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On the Number of 4-Matchings in Graphs

A.Behmaram

Department of Mathematics, University of Tehran, Tehran, Iran

behmaram@khayam.ut.ac.ir

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Abstract

We establish a formula for the number of 4-matchings in triangular-free graph with respect to the number of vertices, edges, degrees and 4-cycles . Using this formula, it is proved that the Petersen graph is uniquely determined by its matching polynomial. Also, we find other matching unique graphs.

1 Introduction

The graphs considered here are finite, loopless and contains no multiple edges. Let G be such a graph and let n and m be the number of its vertices and edges respectively.

We define a matching in G to be a spanning subgraph of G, whose components are vertices and edges. A k-matching is a matching with k edges. A perfect matching is a matching with edges only.

Let $p(G; k)$ denote the number of k-matchings in G and let $p(G; 0) = 1$.

Define the matching polynomial of graph G as follows:

$$\mu(G,x)=\sum_{k=0}^{\lfloor \frac{n}{2} \rfloor}(-1)^k p(G,k)x^{n-2k}$$

It is obvious that if G and H are isomorphic, then $\mu(G;x)=\mu(H;x)$. But the converse is not true. For example in figure 1, G_1 and G_2 are not isomorphic but we have $\mu(G_1;x)=\mu(G_2;x)=x^4-3x^2$

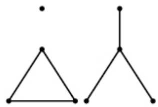

Fig. 1.

Graphs with the same matching polynomial are called co-matching. In figure 1, G_1 and G_2 are co-matching.

A graph which is characterized by its matching polynomial is called matching unique.

2 Preliminaries

It is clear from the definition of $p(G; k)$ that the number of 1-matchings is equal to the number of edges of G. Therefore,

$$p(G,1)=m$$

If the degrees of vertices of G are d_1, d_2, ..., d_n then the number of 2-matchings is obtained by the following lemma(see [2]-p15)

Lemma 2.1.

$$p(G,2)=\binom{m}{2}-\sum_{i=1}^{n}\binom{d_i}{2}$$

Using this formula, Farrell and Guo (see[3]) established a property for the graphs that are co-matching with regular graphs.

Lemma 2.2. Any graph that is co-matching with a regular graph is also regular of the same valency.

For the forth coefficient of matching polynomial, Farrell and Guo (see [3]) found a formula that calculates the number of 3-matchings in graphs using the degrees of vertices and the number of vertices, edges and triangles in graph.

Theorem 2.3.

$$p(G,3)=\binom{m}{3}-(m-2)\sum_i\binom{d_i}{2}+2\sum_i\binom{d_i}{3}+\sum_{ij}(d_i-1)(d_j-1)-N_T$$

where N_T is the number of triangles in *G*.

For regular graphs, we get the following result which is an immediate consequence of the theorem.

Corollary 2.4. Let *G* be a regular graph of degree *d* with *n* vertices. Then,

$$p(G,3)=\binom{m}{3}-(m-2)\sum_i\binom{d_i}{2}+2\sum_i\binom{d_i}{3}+\sum_{ij}(d_i-1)(d_j-1)-N_T$$

where $m=\frac{nd}{2}$ and $p(G,2)=\frac{nr(nr-4r+2)}{8}$

By Corollary 2-4 and lemma 2-2 we obtain the following result:

Corollary 2.5. Suppose that *G* and *H* are two regular graphs which are co-matching, then the number of triangles in *G* is equal to the number of triangles in *H*.

3 The number of 4-matchings

In the following Theorem we will obtain a formula for the fifth coefficient, i.e. p(*G*; 4), of the matching polynomial in triangular-free graphs.

Theorem 3.1. Let *G* be a triangular-free graph, with V(*G*) = {1,2,..,n} and let the degree of vertex i is di. Also, let N(i) be the set of neighbors of i in *G*. Hence, the number of 4-matchings is:

$$p(G,4)=\binom{m}{4}+(m-2)\sum_{ij}(d_i-1)(d_1-1)-\sum_i\binom{d_i}{4}-\sum_{\{i,j\}\subset V}\binom{d_i}{2}\binom{d_j}{2}$$

$$-\sum_i\sum_{\{k,t\}\subset N(i)}(d_k-1)(d_t-1)-\sum_i\binom{d_i}{2}p(G-i,2)-\sum_i\binom{d_i}{3}(m-d_i)$$

$$-\sum_i\sum_{\{k,s,t\}\in N(i)}(d_k+d_t+d_s-3)-2N_q$$

where N_q is the number of 4-cycles in *G*.

Proof.

