**MATCH**
*Communications in Mathematical and in Computer Chemistry*

MATCH Commun. Math. Comput. Chem. **62** (2009) 573-580

ISSN 0340 - 6253

# On Incidence Energy of Graphs

**Ivan Gutman,**[a] **Dariush Kiani**[b,c] and **Maryam Mirzakhah**[b]

[a]*Faculty of Science, University of Kragujevac,*
*P. O. Box 60, 34000 Kragujevac, Serbia*
e-mail: gutman@kg.ac.rs

[b]*Faculty of Mathematics and Computer Science,*
*Amirkabir University of Technology,*
*P. O. Box 15875–4413, Tehran, Iran*
e-mail: dkiani@aut.ac.ir ; mirzakhah@aut.ac.ir

[c]*School of Mathematics, Institute for Research in Fundamental Sciences (IPM), P. O. Box 19395–5746, Tehran, Iran*

(Received January 14, 2009)

**Abstract**

It is shown that in the case of bipartite graphs, the incidence energy $IE$, introduced in a recent work [M. R. Jooyandeh, D. Kiani, M. Mirzakhah, *MATCH Commun. Math. Comput. Chem.*, preceding article] coincides with the previously studied Laplacian–energy like invariant, $LEL$ [ J. Liu, B. Liu, *MATCH Commun. Math. Comput. Chem.* **59** (2008) 355–372]. In the case of non-bipartite graphs, $IE$ is equal to the quantity $LEL^+$, calculated in an analogous manner as $LEL$, but from the eigenvalues of the signless Laplacian matrix. Some other relations for $IE$ are pointed out.

## 1. INTRODUCTION

Two of the present authors, together M. R. Jooyandeh [1], conceived recently a graph–energy like quantity, called *incidence energy*, $IE(G)$, calculated from the incidence matrix $\mathbf{I}(G)$ of a graph $G$.

The energy $E(G)$ of a graph $G$ is equal to the sum of the absolute values of the graph eigenvalues, i. e., of the eigenvalues of the adjacency matrix $\mathbf{A}(G)$ of $G$. (For details on graph energy see the reviews [2, 3].) Nikiforov [4] recently extended the concept of energy to all (not necessarily square) matrices, defining the energy of a matrix $\mathbf{M}$ as the sum of the singular values of $\mathbf{M}$. Recall that the singular values of a matrix $\mathbf{M}$ are equal to the square roots of the eigenvalues of the (square) matrix $\mathbf{M}\,\mathbf{M}^t$.

Let $G = (V, E)$ be a simple graph, where $V = \{v_1, v_2, \ldots, v_n\}$ is its vertex set, and $E = \{e_1, e_2, \ldots, e_m\}$ its edge set. Thus $n$ and $m$ denote, respectively, the number of vertices and edges of $G$. In [1] as well as in the present paper, $\mathbf{I}(G)$ is the *vertex–edge incidence matrix*, an $(n \times m)$-matrix whose $(i, j)$-element is equal to 1 if the vertex $v_i$ is incident to the edge $e_j$, and is equal to 0 otherwise (see, for instance [5], p. 16).

In line with Nikiforov's idea, the incidence energy of a graph $G$ was defined [1] as the sum of the singular values of the incidence matrix $\mathbf{I}(G)$, that, in turn, are equal to the square roots of the eigenvalues of $\mathbf{I}(G)\,\mathbf{I}(G)^t$. Recall that $\mathbf{I}(G)\,\mathbf{I}(G)^t$ is a square matrix of order $n$.

Let $\mathbf{D}(G)$ be the diagonal matrix of order $n$ whose $(i, i)$-entry is the degree (= number of first neighbors) of the vertex $v_i$ of the graph $G$. Then the matrix $\mathbf{L}(G) = \mathbf{D}(G) - \mathbf{A}(G)$ is the *Laplacian matrix* of the graph $G$, for details see [6, 7]. The matrix $\mathbf{L}^+(G) = \mathbf{D}(G) + \mathbf{A}(G)$ is the *signless Laplacian matrix*, for details see [8].

