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Note on Hypoenergetic Graphs

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Abstract

The energy $E(G)$ of a graph G is the sum of the absolute values of the eigenvalues of G. An n-vertex graph G is said to be hypoenergetic if $E(G) < n$. Earlier were reported results on hypoenergetic trees, unicyclic, and bicyclic graphs. In this paper we show that there exist n-vertex hypoenergetic tricyclic graphs with maximum vertex degree Δ for $\lfloor (n+3)/2 \rfloor \leq \Delta \leq n-1$. Some complete bipartite graphs and complete bipartite graphs with attached pendent vertices are hypoenergetic. A general construction of hypoenergetic graphs is provided, implying that there exist hypoenergetic k-cyclic graphs for any k.

1. INTRODUCTION

Let G be a simple graph with n vertices and m edges. The cyclomatic number of a connected graph is defined as $c(G) = m - n + 1$. A graph G with $c(G) = k$ is said to be k-cyclic. Denote by Δ the maximum degree of a graph. The eigenvalues λ_1, $\lambda_2, \ldots, \lambda_n$ of the adjacency matrix of the graph G are said to be the eigenvalues of G and form its spectrum [1]. The nullity of the graph G, denoted by $\eta(G)$, is the multiplicity of zero in the spectrum.

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The energy of G is defined as [2]

$$E = E(G) = \sum_{i=1}^{n} |\lambda_i| \ .$$

For details on graph energy see the reviews [3, 4], the recent papers [5–16] and the references cited therein.

In [5] Nikiforov showed that for almost all graphs,

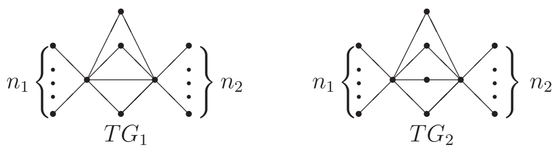
$$E = \left(\frac{4}{3\pi} + o(1) \right) n^{3/2} \ .$$

Thus the number of graphs satisfying the condition $E < n$ is relatively small. In 2007 one of the present authors and Radenković [9] proposed the definition of hypoenergetic graphs, i. e., the graphs whose energy is less than the number of vertices. Recently, the present author et al. [14] obtained results about hypoenergetic trees. Two of the present authors [15] showed that there exist hypoenergetic unicyclic graphs for all $n \geq 7$ and bicyclic graphs for all $n \geq 8$. In this paper we demonstrate that there exist hypoenergetic tricyclic, complete bipartite graphs, and complete bipartite graphs with attached pendent vertices. Finally we offer constructions of hypoenergetic k-cyclic graphs for any k.

2. HYPOENERGETIC TRICYCLIC GRAPHS

Lemma 2.1. [17] *If G is a connected tricyclic n-vertex graph, then for sufficiently large n, $\eta(G) \leq n - 4$.*

The equality $\eta(G) = n - 4$ in Lemma 2.1 is attained for graphs whose structure is determined in [17]. Among these are TG_1 and TG_2, depicted below:



By Lemma 2.1, the maximum nullity of a connected n-vertex tricyclic graph is $n - 4$.

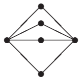
Lemma 2.2. [18] *If the nullity of G is n_0, then $E(G) \leq \sqrt{2m(n-n_0)}$.*

Theorem 2.3. *If $n = 6$ or $n \geq 8$, then there exist n-vertex hypoenergetic connected tricyclic graphs.*

Proof. We consider four cases:

Case 1. $n = 6$.

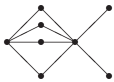
The graph G_6 is a connected tricyclic graph with $n = 6$. By direct calculation, $E(G_6) = 5.65685 < 6$.



G_6

Case 2. $n = 8$.

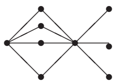
The graph G_8 is a connected tricyclic graph with $n = 8$. By direct calculation, $E(G_8) = 7.91375 < 8$.



G_8

Case 3. $n = 9$.

The graph G_9 is a connected tricyclic graph with $n = 9$. By direct calculation, $E(G_9) = 8.46834 < 9$.



G_9

Case 4. $n \geq 10$.

By Lemma 2.2, $E(G) \leq \sqrt{2(n+2)(n-n_0)}$. Now, if $\sqrt{2(n+2)(n-n_0)} < n$, then $\eta(G) > n - n^2/[2(n+2)]$. By Lemma 2.1, the maximum nullity is $n-4$. Thus $n-4 > n - n^2/[2(n+2)]$ implying $n^2 - 8n - 16 > 0$. The latter inequality is obeyed by all $n \geq 10$. $\square$

Theorem 2.4. *If $n = 4, 5, 7$, then there exist no hypoenergetic tricyclic graphs.*

Proof. In the books [1, 19] all connected graphs with $n = 4, 5, 7$ are listed. Theorem 2.4 is obtained by systematic examination of their energies. $\square$

In the next theorem we consider hypoenergetic tricyclic graphs with specified values of Δ.

Theorem 2.5. *For $\lfloor (n+3)/2 \rfloor \leq \Delta \leq n-1$, there exist connected hypoenergetic tricyclic graphs with n vertices and maximum vertex degree Δ.*

Proof. If n is even and $\Delta \in \left[\frac{n}{2}+1, n-2\right]$, let $G \cong TG_2$ with $n_1 = \Delta - 4$ and $n_2 = n - \Delta - 2$. By Lemma 2.1, then $\eta(G) = n_0 = n-4$ and $E(G) \leq \sqrt{2(n+2)(n-n_0)} = \sqrt{2(n+2) \times 4} < n$ $(n \geq 10)$.

