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COMPUTATION OF THE FIRST EDGE-WIENER INDEX OF TUC_4C_8(S) NANOTUBE

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Abstract

Wiener index was introduced by Harold Wiener in 1947. This index is the sum of distances between all vertices of a graph. The edge versions of Wiener index was introduced by Iranmanesh et al. in [18]. In this paper, the first edge Wiener index of TUC_4C_8(S) nanotube is computed.

1. Introduction

We denote the set of vertices of connected graph G with $V(G)$ and set of edges with $E(G)$. In a molecular graph, each vertex denotes an atom and edges denote the bond of between atoms. A topological index is a real number which describes the molecular graph.

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The oldest topological index which is vertex-Wiener index was introduced by Harold Wiener [1]. He introduced this index for comparing and describing the relation between Physical-Chemical properties.

The definition of this index is as follows:

If $u,v \in V(G)$ and $d(u,v)$ is the shortest distance between them, then

$$W(G)=\frac{1}{2}\sum_{u,v\in V(G)} d(u,v)$$

The Wiener index of many nanotubes has been computed. For example see [2-17].

The edge-Wiener index was introduced by Iranmanesh et al. in [18] as follow:

Suppose $e,f \in E(G)$ where $e=(u,v), f=(x,y)$. Set

$d_1(e,f)=\min\{d(u,x),d(u,y),d(v,x),d(v,y)\}$ and

$$d_0(e,f)=\begin{cases} d_1(e,f)+1 & ,e\neq f \\ 0 & ,e=f \end{cases}$$

The first edge-Wiener index is denoted by $W_{e0}(G)$ and define by:

$$W_{e0}(G)=\frac{1}{2}\sum_{e,f\in E(G)} d_0(e,f)$$

The aim of this paper is computing of the edge Wiener index of $TUC_4C_8(S)$ nanotube.

2. The first edge Wiener index of TUC$_4$C$_8$(S)

A C_4C_8 net is a trivalent decoration made by alternating squares and octagons C_8. Let $T(p,q)=TUC_4C_8(S)$ where p is denoted the number of octagonal in rows and q is the number of octagons in columns. We consider j periods, where $1\leq j\leq q$, for this nanotube that each period has an upper row and a lower row. For example in Fig.1, we show that $T(6,4)$ nanotube.

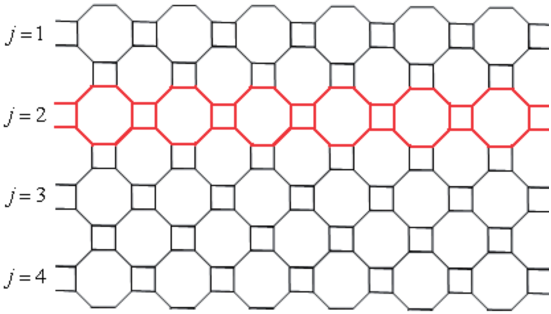

Fig. 1. $T(6,4)$ nanotube with $1 \le j \le 4$ periods.

Definition 2-1. Suppose $e \in E(G)$. Set:

$$A_u = \bigcup_{j=1}^{q} \{ e \in E(G) \mid e \text{ is an edge over octagones in upper row of } j-th \text{ period} \}$$

$$A_d = \bigcup_{j=1}^{q} \{ e \in E(G) \mid e \text{ is an edge over squares in lower row of } j-th \text{ period} \}$$

$$B_u = \bigcup_{j=1}^{q} \{ e \in E(G) \mid e \text{ is an edge under octagones in upper row of } j-th \text{ period} \}$$

$$B_d = \bigcup_{j=1}^{q} \{ e \in E(G) \mid e \text{ is an edge under squares in lower row of } j-th \text{ period} \}$$

$$C_u = \bigcup_{j=1}^{q} \{ e \in E(G) \mid e \text{ is an oblique edge in upper row of } j-th \text{ period} \}$$

$$C_d = \bigcup_{j=1}^{q} \{ e \in E(G) \mid e \text{ is an oblique edge in lower row of } j-th \text{ period} \}$$

$$D = \bigcup_{j=1}^{q} \{ e \in E(G) \mid e \text{ is a vertical edge that located between upper and lower rows} \}$$

$$E = \bigcup_{j=1}^{q-1} \{ e \in E(G) \mid e \text{ is vertical edge that located between } j-th \text{ period and } j+1-th \text{ period} \}$$

