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Constructing Graphs with Energy $\sqrt{\mathbf{r}}\,\mathbf{E}(\mathbf{G})$ where G is a Bipartite Graph

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Abstract

The energy $E(G)$ of a graph G is the sum of the absolute values of the eigenvalues of its adjacency matrix. If G is a bipartite graph and r is any positive integer, we construct graphs with energy $\sqrt{r}\,E(G)$.

1. INTRODUCTION

Let M be an $n \times n$ complex matrix. Here, as usual, $\lambda_1(M), \lambda_2(M), \ldots, \lambda_n(M)$ are the eigenvalues of M. If $\alpha_1, \alpha_2, \ldots, \alpha_n$ are nonnegative numbers then $\sum \alpha_j$ denotes the sum over the all positive α_j.

Let B be an $m \times n$ complex matrix. Let $q = \min\{m, n\}$. Let

$$\sigma_1(B) \geq \sigma_2(B) \geq \cdots \geq \sigma_q(B)$$

be the singular values of B. It is well known that if $m \leq n$ then, for $j = 1, 2, \ldots, m$, $\sigma_j(B)$ are the square roots of the eigenvalues of BB^* and if $m > n$ then, for $j = 1, 2, \ldots, n$, $\sigma_j(B)$ are the square roots of the eigenvalues of B^*B.

Nikiforov [1] defines the energy of B, denoted by $E(B)$, as

$$E(B) = \sum \sigma_j(B) \ .$$

Since the positive semidefinite matrices BB^* and B^*B have the same positive eigenvalues

$$E(B) = \sum \sqrt{\lambda_j(BB^*)} = \sum \sqrt{\lambda_j(B^*B)} \ .$$

Let G be a simple graph on n vertices. Let $A(G)$ be the adjacency matrix of G. The eigenvalues of $A(G)$ are called the eigenvalues of G. The energy $E(G)$ of G was first introduced by Gutman in 1978 as

$$E(G) = \sum_{j=1}^{n} |\lambda_j(A(G))| \ .$$

The energy of a graph is intensively studied in chemistry and it is used to approximate the total π-electron energy of a molecule [2, 3]. Since $A(G)$ is a real symmetric matrix, its singular values are the modulus of its eigenvalues. Then

$$E(G) = E(A(G)) .$$

Let 0 and I be the all zeros matrix and the identity matrix of the appropriate sizes, respectively.

Let $r \geq 1$ be an integer. Given an $m \times n$ matrix B, we denote by $B^{(r+1)}$ the $(r+1) \times (r+1)$ block bordered matrix

$$B^{(r+1)} = \begin{bmatrix} 0 & B & \cdots & B \\ B^* & 0 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ B^* & 0 & \cdots & 0 \end{bmatrix} .$$

Observe that $B^{(r+1)}$ is an Hermitian matrix of order $(m + rn)$ in which there are r copies de B.

Lemma 1.

$$E\left(B^{(r+1)}\right) = 2\sqrt{r}E(B) \ .$$

Proof. The singular values of $B^{(r+1)}$ are the square roots of the eigenvalues of the matrix

$$B^{(r+1)}B^{(r+1)} = \begin{bmatrix} rBB^* & 0 & \cdots & 0 \\ 0 & B^*B & \cdots & B^*B \\ \vdots & \vdots & \ddots & \vdots \\ 0 & B^*B & \cdots & B^*B \end{bmatrix}.$$

At this point, we recall that the Kronecker product [4] of two matrices $A = (a_{i,j})$ and $B = (b_{i,j})$ of sizes $m \times m$ and $n \times n$, respectively, is defined to be the $(mn) \times (mn)$ matrix $A \otimes B = (a_{i,j}B)$. It is known that the eigenvalues of $A \otimes B$ are $\lambda_i(A)\,\lambda_j(B)$ with $1 \le i \le m$ and $1 \le j \le n$. We have

$$\begin{bmatrix} B^*B & \cdots & B^*B \\ \vdots & \ddots & \vdots \\ B^*B & \cdots & B^*B \end{bmatrix} = (B^*B) \otimes \begin{bmatrix} 1 & \cdots & 1 \\ \vdots & \ddots & \vdots \\ 1 & \cdots & 1 \end{bmatrix}.$$

The eigenvalues of all ones matrix of order $r \times r$ are the simple eigenvalue r and 0 with multiplicity $(r-1)$. Then the positive eigenvalues of $B^{(r+1)}B^{(r+1)}$ are the positive eigenvalues of rBB^* and the positive eigenvalues of rB^*B. In addition, BB^* and B^*B have the same positive eigenvalues. Therefore

$$E\left(B^{(r+1)}\right) = \sum \sigma_j\left(B^{(r+1)}\right) = \sum 2\sqrt{r\lambda_j\left(BB^*\right)} = 2\sqrt{r}\,E(B)\,.$$