Denote by $\mu_1, \mu_2, \ldots, \mu_n$ the eigenvalues of the Laplacian matrix $\mathbf{L}(G)$ and by $\mu_1^+, \mu_2^+, \ldots, \mu_n^+$ the eigenvalues of the signless Laplacian matrix $\mathbf{L}^+(G)$. All eigenvalues of both $\mathbf{L}(G)$ and $\mathbf{L}^+(G)$ are real and non-negative. If the graph $G$ is connected, then $n-1$ eigenvalues of $\mathbf{L}(G)$ are positive, and one is equal to zero [6, 7]. If $G$ is a connected non-bipartite graph, then all eigenvalues of $\mathbf{L}^+(G)$ are positive; if $G$ is connected and bipartite, then exactly one eigenvalue of $\mathbf{L}^+(G)$ is equal to zero [8].

The following result is well known [6, 7, 8]:

**Lemma 1.** The spectra of $\mathbf{L}(G)$ and $\mathbf{L}^+(G)$ coincide if and only if the graph $G$ is bipartite.

Short time ago, J. Liu and B. Liu [9] introduced the so-called *Laplacian–energy like invariant*, $LEL(G)$ of a graph $G$, as the sum of the square roots of the eigenvalues of the Laplacian matrix of $G$, i. e.,

$$LEL(G) := \sum_{i=1}^{n} \sqrt{\mu_i} \ . \tag{1}$$

In [9] and in the subsequent papers [10, 11, 12] a number of properties of $LEL$ were established.

## 2. RELATION BETWEEN INCIDENCE ENERGY AND LAPLACIAN–ENERGY LIKE INVARIANT

A well known identity for the incidence matrix of any graph $G$ is (see, for instance, [5], p. 16):

$$\mathbf{I}(G)\,\mathbf{I}(G)^t = \mathbf{A}(G) + \mathbf{D}(G)$$

i. e.,

$$\mathbf{I}(G)\,\mathbf{I}(G)^t = \mathbf{L}^+(G) \ . \tag{2}$$

Then an immediate consequence of Eq. (2) and the definition of the incidence energy is:

**Theorem 2.** If $IE(G)$ is the incidence energy of an $n$-vertex graph $G$, and if $\mu_1^+, \mu_2^+, \ldots, \mu_n^+$ are the eigenvalues of the signless Laplacian matrix of $G$, then

$$IE(G) = \sum_{i=1}^{n} \sqrt{\mu_i^+} \ .$$

In view of Lemma 1 and Eq. (1), we arrive at the following noteworthy:

**Corollary 3.** If $IE(G)$ is the incidence energy of a bipartite graph $G$, and $\mu_1$, $\mu_2, \ldots, \mu_n$ are the eigenvalues of the Laplacian matrix of $G$, then

$$IE(G) = \sum_{i=1}^{n} \sqrt{\mu_i} \equiv LEL(G) \ .$$

In other words, for bipartite graphs the incidence energy $IE$ and the Laplacian–energy like invariant $LEL$ coincide.

In analogy with Eq. (1), we define

$$LEL^+(G) := \sum_{i=1}^{n} \sqrt{\mu_i^+}$$

which immediately implies:

**Corollary 4.** If $G$ is any graph, then $IE(G) \equiv LEL^{+}(G)$.

## 3. ON A RELATION BETWEEN INCIDENCE ENERGY AND GRAPH ENERGY

In [1] it was demonstrated that if $G$ is any graph, it is possible to find a bipartite graph $\hat{G}$, such that

$$IE(G) = \frac{1}{2}\,E(\hat{G})$$

where $E(\hat{G})$ is the ordinary energy of the graph $\hat{G}$. In [1] the graph $\hat{G}$ is constructed so that its adjacency matrix is of the form:

$$\mathbf{A}(\hat{G}) = \begin{bmatrix} \mathbf{0} & \mathbf{I}(G) \\ \mathbf{I}(G)^t & \mathbf{0} \end{bmatrix} . \tag{3}$$

In graph theory it is well known that the graph defined via Eq. (3) is the *subdivision graph* $S(G)$ of the graph $G$, obtained by inserting an additional vertex into each edge of $G$ (see, for instance, [5], p. 16). Thus $\hat{G}$ is just the subdivision graph of $G$.