So we have

$$W_{e0}(G) = W_{e0}(A_u,G) + W_{e0}(A_d,G) + W_{e0}(B_u,G) + W_{e0}(B_d,G) + W_{e0}(C_u,G) + W_{e0}(C_d,G) + W_{e0}(D,G) + W_{e0}(E,G)$$

We use the notation ${}_XW_{e0}(e_Y, G)_j$ for W_{e0} if e is fix edge from set Y in region X and period j in graph G. Also we use the notations G_1 and G_2 for the sub-graphs $G_1 = T(p, j)$ and $G_2 = T(p, q-j+1)$ respectively.

In addition, we denoted $WYn_j = W_{e0}(e_{Yn}, G_1)_1 + W_{e0}(e_{Zs}, G_2)_1 - m$ where e is a fix edge from set Y_n in $j-th$ period and m is equal to the sum of distances between edges which located in the common region between graph G_1 and G_2.

For compute the first edge index, we need three cases: $q \prec \left[\frac{p}{2}\right]+1$, $q = \left[\frac{p}{2}\right]+1$ and

$$q \succ \left[\frac{p}{2}\right]+1.$$

Case 1. $q \prec \left[\frac{p}{2}\right]+1$

In this case we have two sub cases:

(i): p is even.

Lemma 2.2. Suppose $e \in A_u$, then there are two regions R *and* R' in Fig.2, such that

$$
{}_RW_{e_0}(e,G)_1 = 2\left(\sum_{j=0}^{q-2}\left(\left(\frac{p}{2}-j\right)(6j+2)+\sum_{i=0}^{\frac{p}{2}-j-1}4i\right)\right)-\left(\sum_{j=0}^{q-2}(6j+2)\right)+\left(\sum_{j=0}^{q-2}\sum_{i=3j}^{p+j-1}(2i+3)\right)+\left(\left(\frac{p}{2}-q+1\right)(6q-4)+\sum_{i=0}^{\frac{p}{2}-q}4i\right)
$$

$$
+\left(\left(\frac{p}{2}-q+1\right)(6q-3)+\sum_{i=0}^{\frac{p}{2}-q}4i\right)+\left(\sum_{k=0}^{2q-1}\sum_{i=3k}^{2p+k}i\right)
$$

$$
{}_{R'}W_{e_0}(e,G)_1 = 2\left(\sum_{j=1}^{q-1}\left((q-j)(6j)+\sum_{i=0}^{q-j-1}4i\right)\right)+\left(\sum_{j=1}^{q-1}\sum_{i=3j}^{2q+j-1}(2i+1)\right)+\left(\sum_{j=1}^{q}(6j-3)\right)+\left(\sum_{k=1}^{2q-2}\sum_{i=3k+1}^{4q+j-2}i\right).
$$

Proof. The regions R *and* R' are shown in Fig.2. For computing ${}_RW_{e_0}(e,G)_1$ and ${}_{R'}W_{e_0}(e,G)_1$, we consider the rows k, j, where indicated in Fig.2. Due to the rows and distances between edge $e \in A_u$ and other edges in region R, ${}_RW_{e_0}(e,G)_1$ can compute easily. Similarly, the ${}_{R'}W_{e_0}(e,G)_1$ will be computed. ■

