds if and only if G is completely balanced Turán's graph.

This theorem can be obviously reformulated as:

Theorem B. If G is K_{r+1}-free graph with n vertices, then $\sum_{v\in V(G)} d_G(v) \le \left(1-\frac{1}{r}\right)n^2$ and the equality holds if and only if G is completely balanced Turán's graph.

In paper [1] it has been established that for every graph G with $n = xr$ vertices, it holds $M_1(G) \le M_1(T_{r+1}(n))$ and $M_2(G) \le M_2(T_{r+1}(n))$.

Obviously, completely balanced Turán's graph in both cases appears to be an extremal graph. In this paper, we propose the following questions:

1) For which λ it holds that ${}^{\lambda}M_1(G) \le {}^{\lambda}M_1(T_{r+1}(n))$?
2) For which λ it holds that ${}^{\lambda}M_2(G) \le {}^{\lambda}M_2(T_{r+1}(n))$?

Obviously, Theorem B gives affirmative answer to the first question for $\lambda = 1/2$ and the paper [1] for $\lambda = 1$. Also, paper [1] gives an affirmative answer for the second question in the case $\lambda = 1$. Here we prove:

Theorem 1. Let $r \ge 2$. If G is K_{r+1}-free graph with $n = xr$ vertices, then ${}^{\lambda}M_1(G) \le {}^{\lambda}M_1(T_{r+1}(n))$ for all $\lambda \in [0, 3/2]$. Moreover, this statement can not be extended to any $\lambda \in R \setminus [0, 3/2]$ (for $\lambda < 0$, it is assumed that G has no isolated points). More precisely, for each $\lambda \in R \setminus [0, 3/2]$, there are $r \ge 2$ and G such that ${}^{\lambda}M_1(G) > {}^{\lambda}M_1(T_{r+1}(n))$.

Theorem 2. Let $r \ge 2$. If G is K_{r+1}-free graph with $n = xr$ vertices, then ${}^{\lambda}M_2(G) \le {}^{\lambda}M_2(T_{r+1}(n))$ for all $\lambda \in [-1/2, 1]$. Moreover, this statement can not be extended to any $\lambda \in \langle -\infty, -1/2 \rangle \cup [5.4496, +\infty\rangle$ (for $\lambda < 0$, it is assumed that G has no isolated points). More precisely, for each $\lambda \in \langle -\infty, -1/2 \rangle \cup [5.4496, +\infty\rangle$, there are $r \ge 2$ and G such that ${}^{\lambda}M_2(G) > {}^{\lambda}M_2(T_{r+1}(n))$.

It also holds:

Theorem 3. If G is K_3-free graph with $n=2x$ vertices, then ${}^{\lambda}M_2(G) \leq {}^{\lambda}M_2(T_3(n))$ for all $\lambda \in [-1/2, +\infty]$. Moreover, this statement can not be extended to any $\lambda \in \langle -\infty, -1/2 \rangle$ (it is assumed that G has no isolated points).

Theorem 4. Let $r \geq 2$. If G is r-partite graph with $n = xr$ vertices, then ${}^{\lambda}M_2(G) \leq {}^{\lambda}M_2(T_{r+1}(n))$ for all $\lambda \in [-1/2, 3]$.

2. Proof of the Theorem 1

In the proof of the auxiliary Lemma, we shall use Taylor's theorem:

Theorem C Let function $y = f(x)$ be continuous on $[a, a+h]$ and be continuously derivable $n-1$ times on $[a, a+h]$ and has n-th derivation on $\langle a, a+h \rangle$, then it holds:

$$f(a+h) = f(a) + \frac{h}{1!} f'(a) + \frac{h^2}{2!} f''(a) + \ldots + \frac{h^{n-1}}{(n-1)!} f^{(n-1)}(a) + \frac{h^n}{n!} f^{(n)}(a + \phi h),$$

Where ϕ is some number in the interval $\langle 0,1 \rangle$.

Remark In the previous Theorem h can be positive or negative number.