If $G$ is a graph with $n$ vertices and $m$ edges, then its subdivision graph $S(G)$ has $n+m$ vertices and $2m$ edges.

A connection between the spectrum of $S(G)$ and the Laplacian spectrum of $G$ was recently communicated [13]. The main result in [13] is the following:

**Theorem 5.** Let $G$ be a bipartite graph with $n$ vertices and $m$ edges, and let $S(G)$ be its subdivision graph. If $\mu_i$ , $i = 1, \ldots, h$, are the non-zero eigenvalues of the Laplacian matrix of $G$, then the ordinary spectrum of $S(G)$ consists of the numbers $\pm\sqrt{\mu_i}$ , $i = 1, \ldots, h$, and of $n+m-2h$ zeros.

Although not stated in [13], the following extension of Theorem 5 to all graphs is evident (and is proven in the same way as Theorem 5 itself):

**Theorem 6.** Let $G$ be a graph with $n$ vertices and $m$ edges, and let $S(G)$ be its subdivision graph. If $\mu_i^+$ , $i = 1, \ldots, h$, are the non-zero eigenvalues of the signless Laplacian matrix of $G$, then the ordinary spectrum of $S(G)$ consists of the numbers $\pm\sqrt{\mu_i^+}$ , $i = 1, \ldots, h$, and of $n + m - 2h$ zeros.

From Theorem 5 it immediately follows (but is not stated in [13]) that for a bipartite graph $G$, $E(S(G)) = 2\,LEL(G)$. Analogously, Theorem 6 implies that for any graph $G$, $E(S(G)) = 2\,LEL^+(G)$. In view of Corollaries 3 and 4, we then get

**Corollary 7.** For any graph $G$ whose subdivision graph is $S(G)$,

$$IE(G) = \frac{1}{2}\,E(S(G)) \ .$$

Corollary 7 is, of course, precisely identical to Theorem 1 in [1].

## 4. TREES WITH MINIMAL AND MAXIMAL INCIDENCE ENERGY

Let $\mathcal{T}_n$ be the set of all $n$-vertex trees. Let $S_n$ and $P_n$ be the $n$-vertex star and $n$-vertex path, respectively. In [11] it was shown that for any tree $T \in \mathcal{T}_n$,

$$LEL(S_n) \leq LEL(T) \leq LEL(P_n) \tag{4}$$

with equality if and only if $T \cong S_n$ and $T \cong P_n$, respectively. Because trees are bipartite graphs, by means of Corollary 3 the above result can be re-stated as:

**Theorem 8.** For any tree $T \in \mathcal{T}_n$,

$$IE(S_n) \leq IE(T) \leq IE(P_n)$$

with equality if and only if $T \cong S_n$ and $T \cong P_n$, respectively.

In view of the result reported in [11], no proof of Theorem 8 would be needed. We nevertheless offer an alternative proof.

**Proof.** Denote by $\psi(G, \lambda)$ the characteristic polynomial of the Laplacian matrix of the graph $G$. Its zeros are $\mu_1, \mu_2, \ldots, \mu_n$. It is known [5, 13] that this polynomial is of the form

$$\psi(G,\lambda) = \sum_{k\geq 0} (-1)^k \, c_k(G) \, \lambda^{n-k}$$

where $c_k(G) \geq 0$ .[1]

Let $\Psi(G,\lambda)$ be an auxiliary polynomial defined as

$$\Psi(G,\lambda) = \psi(G,\lambda^2) = \sum_{k\geq 0} (-1)^k \, c_k(G) \, \lambda^{2n-2k} \; .$$

Then the zeros of $\Psi(G,\lambda)$ are $\pm\sqrt{\mu_1}, \pm\sqrt{\mu_2}, \ldots, \pm\sqrt{\mu_n}$ . Therefore, the sum of the positive zeros of $\Psi(G,\lambda)$ is just $LEL(G)$ .