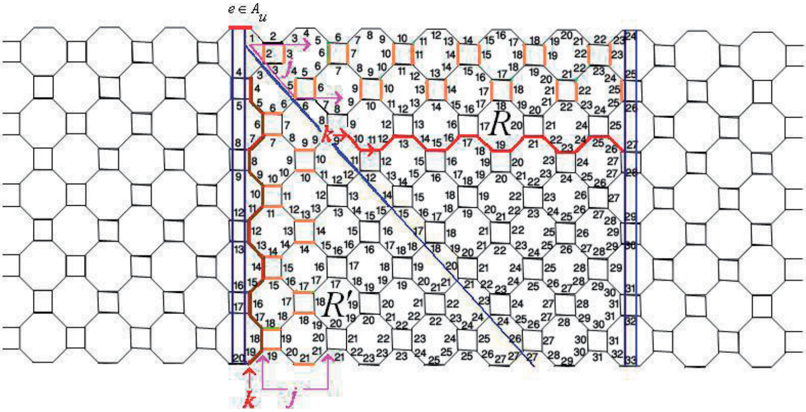

Fig. 2. The regions R and R' in T(12,5) for $e \in A_u$ where $q \prec \left[\frac{p}{2}\right]+1$.

Lemma 2.3. Suppose $e \in A_u$. Then

$W_{e0}(e,G)_1 = 2\left(\left({}_{R}W_{e0}(e,G)_1\right)+\left({}_{R'}W_{e0}(e,G)_1\right)\right)-(t_{11})+(t_{21})$ where $t_{11} = \sum_{i=2p}^{2p+2q-1} i$ and

$t_{21} = \left(\sum_{i=1}^{q-1}(8i+1)\right)+4q$.

Proof. In Fig.2, $W_{e0}(e_{Au},G)_1$ is equal to $2\left(\left({}_{R}W_{e0}(e,G)_1\right)+\left({}_{R'}W_{e0}(e,G)_1\right)\right)-(t_{11})+(t_{21})$, where t_{11} is the sum of distances between edges on symmetry line in right side of region R and t_{21} is the sum of distances between edges on symmetry line in left side of region R'.

■

Lemma 2.4. Suppose $e \in A_d$. Then

$$W_{e0}\left(e_{A_d},G\right)_1 = W_{e0}\left(e_{A_u},G\right)_1 + (u_1) - (d_1) \text{ where } u_1 = 2(1+(\sum_{i=1}^{\frac{p}{2}-1}(8i+1))+2p)+(2+(2\sum_{i=2}^{2p} i)+(2p+1))$$

and $$d_1 = (4q)+2\begin{pmatrix} (\sum_{i=4q-2}^{6q-5} i)+((\frac{p}{2}-q+1)(6q-4)+\sum_{i=0}^{\frac{p}{2}-q}4i)+((\frac{p}{2}-q+1)(6q-3)+\sum_{i=0}^{\frac{p}{2}-q}4i) \\ +(\sum_{i=2q-1}^{3q-2}(2i+1))+(\sum_{i=4q-1}^{2q+2p-2} i)+(\sum_{i=2q}^{3q-2}(2i+1)) \end{pmatrix}+(2q+2p-1).$$

Proof. Let $e \in A_d$ be a fix and grey edge in Fig.3. According to this figure, for computing $W_{e0}(e_{Ad},G)_1$, at first we need obtain the sum of distances between edges on green rectangular. This quantity is equal to the first term of u_1. Then by the commute of the graph such that the grey edge matches on the upper horizontal edge (red edge). The sum of distances from $e \in A_d$ to the other edges is equal to $W_{e0}(e_{Au},G)_1$ minus the sum of distances between edges on below green rectangular in Fig.3, that is, d_1. Therefore, we can get $W_{e0}(e_{Ad},G)_1$ with add the summation of distances between edges on upper green rectangular to the computation. ■