This completes the proof. $\square$

2. CONSTRUCTING GRAPHS WITH ENERGY $\sqrt{\mathbf{r}}\,\mathbf{E}(\mathbf{G})$

From now on, G is a given bipartite graph on n vertices. Then the vertex set of G can be divided into two disjoint sets V_1 with n_1 vertices and V_2 with n_2 vertices, such that every edge of G connects a vertex in V_1 to one in V_2. Clearly $n = n_1 + n_2$. Labelling the vertices in V_1 by $1, 2, \ldots, n_1$ and the vertices in V_2 by $n_1+1, n_1+2, \ldots, n_1+n_2$, the adjacency matrix of G becomes of the form

$$A(G) = \begin{bmatrix} 0 & B \\ B^T & 0 \end{bmatrix} = B^{(2)}$$

where B is an $n_1 \times n_2$ matrix. Similarly, labelling the vertices in V_2 by $1, 2, \ldots, n_2$ and the vertices in V_1 by $n_2+1, n_2+2, \ldots, n_2+n_1$, the adjacency matrix of G is of the form

$$A(G) = \begin{bmatrix} 0 & C \\ C^T & 0 \end{bmatrix} = C^{(2)}$$

where C is an $n_2 \times n_1$ matrix.

Lemma 2. If G is a bipartite graph then

$$E(G) = 2\,E(B) = 2\,E(C) \qquad \text{and} \qquad E(B) = E(C)\ .$$

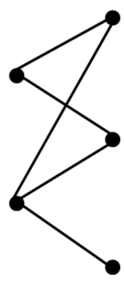
Proof. We know that $E\,(G) = E\,(A\,(G)) = E\left(B^{(2)}\right) = E\left(C^{(2)}\right)$. We apply Lemma 1 to obtain

$$E\,(G) = E\left(B^{(2)}\right) = 2E\,(B) \qquad \text{and} \qquad E\,(G) = E\left(C^{(2)}\right) = 2E\,(C)\ .$$

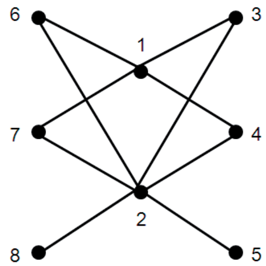
Consequently, $E\,(B) = E\,(C)$. □

Let $G_1^{(2)}$ be the graph obtained from two copies of G by identifying the vertices in V_1. In this case, we label the vertices in V_1 by $1, 2, \ldots, n_1$. Similarly, let $G_2^{(2)}$ be the graph obtained from two copies of G by identifying the vertices in V_2. In this last case, we label the vertices in V_2 by $1, 2, \ldots, n_2$.

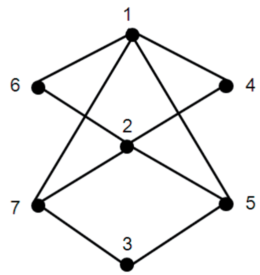
Example 1. Let G be the bipartite graph in which V_1 has two vertices and V_2 has three vertices as we show below:



Then $G_1^{(2)}$:



and $G_2^{(2)}$:



Observe that $G_1^{(2)}$ is a bipartite graph on $n_1 + 2n_2$ vertices and $G_2^{(2)}$ is a bipartite graph on $n_2 + 2n_1$ vertices. By labelling the vertices as in Example 1, we have

$$A\left(G_1^{(2)}\right) = \begin{bmatrix} 0 & B & B \\ B^T & 0 & 0 \\ B^T & 0 & 0 \end{bmatrix} \qquad \text{and} \qquad A\left(G_1^{(2)}\right) = \begin{bmatrix} 0 & C & C \\ C^T & 0 & 0 \\ C^T & 0 & 0 \end{bmatrix}$$

in which $C = B^T$. In the Example 1, $B = \begin{bmatrix} 1 & 1 & 0 \\ 1 & 1 & 1 \end{bmatrix}$ and $C = \begin{bmatrix} 1 & 1 \\ 1 & 1 \\ 0 & 1 \end{bmatrix}$.

Definition 1. Let $G_1^{(r)}$ be the graph obtained from r copies of G by identifying the vertices in $V_1 = \{1, 2, \ldots, n_1\}$ and let $G_2^{(r)}$ be the graph obtained from r copies of G by identifying the vertices in $V_2 = \{1, 2, \ldots, n_2\}$.

Observe that $G_1^{(r)}$ is a bipartite graph on $n_1 + rn_2$ vertices and $G_2^{(r)}$ is a bipartite graph on $n_2 + rn_1$ vertices.