The Coulson integral formula (see [2, 14, 15] and the references cited therein) makes it possible to compute the sum of the positive zeros of a polynomial without knowing the actual values of these zeros. Applying to $\Psi(G,\lambda)$ a variant of the Coulson integral formula from the work [16], we get

$$LEL(G) = \frac{1}{\pi} \int_0^{+\infty} \ln \left[ \sum_{k\geq 0} c_k(G) \, x^{2k} \right] \frac{dx}{x^2} \; . \tag{5}$$

From formula (5) we see that $LEL(G)$ is a monotonically increasing function of each of the coefficients $c_k(G)$ .

For $n$-vertex trees it has been shown [13] that for all $k \geq 0$ and for all $T \in \mathcal{T}_n$ ,

$$c_k(S_n) \leq c_k(T) \leq c_k(P_n) \tag{6}$$

and that equality for all values of $k$ occurs if and only if $T \cong S_n$ and $T \cong P_n$ , respectively. Combining (5) and (6) we immediately obtain the relations (4), from which Theorem 8 follows. □

*Acknowledgement.* The research of the second author was in part supported by IPM (Grant No. 87200116). I. G. thanks for support by the Serbian Ministry of Science (Grant No. 144015G).

[1] The dependence of the coefficients $c_k(G)$ on the structure of the graph $G$ is also fully understood [5], but is irrelevant for the present considerations.

## References

[1] M. R. Jooyandeh, D. Kiani, M. Mirzakhah, Incidence energy of a graph, *MATCH Commun. Math. Comput. Chem.* **62** (2009) 000-000, preceding paper.

[2] I. Gutman, The energy of a graph: Old and new results, in: A. Betten, A. Kohnert, R. Laue, A. Wassermann (Eds.), *Algebraic Combinatorics and Applications*, Springer–Verlag, Berlin, 2001, pp. 196–211.

[3] I. Gutman, X. Li, J. Zhang, Graph energy, in: M. Dehmer, F. Emmert–Streib (Eds.), *Analysis of Complex Networks. From Biology to Linguistics*, Wiley–VCH, Weinheim, 2009, in press.

[4] V. Nikiforov, The energy of graphs and matrices, *J. Math. Anal. Appl.* **326** (2007) 1472–1475.

[5] D. M. Cvetković, M. Doob, H. Sachs, *Spectra of Graphs – Theory and Applications*, Academic Pres, New York, 1980.

[6] R. Merris, Laplacian matrices of graphs: A survey, *Lin. Algebra Appl.* **197–198** (1994) 143–176.

[7] R. Merris, A survey of graph Laplacians, *Lin. Multilin. Algebra* **39** (1995) 19–31.

[8] D. Cvetković, P. Rowlinson, S. Simić, Signless Laplacians of finite graphs, *Lin. Algebra Appl.* **423** (2007) 155–171.

[9] J. Liu, B. Liu, A Laplacian–energy like invariant of a graph, *MATCH Commun. Math. Comput. Chem.* **59** (2008) 355–372.

[10] J. Liu, B. Liu, S. Radenković, I. Gutman, Minimal LEL–equienergetic graphs, *MATCH Commun. Math. Comput. Chem.* **61** (2009) 471–478.

[11] D. Stevanović, Laplacian–like energy of trees, *MATCH Commun. Math. Comput. Chem.* **61** (2009) 407–417.

[12] D. Stevanović, A. Ilić, Maximum Laplacian–like energy of unicyclic graphs, *Lin. Algebra Appl.*, in press.

[13] B. Zhou, I. Gutman, A connection between ordinary and Laplacian spectra of bipartite graphs, *Lin. Multilin. Algebra* **56** (2008) 305–310.

[14] I. Gutman, M. Mateljević, Note on the Coulson integral formula, *J. Math. Chem.* **39** (2006) 259–266.

[15] M. Mateljević, I. Gutman, Note on the Coulson and Coulson–Jacobs integral formulas, *MATCH Commun. Math. Comput. Chem.* **59** (2008) 257–268.

[16] I. Gutman, Acyclic systems with extremal Hückel $\pi$-electron energy, *Theoretica Chimica Acta* **45** (1977) 79–87.