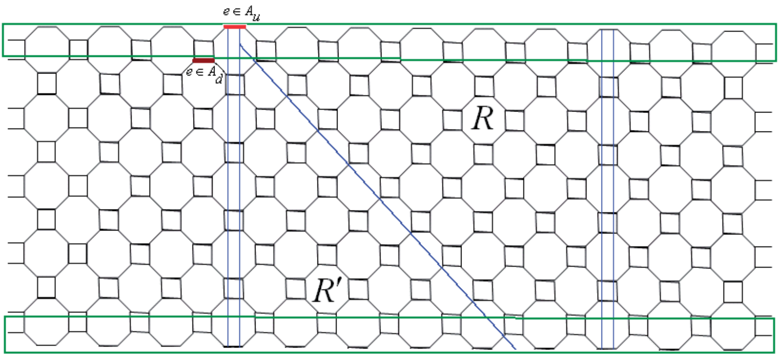

Fig. 3. Computing $W_{e0}\left(e_{A_d},G\right)_1$ for $e \in A_d$ in T(12,5) where $q \prec \left[\frac{p}{2}\right]+1$.

Lemma 2.5. Suppose $e \in B_u$, then there are two regions R and R' in Fig.4, such that

$$_{R}W_{e_0}(e,G)_1 = 2\left(\sum_{j=0}^{q-2}((\frac{p}{2}-j+1)(6j)+\sum_{i=0}^{\frac{p}{2}-j}4i)\right)-(\sum_{j=0}^{q-2}6j)+(\sum_{j=0}^{q-2}\sum_{i=3j}^{p+j-1}(2i+1))+((\frac{p}{2}-q+2)(6q-6)+\sum_{i=0}^{\frac{p}{2}-q+1}4i))$$

$$+((\frac{p}{2}-q+1)(6q-5)+\sum_{i=0}^{\frac{p}{2}-q}4i))+(1+\sum_{k=0}^{2q-1}\sum_{i=3k-2}^{2p+k-1}i)$$

$$_{R'}W_{e_0}(e,G)_1 = 2\left(\sum_{j=1}^{q-1}((q-j)(6j+1)+\sum_{i=0}^{q-j-1}4i)\right)+(\sum_{j=1}^{q-2}\sum_{i=3j}^{2q+j-3}(2i+2))+(\sum_{j=1}^{q-1}(6j-2))+(\sum_{k=1}^{2q-3}\sum_{i=3k+2}^{4q+j-3}i)$$

Proof. Similarly to the proof of Lemma 2.2, we obtain the desire result. ■

Lemma 2.6. Suppose $e \in B_u$. Then

$W_{e0}(e,G)_1 = 2\left(\left({}_{R}W_{e0}(e,G)_1\right)+\left({}_{R'}W_{e0}(e,G)_1\right)\right)+(t_{12})+(t_{22})$ where

$t_{12} = \sum_{i=2p}^{2p+2q-1} i$ and $t_{22} = 2+(\sum_{i=1}^{q-1}(8i+3))$.

Proof. In Fig.4, $W_{e0}(e_{Bu},G)_1$ is equal to $2\left(\left({}_{R}W_{e0}(e,G)_1\right)+\left({}_{R'}W_{e0}(e,G)_1\right)\right)+(t_{12})+(t_{22})$, where t_{12} is sum of the distances between edges on symmetry line in right side of region R and

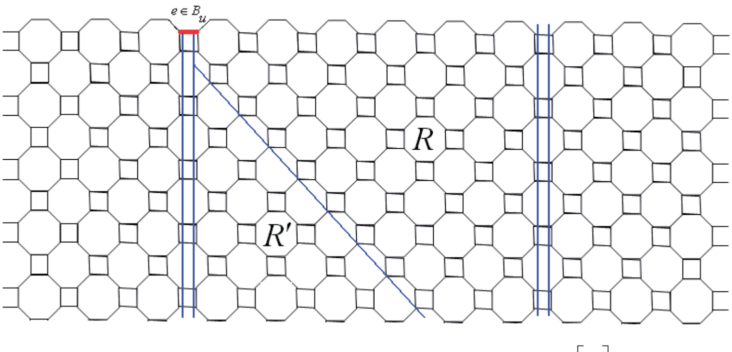

Fig. 4. The regions R and R' in T(12,5) for $e \in B_u$ where $q \prec \left[\frac{p}{2}\right]+1$.