To find p(G; 4), first we find the number of subsets of edges in G that have4 edges, i.e., The possible subgraphs which does not form a 4-matching are shown in figure 2.

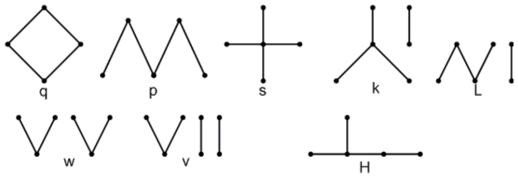

Fig. 2.

Let Nq; N_P; N_S; N_K; N_L; N_W; N_V and N_H denote the number of subgraphs of G that are isomorphic to *q; P; S; K; L; W; V* and H respectively. Now, we calculate each of these numbers, as follows:

N_S: For counting the number of graphs which are isomorphic to S, we choose one vertex and then four edges adjacent to this vertex. Therefore we have:

$$N_S = \sum_{i=1}^{n} \binom{d_i}{4}$$

- N_K: For counting N_K choose one vertex. Then select 3 edges adjacent to this vertex and choose another edge that is not adjacent to the selected vertex.

But these four edges may all be connected to each other and so be isomorphic to H. Therefore, we subtract N_H. So we have:

$$N_K = \sum_{i=1}^{n} \binom{d_i}{3} (m - d_i) - N_H$$

N_L: For counting N_L, first select a path of length three ($p4$). The number of subgraphs in G that are isomorphic to $p4$ is $\sum_{ij} (d_i - 1)(d_j - 1)$.

Because it is enough to select an edge ij, then choose two edges that are adjacent to ij. After selecting $p4$, we choose one edge of graph that does not belong to $p4$.

This edge should not be connected to the edges of $p4$. Therefore we subtract the number of graphs in which this single edge is connected to $p4$. We show this statement in figure 3.

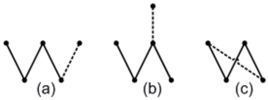

Fig. 3.

In mentioned process, we count N_P and N_H two times(fig3-a,b) and N_q four times (fig3-c). Thus:

$$N_L = (m-3)\sum_{ij}(d_i-1)(d_j-1) - 2N_H - 2N_P - 4N_q$$

N_W: For counting N_W, first select a subset $\{i,j\}$ from $V(G)$, then choose two edges from each i and j. Then, subtract the number of cases in which they do not form graphs isomorphic to W. These cases are as follows:

a: If one of the edges of i is connected to one of j , a path of length 4 is produced.

b: If one edge of i is connected to j it makes a graph isomorphic to H.

c: If we select an edge that connects i to j then it produces $P4$. The number of these graphs is $\sum_{ij}(d_i-1)(d_j-1)$

d: If two edges of i, are connected to two edges of j, then they produce one 4-cycle. In this counting, each 4 cycle is counted two times. Therefore N_W is:

$$N_W = \sum_{\{i,j\}}\binom{d_i}{2}\binom{d_j}{2} - \sum_{ij}(d_i-1)(d_j-1) - N_P - N_H - 2N_q$$

N_V: For counting N_V, choose a vertex i of $V(G)$ and select two edges adjacent to i. Then select a 2-matching from G-i. Now subtract the number of cases in which edges of the 2-matching are connected to edges of i. see figure 4.

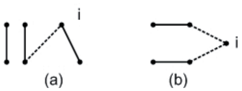

Fig. 4.

In the above counting we count N_L twice (figure 4-a) and N_P once. Therefore, N_V is:

$$N_V = \sum_i \binom{d_i}{2} p(G-i,2) - 2N_L - N_P$$

N_H: In this case, we choose a vertex i from $V(G)$ and then select a subset $\{k,s,t\}$ from $N(i)$. Then we select an edge which adjacent to k or s or t, other than edges connecting k, s, t to i. Therefore, N_H is:

$$N_H = \sum_i \sum_{\{k,s,t\}\subset N(i)} \left(d_k + d_t + d_s - 3\right)$$

Now, the number of 4-matchings is:

$$p(G,4) = \binom{m}{4} - N_P - N_S - N_W - N_K - N_H - N_L - N_q$$

The result is obtained by direct substitution into above formula.