Now, let us prove an auxiliary Lemma:

Lemma 1. For each $\mu > 3$, there is a rational number $x \in \langle -0.1, 0.1 \rangle$ such that

$$\left(\frac{1}{2}+x\right) \cdot \left(\frac{1}{2}-x\right)^{\mu} + \left(\frac{1}{2}-x\right) \cdot \left(\frac{1}{2}+x\right)^{\mu} > \left(\frac{1}{2}\right)^{\mu}.$$

Proof: Suppose to the contrary that:

$$\left(\frac{1}{2}+x\right) \cdot \left(\frac{1}{2}-x\right)^{\mu} + \left(\frac{1}{2}-x\right) \cdot \left(\frac{1}{2}+x\right)^{\mu} \leq \left(\frac{1}{2}\right)^{\mu} \quad \text{for all } x \in \langle -0.1, 0.1 \rangle.$$

Last inequality can be rewritten as:

$$\left(\frac{1}{4}-x^2\right)\left(\left(\frac{1}{2}+x\right)^{\mu-1} + \left(\frac{1}{2}-x\right)^{\mu-1}\right) \leq \left(\frac{1}{2}\right)^{\mu} \tag{1}$$

Applying Taylor's theorem to the function $f(x) = \left(\frac{1}{2}+x\right)^{\mu-1}$, we get:

$$\left(\frac{1}{2}+x\right)^{\mu-1}=\left(\frac{1}{2}\right)^{\mu}+\frac{x}{1}(\mu-1)\left(\frac{1}{2}\right)^{\mu-2}+\frac{x^2}{2}(\mu-1)(\mu-2)\left(\frac{1}{2}+\phi_1 x\right)^{\mu-3}; \tag{2}$$

$$\left(\frac{1}{2}-x\right)^{\mu-1}=\left(\frac{1}{2}\right)^{\mu}-\frac{x}{1}(\mu-1)\left(\frac{1}{2}\right)^{\mu-2}+\frac{x^2}{2}(\mu-1)(\mu-2)\left(\frac{1}{2}-\phi_2 x\right)^{\mu-3}, \tag{3}$$

Where $\phi_1,\phi_2\in\langle 0,1\rangle$. Putting (2) and (3) in (1), we get:

$$\left(\frac{1}{4}-x^2\right)\left[2\left(\frac{1}{2}\right)^{\mu-1}+\frac{x^2}{2}(\mu-1)(\mu-2)\left(\left(\frac{1}{2}+\phi_1 x\right)^{\mu-3}+\left(\frac{1}{2}-\phi_2 x\right)^{\mu-3}\right)\right]\le\left(\frac{1}{2}\right)^{\mu}$$

Dividing both hand-sides of the last inequality by $\left(\frac{1}{2}\right)^{\mu}$, we get:

$$\left(1-4x^2\right)\left[1+\frac{\frac{x^2}{2}(\mu-1)(\mu-2)}{2\left(\frac{1}{2}\right)^{\mu-1}}\left(\left(\frac{1}{2}+\phi_1 x\right)^{\mu-3}+\left(\frac{1}{2}-\phi_2 x\right)^{\mu-3}\right)\right]\le 1$$

The last inequality can be reformulated as:

$$-4+\frac{\frac{1}{2}(\mu-1)(\mu-2)}{2\left(\frac{1}{2}\right)^{\mu-1}}\left(\left(\frac{1}{2}+\phi_1 x\right)^{\mu-3}+\left(\frac{1}{2}-\phi_2 x\right)^{\mu-3}\right)\le$$

$$\le 4\frac{\frac{x^2}{2}(\mu-1)(\mu-2)}{2\left(\frac{1}{2}\right)^{\mu-1}}\left(\left(\frac{1}{2}+\phi_1 x\right)^{\mu-3}+\left(\frac{1}{2}-\phi_2 x\right)^{\mu-3}\right)$$

Since, the last inequality holds for all $x\in\langle -0.1,0.1\rangle$, it holds when x on both hand sides tends to 0, i.e.

$$\lim_{x\to 0}\left[-4+\frac{\frac{1}{2}(\mu-1)(\mu-2)}{2\left(\frac{1}{2}\right)^{\mu-1}}\left(\left(\frac{1}{2}+\phi_1 x\right)^{\mu-3}+\left(\frac{1}{2}-\phi_2 x\right)^{\mu-3}\right)\right]\le$$

$$\le\lim_{x\to 0}\left[4\frac{\frac{x^2}{2}(\mu-1)(\mu-2)}{2\left(\frac{1}{2}\right)^{\mu-1}}\left(\left(\frac{1}{2}+\phi_1 x\right)^{\mu-3}+\left(\frac{1}{2}-\phi_2 x\right)^{\mu-3}\right)\right].$$