t_{22} is sum of the distances between edges on symmetry line in left side of region R'. ■

Lemma 2.7. Suppose $e \in B_d$. Then

$W_{e0}\left(e_{B_d},G\right)_1 = W_{e0}\left(e_{B_u},G\right)_1 + (u_2) - (d_2)$ where $u_2 = \left(2\sum_{i=1}^{\frac{p}{2}}(8i-3)\right) + \left(4 + 2\left(3 + \sum_{i=3}^{2p} i\right) + (2p+1)\right)$ and

$$d_2 = (4q-2) + 2\left(\begin{array}{l}(\sum_{i=4q-3}^{6q-7} i) + ((\frac{p}{2}-q+2)(6q-6) + \sum_{i=0}^{\frac{p}{2}-q+1} 4i) + ((\frac{p}{2}-q+1)(6q-5) + \sum_{i=0}^{\frac{p}{2}-q} 4i) + \\ (2\sum_{i=2q-1}^{3q-3}(2i+1)) + (\sum_{i=4q-2}^{2q+2p-2} i))\end{array}\right) + (2q+2p-1) \, .$$

Proof. Similarly to the proof of Lemma 2.4, we obtain the desire result. ■

Lemma 2.8. For the set A_u, we have:

$$W_{e0}\left(A_u\right) = \frac{1}{2}\sum_{j=1}^{q} p\left(WAu_j\right) + \frac{1}{2} p W_{e0}\left(e_{A_u},G\right)_1$$

Proof. Let $e \in A_u$ be an edge on $j-th$ period. We divide the graph $G = T(p,q)$ into two sub-graphs G_1 and G_2 which have been indicated in Figure 5. In this case, we have: $WAu_j = W_{e0}(e_{Bd},G_1)_1 + W_{e0}(e_{Au},G_2)_1 - m_1$ where m_1 is equal to the sum of distances between edges which located in the common region between graph G_1 and G_2, that is,

$$m_1 = 4 + 2\left((\sum_{i=1}^{2p-1} i) + (\sum_{i=3}^{2p} i) + 3 + (\sum_{i=1}^{\frac{p}{2}}(8i-3))\right) + (2p) + (2p+1)$$

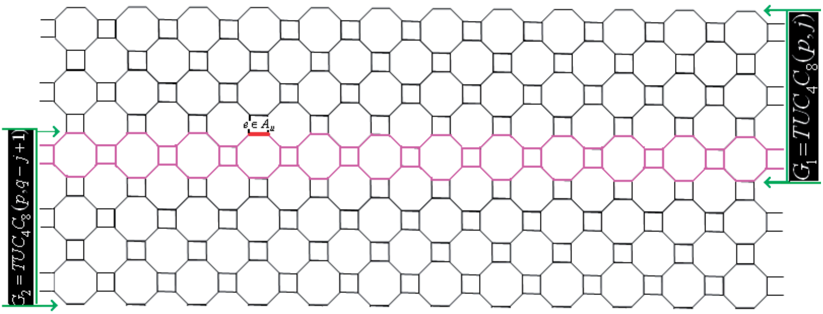

Fig. 5. Dividing the graph T(12,5) in two sub-graph T(12,3) and T(12,3) for $e \in A_u$ where $q \prec \left[\frac{p}{2}\right]+1$.