Corollary 3.2. Let G be a triangular-free graph with n vertices which is regular of valency d. Then,

$$p(G,4) = \binom{m}{4} + m(d-1)^2(m-2) - n\binom{d}{4} - \binom{n}{2}\binom{d}{2}^2 - n\binom{d}{2}(d-1)^2$$

$$-p(G,2)(n-4)\binom{d}{2} - n\binom{d}{3}(m+2d-3) + 2N_q$$

Proof. In this case we have

$$-\ N_S = n\binom{d}{4}$$

$$-\ N_K = n\binom{d}{3}(m-d) - N_H$$

$$-\ N_L = m(d-1)^2(m-3) - 2N_H - 2N_P - 4N_q$$

$$-\ N_W = \binom{n}{2}\binom{d}{2}^2 - m(d-1)^2 - N_P - N_H - 2N_q$$

$$-\ N_P = n\binom{d}{2}(d-1)^2 - 4N_q$$

$$-\ N_V = p(G,2)(n-4)\binom{d}{2} - 2N_L - N_P$$

$$-\ N_H = 3n\binom{d}{2}(d-1)$$

The result is obtained by direct substitution into the formula for p(G; 4), given in the Theorem.

Corollary 3.3. Let G and H be triangular-free regular graphs and suppose that G and H are co-matching graphs. Then the number of 4-cycles in G and H are equal.

Proof. From $\mu(G; x) = \mu(H;x)$, we deduce that p(G; 4) = p(H; 4) and it follows from Corollary 3.2 that Nq(G) = Nq(H).

4 The matching polynomial of Petersen graphs

Petersen graph is a 3-regular graph with 10 vertices, as shown in fig5. The girth of Petersen graph is five.

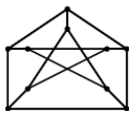

Fig. 5.

Let G be the Petersen graph. We obtain the matching polynomial of G, using the formula that we have found for $p(G;r)(1\leq r\leq 4)$. We have: $p(G,1)=15$, $p(G,2)=75$, $p(G,3)=185$, $p(G,4)=90$.

It is easily seen that, the number of perfect matching in this graph is 6. Therefore the matching polynomial of Petersen graph is:

$$\mu(G,x) = x^{10} - 15x^8 + 75x^6 - 185x^4 + 90x^2 - 6$$

Lemma 4.1. The Petersen graph is the unique 3-regular graph of ten vertices that has girth 5.

Proof. Let G be the Petersen graph and suppose that H is another graph with properties mentioned in Lemma. Let the vertices of G and H be $\{v_1, v_2, \ldots, v_{10}\}$

Assume that $v_1v_2\ldots v_5v_1$ is a 5-cycle in H. Then every vertex of the cycle is connected to one vertex not in the cycle, because the 5-cycle doesn't have diameter. Now, the induced subgraph of vertices $v_6\ldots v_{10}$ is a 2-regular graph. Then $v_6\ldots v_{10}v_6$ is a 5-cycle. Each vertex of a 5-cycle is connected to one of the vertices of the other 5-cycle such that there is not any 4-cycles in the H. Therefore, H is isomorphic to G.

Theorem 4.2. The Petersen graph is matching unique.

Proof. Let *G* be the Petersen graph and *H* be another graph which is co-matching to *G*. Using Lemma 2-2 we follows that *H* is a 3-regular graph and since $\mu(G;x) = \mu(H;x)$ we have:

– $p(G; 3) = p(H; 3)$ therefore by lemma 2-5, *H* is triangular-free

– $p(G; 4) = p(H; 4)$ therefore by lemma 3-3, *H* is 4-cycle-free

Then the girth of *H* is greater than 4. Let *m* be the girth of *H*. If $m>5$, Then each vertex of *m*-cycle must be connected to a vertex not in the *m*-cycle. Therefore, we have at least $2m$ vertices, which is contradiction. Thus, the girth of *H* is 5.Using previous lemma, we follows that *H* is isomorphic to *G*. Therefore, *G* is matching unique.

Theorem 4.3. Every triangular free 3-regular graphs with ten vertices is matching unique.

Proof. There exists only 6-graphs with this property (see [4]). One of them is the Petersen graph which we have proven that is matching unique. Five remaining graphs are shown in figure 6, and the number of their 4-cycles is written below them:

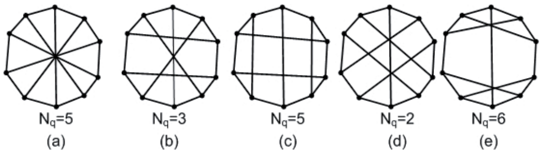

Fig. 6.

The graphs shown in (b), (d) and (e) are matching unique, because by Lemma 3.3, there is not any 3-regular graph which is triangular-free and have the same 4-cycles.

The graphs shown in (a) and (c) have the same number of 4-cycle but the number of perfect matching of (a) is 13 where that of (c) is 11. So (a) and (c) have different number of 5-matchings. Therefore all graphs above are matching unique.

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