This is equivalent to:

$$-4+\frac{\frac{1}{2}(\mu-1)(\mu-2)}{2\left(\frac{1}{2}\right)^{\mu-1}}\left(2\left(\frac{1}{2}\right)^{\mu-3}\right)\leq 0$$

$$-4+\frac{1}{2}(\mu-1)(\mu-2)\left(\frac{1}{2}\right)^{-2}\leq 0$$

$$\frac{1}{2}(\mu-1)(\mu-2)\leq 1$$

Solving the last inequality, we get $\mu\in[0,3]$, which is contradiction. ■

Now, let us prove the theorem. First, we prove that claim holds for $\lambda\in\{0,1/2,1,3/2\}$. Note that ${}^{0}M_1(G)=n$ for every graph with n vertices, hence the claim holds and for $\lambda=1/2$ and $\lambda=1$, the claim is proved in [2] and [1]. Let us prove it for $\lambda=3/2$. Denote by N_u the set of neighbors of vertex u and denote by $d_v^{(u)}$ degree of vertex $v\in N_u$ in graph $G[N_u]$ induced by the set of vertices in N_u. We have:

$$\begin{aligned}\sum_{u\in V}d_u^{\,3}&=\sum_{u\in V}\sum_{v\in N_u}d_v^{\,2}\leq\sum_{u\in V}\sum_{v\in N_u}\left(n-d_u+d_v^{(u)}\right)^2=\\&=\sum_{u\in V}\sum_{v\in N_u}(n-d_u)^2+2\sum_{u\in V}\sum_{v\in N_u}(n-d_u)d_v^{(u)}+\sum_{u\in V}\sum_{v\in N_u}\left(d_v^{(u)}\right)^2\\&=\sum_{u\in V}(n-d_u)^2\cdot d_u+2\sum_{u\in V}(n-d_u)\left(\sum_{v\in N_u}d_v^{(u)}\right)+\sum_{u\in V}\left(\sum_{v\in N_u}\left(d_v^{(u)}\right)^2\right).\end{aligned}\tag{4}$$

Note that $G[N_u]$ is K_r-free graph, hence

$$\sum_{v\in N_u}d_v^{(u)}={}^{1/2}M_1(G[N_u])\leq\left(1-\frac{1}{r-1}\right)d_u^{\,2};\tag{5}$$

$$\sum_{v\in N_u}\left(d_v^{(u)}\right)^2={}^{1}M_1(G[N_u])\leq\left(1-\frac{1}{r-1}\right)^2d_u^{\,3}.\tag{6}$$

Putting (5) and (6) in (4), we get:

$$\sum_{u\in V}d_u^{\,3}\leq\sum_{u\in V}(n-d_u)^2\cdot d_u+2\sum_{u\in V}(n-d_u)\left(1-\frac{1}{r-1}\right)d_u^{\,2}+\sum_{u\in V}\left(1-\frac{1}{r-1}\right)^2d_u^{\,3}.$$

Putting all summands that contain $\sum_{u\in V}d_u^{\,3}$ on the left hand-side of the inequality and all other summands on the right-hand side of the inequality, we get:

$$\left(2\cdot\frac{r-2}{r-1}-\left(\frac{r-2}{r-1}\right)^2\right)\sum_{u\in V}d_u^{\ 3}\le n^2\sum_{u\in V}d_u+\left(-2n+2n\frac{r-2}{r-1}\right)\sum_{u\in V}d_u^{\ 2}.$$

Since $\sum_{u\in V}d_u\le\left(1-\frac{1}{r}\right)n^2$ and $\sum_{u\in V}d_u^{\ 2}\le\left(1-\frac{1}{r}\right)^2 n^3$, it follows that:

$$\left(2\cdot\frac{r-2}{r-1}-\left(\frac{r-2}{r-1}\right)^2\right)\sum_{u\in V}d_u^{\ 3}\le n^2\cdot\left(1-\frac{1}{r}\right)n^2+\left(-2n+2n\frac{r-2}{r-1}\right)\left(1-\frac{1}{r}\right)^2 n^3$$

$$\sum_{u\in V}d_u^{\ 3}\le\frac{n^2\cdot\left(1-\frac{1}{r}\right)n^2+\left(-2n+2n\frac{r-2}{r-1}\right)\left(1-\frac{1}{r}\right)^2 n^3}{2\cdot\frac{r-2}{r-1}-\left(\frac{r-2}{r-1}\right)^2}$$

$$\sum_{u\in V}d_u^{\ 3}\le\left(1-\frac{1}{r}\right)^3 n^4$$

$${}^{3/2}M_1(G)\le{}^{3/2}M_1\left(T_{r+1}(n)\right).$$

Now, assume that $\lambda\in\langle 0,3/2\rangle\setminus\{1/2,1\}$. Let $x_1=\lfloor 2\lambda\rfloor$ and $x_2=\lceil 2\lambda\rceil$.