Now, since there are p edges in the set A_u of each period and also there are p edges in the lower row of $j-th$ period (that the sum of their distances is equal to $\frac{1}{2}pW_{e0}\left(e_{A_u},G\right)_1$), we obtain the desire result. ■

Lemma 2.9. For the set A_d, we have:

$$W_{e0}\left(A_d\right)=\frac{1}{2}\sum_{j=1}^{q-1}p\left(WAd_j\right)+\frac{1}{2}pW_{e0}\left(e_{B_u},G\right)_1$$

Proof. The proof is similar to the proof of Lemma 2.8, but we have $WAd_j = W_{e0}(e_{Bu},G_1)_1 + W_{e0}(e_{Ad},G_2)_1 - m_2$ where m_2 is equal to the sum of distances between edges which located in the common region between graph G_1 and G_2, that is,

$$m_2 = 2+2\left((\sum_{i=1}^{2p-1} i)+(\sum_{i=2}^{2p} i)+(1+\sum_{i=1}^{\frac{p}{2}-1}(8i+1)+2p) \right)+(2p)+(2p+1)$$

Since the number of periods contain these edges are $q-1$ and there are p edges in the lower row of last period, we obtain the desired results. ■

Lemma 2.10. For the set B_u, we have:

$$W_{e0}\left(B_u\right)=\frac{1}{2}\sum_{j=1}^{q}p\left(WBu_j\right)+\frac{1}{2}pW_{e0}\left(e_{B_u},G\right)_1$$

Proof. The proof is similar to the proof of Lemma 2.8, but we have $WBu_j = W_{e0}(e_{Ad}, G_1)_1 + W_{e0}(e_{Bu}, G_2)_1 - m_3$ where m_3 is equal to the sum of distances between edges which located in the common region between graph G_1 and G_2, that is,

$$m_3 = 2+2\left((\sum_{i=1}^{2p-1} i)+(\sum_{i=2}^{2p} i)+(1+\sum_{i=1}^{\frac{p}{2}-1}(8i+1)+2p)\right)+(2p)+(2p+1)\,.$$ ■

Lemma 2.11. For the set B_d, we have:

$$W_{e0}\left(B_d\right)=\frac{1}{2}\sum_{j=1}^{q-1}p\left(WBd_j\right)+\frac{1}{2}pW_{e0}\left(e_{A_u},G\right)_1$$

Proof. This is similar to the proof of lemma 2.9, but we have $WBd_j = W_{e0}(e_{Au}, G_1)_1 + W_{e0}(e_{Bd}, G_2)_1 - m_4$ where m_4 is equal to the sum of distances between edges which located in the common region between graph G_1 and G_2, that is,

$$m_4 = 4+2\left((\sum_{i=1}^{2p-1} i)+(\sum_{i=3}^{2p} i)+3+(\sum_{i=1}^{\frac{p}{2}}(8i-3))\right)+(2p)+(2p+1)\,.$$ ■

Lemma 2.12. Let $e \in C_u$. According to Fig.6, there are 4 regions for $e \in C_u$ in $T(p,q)$ that satisfy the following relations:

$${}_{R_1}W_{e_0}(e,G)_1 = 2\left(\sum_{j=0}^{q-2}((\frac{p}{2}-j+1)(6j+1)+\sum_{i=0}^{\frac{p}{2}-j}4i)\right)-(2+\sum_{j=0}^{q-2}(6j+1))+(\sum_{j=0}^{q-2}\sum_{i=3j}^{p+j-1}(2i+2))$$

$$+((\frac{p}{2}-q+2)(6q-5)+\sum_{i=0}^{\frac{p}{2}-q+1}4i))+((\frac{p}{2}-q+1)(6q-4)+\sum_{i=0}^{\frac{p}{2}-q}4i))+(1+(\sum_{k=0}^{2q-1}\sum_{i=3k-1}^{2p+k-1}i)+(\sum_{i=p}^{p+q-1}2i))$$