As we illustrated in Example 1, there is a labelling for the vertices of $G_1^{(r)}$ such that

$$A\left(G_1^{(r)}\right) = \begin{bmatrix} 0 & B & \cdots & B \\ B^T & 0 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ B^T & 0 & \cdots & 0 \end{bmatrix} \tag{1}$$

and there is a labelling for the vertices of $G_2^{(r)}$ such that

$$A\left(G_2^{(r)}\right) = \begin{bmatrix} 0 & C & \cdots & C \\ C^T & 0 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ C^T & 0 & \cdots & 0 \end{bmatrix} \tag{2}$$

with

$$C = B^T. \tag{3}$$

Theorem 1. Let G be a bipartite graph. Then

$$E\left(G_1^{(r)}\right) = E\left(G_2^{(r)}\right) = \sqrt{r}\, E(G) \ .$$

Proof. From (1) and (2)

$$A\left(G_1^{(r)}\right) = B^{(r+1)} \qquad \text{and} \qquad A\left(G_2^{(r)}\right) = C^{(r+1)} \ .$$

We apply Lemma 1 and Lemma 2 to obtain

$$E\left(A\left(G_1^{(r)}\right)\right) = E\left(B^{(r+1)}\right) = 2\sqrt{r}\, E(B) = \sqrt{r} E(G)$$

and

$$E\left(A\left(G_2^{(r)}\right)\right) = E\left(C^{(r+1)}\right) = 2\sqrt{r}\, E(C) = \sqrt{r}\, E(G) \ .$$

The proof is complete. □

We have constructed two graphs $G_1^{(r)}$ and $G_2^{(r)}$ with the same energy $\sqrt{r}E(G)$ from a given bipartite graph G. Clearly, if $n_1 \neq n_2$ then $G_1^{(r)}$ and $G_2^{(r)}$ are graphs of different orders.

Corollary 1. If $n_1 = n_2$, then the graphs $G_1^{(r)}$ and $G_2^{(r)}$ are cospectral.

Proof. Since $n_1 = n_2$, $A\left(G_1^{(r)}\right)$ and $A\left(G_2^{(r)}\right)$ are matrices of the same order. From (3), $C = B^T$. We have

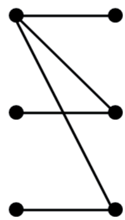
$$\begin{bmatrix} 0 & \cdots & 0 & I \\ \vdots & 0 & \cdots & 0 \\ 0 & I & \ddots & \vdots \\ I & 0 & \cdots & 0 \end{bmatrix} A\left(G_2^{(r)}\right) \begin{bmatrix} 0 & \cdots & 0 & I \\ \vdots & 0 & \cdots & 0 \\ 0 & I & \ddots & \vdots \\ I & 0 & \cdots & 0 \end{bmatrix}$$

$$= \begin{bmatrix} 0 & \cdots & 0 & I \\ \vdots & 0 & \cdots & 0 \\ 0 & I & \ddots & \vdots \\ I & 0 & \cdots & 0 \end{bmatrix} \begin{bmatrix} 0 & C & \cdots & C \\ C^T & 0 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ C^T & 0 & \cdots & 0 \end{bmatrix} \begin{bmatrix} 0 & \cdots & 0 & I \\ \vdots & 0 & \cdots & 0 \\ 0 & I & \ddots & \vdots \\ I & 0 & \cdots & 0 \end{bmatrix}$$

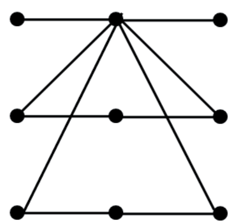
$$= \begin{bmatrix} 0 & C^T & \cdots & C^T \\ C & 0 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ C & 0 & \cdots & 0 \end{bmatrix} = \begin{bmatrix} 0 & B & \cdots & B \\ B^T & 0 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ B^T & 0 & \cdots & 0 \end{bmatrix} = A\left(G_1^{(r)}\right).$$

Therefore the adjacency matrices of the graphs $G_1^{(r)}$ and $G_2^{(r)}$ are unitarily similar. Thus the result follows. $\square$

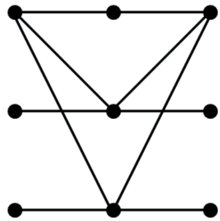
Example 2. Let G be the bipartite graph:



Observe that V_1 and V_2 have both 3 vertices. We have $G_1^{(2)}$:



and $G_2^{(2)}$:



From Corollary 1, the graphs $G_1^{(2)}$ and $G_2^{(2)}$ are cospectral. Observe that they are nonisomorphic. In fact, in $G_1^{(2)}$ the largest vertex degree is 6 whereas in $G_2^{(2)}$ the largest vertex degree is 4.

This example illustrates the following immediate result.

Corollary 2. If $n_1 = n_2$ and if the largest vertex degrees in V_1 and V_2 are different then $G_1^{(r)}$ and $G_2^{(r)}$ are nonisomorhic cospectral graphs.

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