We have:

$${}^{\lambda}M_1(G)=\sum_{v\in V(G)}d_v^{\ 2\lambda}=\sum_{v\in V(G)}\sum_{i=1}^{d_v^{\ x_1}}d_v^{\ 2\lambda-x_1}. \tag{7}$$

Note that $x_1\in\{0,1,2\}$, hence $\sum_{v\in V(G)}d_v^{\ x_1}\le\left(1-\frac{1}{r}\right)^{x_1}\cdot n^{x_1+1}$. Therefore

$$y=\left(1-\frac{1}{r}\right)^{x_1}\cdot n^{x_1+1}-\sum_{v\in V(G)}d_v^{\ x_1}$$

is non-negative integer. Hence, (7) can be rewritten as:

$${}^{\lambda}M_1(G)=\sum_{v\in V(G)}\sum_{i=1}^{d_v^{\ x_1}}d_v^{\ 2\lambda-x_1}+\sum_{j=1}^{y}0^{2\lambda-x_1},$$

where the last summand vanishes if $y=0$. Since, $0<2\lambda-x_1<1$, the function $f(x)=x^{2\lambda}$ is concave function and therefore:

$$ {}^{\lambda}M_1(G) \le \left[\left(1-\frac{1}{r}\right)^{x_1} \cdot n^{x_1+1} \right] \cdot \left[\frac{\sum_{v \in V(G)} \sum_{i=1}^{d_v^{x_1}} d_v}{\left(1-\frac{1}{r}\right)^{x_1} \cdot n^{x_1+1}} \right]^{2\lambda - x_1} $$

Or equivalently,

$$ {}^{\lambda}M_1(G) \le \left[\left(1-\frac{1}{r}\right)^{x_1} \cdot n^{x_1+1} \right] \cdot \left[\frac{\sum_{v \in V(G)} d_v^{x_2}}{\left(1-\frac{1}{r}\right)^{x_1} \cdot n^{x_1+1}} \right]^{2\lambda - x_1} $$

Since, $x_2 \in \{1,2,3\}$, it follows that $\sum_{v \in V(G)} d_v^{x_2} \le \left(1-\frac{1}{r}\right)^{x_2} \cdot n^{x_2+1}$, therefore:

$$ {}^{\lambda}M_1(G) \le \left[\left(1-\frac{1}{r}\right)^{x_1} \cdot n^{x_1+1} \right] \cdot \left[\frac{\left(1-\frac{1}{r}\right)^{x_2} \cdot n^{x_2+1}}{\left(1-\frac{1}{r}\right)^{x_1} \cdot n^{x_1+1}} \right]^{2\lambda - x_1} . $$

Note that $x_2 = x_1 + 1$, hence:

$$ {}^{\lambda}M_1(G) \le \left[\left(1-\frac{1}{r}\right)^{x_1} \cdot n^{x_1+1} \right] \cdot \left[\left(1-\frac{1}{r}\right) \cdot n \right]^{2\lambda - x_1} = \left(1-\frac{1}{r}\right)^{2\lambda} \cdot n^{2\lambda+1} = {}^{\lambda}M_1\left(T_{r+1}(n)\right). $$

This proves the claim for all $\lambda \in [0, 3/2]$. Now, suppose that $\lambda < 0$. The counter-example can be easily constructed even if we restrict our-selves to the connected graphs:

$$ {}^{\lambda}M_1\left(T_3(6)\right) = 6 \cdot 3^{2\lambda} < 6 \cdot 2^{2\lambda} = {}^{\lambda}M_1(C_6). $$

Suppose that $\lambda > 3/2$. From Lemma 1, it follows that there is a rational number $x \in \langle -0.1, 0.1 \rangle$ such that:

$$ \left(\frac{1}{2}+x\right) \cdot \left(\frac{1}{2}-x\right)^{2\lambda} + \left(\frac{1}{2}-x\right) \cdot \left(\frac{1}{2}+x\right)^{2\lambda} > \left(\frac{1}{2}\right)^{2\lambda} . $$