If n is even and $\Delta = n-1$, let $G \cong TG_1$ with $n_1 = n-5$ and $n_2 = 0$. By Lemma 2.1, then $n_0 = n-4$ and $E(G) \leq \sqrt{2(n+2)(n-n_0)} = \sqrt{2(n+2) \times 4} < n$ $(n \geq 10)$.

If n is odd, the proof is fully analogous. $\square$

3. HYPOENERGETIC k-CYCLIC GRAPHS

As usual, by K_{n_1,n_2} we denote the complete bipartite graph on $n_1 + n_2$ vertices.

Lemma 3.1. [20] *Suppose that G is a graph on n vertices and G has no isolated vertices. Then $\eta(G) = n-2$ if and only if $G \cong K_{n_1,n_2}$, where $n_1 + n_2 = n$ and $n_1, n_2 > 0$.*

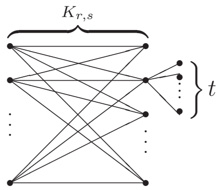
Theorem 3.2. *Let $G \cong K_{n_1,n_2}$, $n_1 \neq n_2$. Then G is hypoenergetic.*

Proof. The spectrum of K_{n_1,n_2} consists of $\pm\sqrt{n_1\, n_2}$ and $n_1 + n_2 - 2$ zeros [1]. Hence, $E(K_{n_1,n_2}) = 2\sqrt{n_1\, n_2}$. Therefore $E(K_{n_1,n_2}) < n_1 + n_2$ because of

$$E(K_{n_1,n_2}) - (n_1 + n_2) = -(\sqrt{n_1} - \sqrt{n_2})^2 < 0 \ .$$

$\square$

Let G^*_{rst} be the complete bipartite graph with attached pendent vertices, having the following structure:



G^*_{rst}

From Theorem 1 in [17] we have the following:

Lemma 3.3. [17] *For* $r, s, t \geq 1$, $\eta(G^*_{rst}) = n - 4$.

Theorem 3.4 *Among the complete bipartite graphs with pendent vertices attached, some are hypoenergetic.*

Proof. Let $G \cong G^*_{rst}$ and $t \geq r > s \geq 5$. By Lemma 3.3, $\eta(G) = n - 4$. By Lemma 2.2,

$$E(G) \leq \sqrt{2m(n - n_0)} = \sqrt{2(rs + t) \times [n - (n - 4)]} = \sqrt{8(rs + t)} \ .$$

Now, if $\sqrt{8(rs + t)} < n = r + s + t$, then

$$r^2 + s^2 - 6rs + 2rt + 2st + t^2 - 8t > 0 \ . \tag{1}$$

The inequality (1) can be transformed into

$$\begin{aligned} r^2 + s^2 - 6rs + 2t(r + s) + t^2 - 8t &\geq r^2 + s^2 - 6rs + 2r(2s + 1) + t^2 - 8t \\ &= (r - s)^2 + 2r + t^2 - 8t \\ &\geq t^2 - 8t + 2r + 1 > 0 \end{aligned}$$

which is obeyed by all $t \geq r \geq 6$.

Thus G^*_{rst} is hypoenergetic for $t \geq r > s \geq 5$. □

Remark. The condition $s \geq 5$ is not necessary. For example, for $r = 3$, $s = 2$, $t \geq 3$, we have $E(G^*_{rst}) < n$. But if $t < r$ or $t < s$, then G^*_{rst} needs not be hypoenergetic. For instance, for $r = 3$, $s = 4$, $t = 2$, $E(G^*_{rst}) = 9.48373 > n = 3 + 4 + 2$.

Theorem 3.5. *There exist hypoenergetic k-cyclic graphs for any $k \geq 0$.*

Proof. Let G be an arbitrary graph on n vertices and with at least one edge. Let v be a non-isolated vertex of G . Construct the graph G_p by attaching p pendent vertices to v . Evidently, if G is k-cyclic, then also G_p is k-cyclic. Since G may be arbitrary, k may be equal to zero or to any positive integer. Theorem 3.5 is now an immediate corollary of the following:

Theorem 3.6. *If* $p \geq \left[1+\sqrt{\frac{n}{2}\,\left(\sqrt{n}-1\right)+1}\,\right]^2$, *then G_p is hypoenergetic.*

Proof. Let X and Y be two graphs with disjoint vertex sets. Let x be a vertex of X and y a vertex of Y . The graph $X \circ Y$ is obtained from X and Y by identifying the vertices x and y . In [16] it was shown that

$$E(X \circ Y) \leq E(X) + E(Y) \tag{2}$$

and that this inequality is strict provided x is not an isolated vertex of X and y is not an isolated vertex of Y .

Applying (2) to the graph G_p we get

$$E(G_p) < E(G) + E(S_{p+1}) \tag{3}$$

where S_{p+1} is the $(p+1)$-vertex star. As well known,

$$E(S_{p+1}) = 2\sqrt{p}\ . \tag{4}$$

An upper bound for the energy of any n-vertex graph G is [21]

$$E(G) \leq \frac{n}{2}\,\left(\sqrt{n}+1\right)\ . \tag{5}$$

Substituting (4) and (5) back into (3) we obtain

$$E(G_p) < \frac{n}{2}\,\left(\sqrt{n}+1\right) + 2\sqrt{p}\ .$$

Therefore, a sufficient condition for G_p being hypoenergetic is

$$\frac{n}{2}\,\left(\sqrt{n}+1\right) + 2\sqrt{p} \leq n+p$$

from which Theorem 3.6 immediately follows. $\square$

Remark. From (2) follows that if X is hypoenergetic, and if the energy of Y is by one less than the number of vertices of Y, then $X \circ Y$ is also hypoenergetic. From this observation, and the fact that the energy of S_5 is by one less than its number of vertices, one can construct arbitrarily many hypoenergetic chemical trees, whose number of vertices belongs to any congruence class modulo 4.

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