Note that the last relation also holds for $-x$, because of the symmetry. Hence, we may assume that x is positive. Therefore x can be written as $x = \frac{p}{q}$, where p and q are positive integers. From $x \in \langle 0, 0.1 \rangle$, it follows that $q - 2p > 0$. Let us observe graphs $K_{q+2p,q-2p}$ and $K_{q,q} = T_3(2q)$. They are both K_3-free graphs with the same number of vertices, but

$$\begin{aligned} {}^{\lambda}M_1\left(K_{q+2p,q-2p}\right) &= (q+2p)(q-2p)^{2\lambda} + (q-2p)(q+2p)^{2\lambda} \\ &= (2q)^{1+2\lambda}\left[\left(\frac{1}{2}+x\right)\left(\frac{1}{2}-x\right)^{2\lambda} + \left(\frac{1}{2}-x\right)\left(\frac{1}{2}+x\right)^{2\lambda}\right] > \\ &> (2q)^{1+2\lambda}\left(\frac{1}{2}\right)^{2\lambda} = 2q \cdot q^{\lambda} = {}^{\lambda}M_1\left(K_{q,q}\right). \end{aligned}$$

This completes the proof of the Theorem 1.

3. Proof of the Theorem 2

First, suppose that $\lambda \in [-1/2, 1]$, then:

$$\begin{aligned} {}^{\lambda}M_2(G) &= \sum_{uv \in E(G)} (d_u d_v)^{\lambda} \le \frac{1}{2} \sum_{uv \in E(G)} \left(d_u^{\,2\lambda} + d_v^{\,2\lambda}\right) \le \\ &\le \frac{1}{2} \sum_{v \in V(G)} d_v^{\,2\lambda+1} = \frac{1}{2} \cdot {}^{(2\lambda+1)/2}M_1(G) \le \{\text{from Theorem 1}\} \le \frac{1}{2} \cdot {}^{(2\lambda+1)/2}M_1\left(T_r(n)\right) = \\ &= \frac{1}{2} \sum_{v \in V(T_r(n))} d_v^{\,2\lambda+1} = \frac{1}{2} \sum_{uv \in E(T_r(n))} \left(d_u^{\,2\lambda} + d_v^{\,2\lambda}\right) = \sum_{uv \in E(T_r(n))} (d_u d_v)^{\lambda} = {}^{\lambda}M_2\left(T_r(n)\right). \end{aligned}$$

Let us prove that the claim can not be extended to $\lambda < -\frac{1}{2}$. Note that C_{2p} and $K_{p,p}$ are K_3-free graphs with $2n$ vertices. Hence it is sufficient to show that

$$\lim_{p \to \infty} \frac{{}^{2}M_{\lambda}\left(K_{p,p}\right)}{{}^{2}M_{\lambda}\left(C_p\right)} = 0 .$$

We have:

$$\lim_{p \to \infty} \frac{{}^{2}M_{\lambda}\left(K_{p,p}\right)}{{}^{2}M_{\lambda}\left(C_p\right)} = \lim_{p \to \infty} \frac{p^2 \cdot p^{2\lambda}}{p \cdot 4^{\lambda}} = \lim_{p \to \infty}\left(4^{-\lambda} p^{1+2\lambda}\right) = 0 .$$

Now, suppose that $\lambda > 5.4496$. It is sufficient to prove that

$${}^{\lambda}M_2\left(K_{1,1,4}\right) > {}^{\lambda}M_2\left(K_{2,2,2}\right)$$

$$2 \cdot 4 \cdot 1 \cdot \left((1+1)(1+4)\right)^{\lambda} + 1 \cdot 1 \cdot \left((1+4)(1+4)\right)^{\lambda} > 3 \cdot 2 \cdot 2 \cdot \left((2+2) \cdot (2+2)\right)^{\lambda}$$

$$8 \cdot \left(\frac{5}{8}\right)^{\lambda} + \left(\frac{25}{16}\right)^{\lambda} - 12 > 0$$

Putting $\lambda = 5.4496$, one gets $0.000218502 > 0$, which is true, hence, it is sufficient to prove that function

$$f(\lambda) = 8 \cdot \left(\frac{5}{8}\right)^{\lambda} + \left(\frac{25}{16}\right)^{\lambda} - 12$$

is increasing when $\lambda \in \left[5.4496, +\infty\right\rangle$, i.e. that $f'(\lambda) > 0$ when $\lambda \in \left[5.4496, +\infty\right\rangle$. We have:

$$f'(\lambda) = -8 \cdot \ln\left(\frac{8}{5}\right) \cdot \left(\frac{8}{5}\right)^{\lambda} + \left(\frac{25}{16}\right)^{\lambda} \cdot \ln\left(\frac{25}{16}\right) \geq -8 \cdot \ln\left(\frac{8}{5}\right) \cdot \left(\frac{8}{5}\right)^{5} \cdot \left(\frac{8}{5}\right)^{\lambda-5} + \left(\frac{25}{16}\right)^{5} \left(\frac{8}{5}\right)^{\lambda-5} \cdot \ln\left(\frac{25}{16}\right) =$$
$$= \left[-8 \cdot \ln\left(\frac{8}{5}\right) \cdot \left(\frac{8}{5}\right)^{5} + \left(\frac{25}{16}\right)^{5} \cdot \ln\left(\frac{25}{16}\right)\right] \cdot \left(\frac{8}{5}\right)^{\lambda-5} > 3 \cdot \left(\frac{8}{5}\right)^{\lambda-5} > 0$$

This proves the claim of the Theorem.

4. Proof of the Theorem 3

From the proof of Theorem 2, it follows that it is sufficient to prove that

$${}^{\lambda}M_2(G) \leq {}^{\lambda}M_2(T_3(n)) \text{ for all } \lambda \in [1, +\infty].$$

Let uv be any edge of G. Since uv is not part of any triangle, it follows that $N_u \cap N_v = \varnothing$, hence $d_u + d_v \leq n$. Therefore contribution of an edge uv can be at most $\left[d_u(n - d_u)\right]^{\lambda}$. Simple analytical method shows that this expressions is maximized for $d_u = \frac{n}{2}$. Form Turán's Theorem, it follows that G has at most $n^2/4$ edges. Hence,

$${}^{\lambda}M_2(G) \leq \frac{n^2}{4} \cdot \left[\frac{n}{2}\left(n - \frac{n}{2}\right)\right]^{\lambda} = \frac{n^2}{4} \cdot \left[\frac{n^2}{4}\right]^{\lambda} = {}^{\lambda}M_2(T_3(n)).$$

This proves the Theorem.

5. Proof of the Theorem 4

From Theorem 2, it follows that it is sufficient to prove that

$${}^{\lambda}M_2(G) \leq {}^{\lambda}M_2(T_{r+1}(n)) \text{ for all } \lambda \in [1, 3].$$

Denote cardinalities of classes of vertices in G by $n_1, \ldots, n_r$. It holds:

$$ {}^{\lambda}M_2(G) \le {}^{\lambda}M_2\left(K_{n_1,\dots,n_r}\right) = \sum_{1\le i<j\le r} n_i n_j \left[(n-n_i)(n-n_j)\right]^{\lambda} = $$

$$ = \sum_{1\le i<j\le r} \left[n_i (n-n_i)^{\lambda}\right]\left[n_j (n-n_j)^{\lambda}\right] $$

$$ = \frac{1}{2}\left[\sum_{i=1}^{r} n_i (n-n_i)^{\lambda}\right]^2 - \frac{1}{2}\sum_{i=1}^{r}\left[n_i (n-n_i)^{\lambda}\right]^2 $$

$$ = \frac{1}{2}\left[\sum_{i=1}^{r} n_i (n-n_i)^{\lambda}\right]^2 - \frac{r}{2}\frac{\sum_{i=1}^{r}\left[n_i (n-n_i)^{\lambda}\right]^2}{r} $$

Using the inequality between quadratic and aritmetic mean, it follows that:

$$ {}^{\lambda}M_2(G) \le \frac{1}{2}\left[\sum_{i=1}^{r} n_i (n-n_i)^{\lambda}\right]^2 - \frac{r}{2}\left(\frac{\sum_{i=1}^{r}\left[n_i (n-n_i)^{\lambda}\right]}{r}\right)^2 = \frac{1}{2}\left(1-\frac{1}{r}\right)\left[\sum_{i=1}^{r} n_i (n-n_i)^{\lambda}\right]^2 . $$

From Theorem 1, it follows that:

$$ {}^{\lambda}M_2(G) \le \frac{1}{2}\left(1-\frac{1}{r}\right)\left[n\cdot\left[\left(1-\frac{1}{r}\right)n\right]^{\lambda}\right]^2 = \frac{1}{2}\left(1-\frac{1}{r}\right)n^2\cdot\left[\left(1-\frac{1}{r}\right)n\right]^{2\lambda} = {}^{\lambda}M_2\left(T_{r+1}(n)\right). $$

This proves the Theorem.

6. Acknowledgment

Partial support of Ministry of Science, Education and Sport of the Republic of Croatia (grants no. 037-0000000-2779 and 177--0000000-0884) is gratefully acknowledged.

7. References

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