$${}_{R_2}W_{e_0}(e,G)_1 = 2\left(\sum_{j=1}^{q-1}((\frac{p}{2}-j+1)(6j-3)+\sum_{i=0}^{\frac{p}{2}-j}4i)\right)-(\sum_{j=1}^{q-1}(6j-3))+(\sum_{j=1}^{q-1}\sum_{i=3j-1}^{p+j-1}2i)+((\frac{p}{2}-q+1)(6q-3)+\sum_{i=0}^{\frac{p}{2}-q}4i))$$

$$+((\frac{p}{2}-q+1)(6q-2)+\sum_{i=0}^{\frac{p}{2}-q}4i))+(\sum_{i=1}^{2q-1}(3i+1))+(\sum_{k=1}^{2q-1}\sum_{i=3k-2}^{2p+k-2}i)$$

$${}_{R'}W_{e_0}(e,G)_1 = 2\left(\sum_{j=1}^{q-1}((q-j)(6j-1)+\sum_{i=0}^{q-j-1}4i)\right)+(\sum_{j=1}^{q-1}\sum_{i=3j}^{2q+j-1}2i)+(\sum_{j=1}^{q-1}(6j+2))+(\sum_{k=1}^{2q-3}\sum_{i=3k+3}^{4q+j-2}i)$$

$${}_{R''}W_{e_0}(e,G)_1 = 2\left(\sum_{j=0}^{q-1}((q-j)(6j+3)+\sum_{i=0}^{q-j-1}4i)\right)+(\sum_{j=1}^{q-1}\sum_{i=3j}^{2q+j-1}(2i-2))-(\sum_{j=0}^{q-1}(6j+3))+(\sum_{k=1}^{2q-2}\sum_{i=3k+1}^{4q+j-2}i)$$

Proof. The proof is similar to the proof of Lemma 2.2. ■

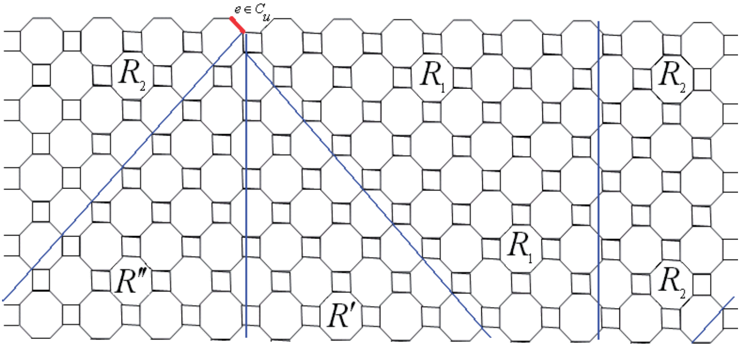

Fig. 6. The regions R', R'', R_1 and R_2 in T(12,5) for $e \in C_u$ where $q \prec \left[\frac{p}{2}\right]+1$.

Lemma 2.13. Let $e \in C_u$ in Fig.6. Then

$$W_{e0}(e,G)_1 = \left({}_{R_1}W_{e0}(e,G)_1\right)+\left({}_{R_2}W_{e0}(e,G)_1\right)+\left({}_{R'}W_{e0}(e,G)_1\right)+\left({}_{R''}W_{e0}(e,G)_1\right)$$

Proof. Straightforward. . ■

Lemma 2.14. Suppose $e \in C_d$ in Fig.7. Then

$W_{e0}\left(e_{C_d}, G\right)_1 = W_{e0}\left(e_{C_u}, G\right)_1 + (u_3) - (d_3)$ where $u_3 = \left(\sum_{i=1}^{2p} i\right) + \left(4 + (2\sum_{i=2}^{2p} i) + (2p+1)\right)$ and

$$d_3 = ((\sum_{i=4q-2}^{6q-6} i) + ((\frac{p}{2}-q+1)(6q-5) + \sum_{i=0}^{\frac{p}{2}-q} 4i) + ((\frac{p}{2}-q+1)(6q-4) + \sum_{i=0}^{\frac{p}{2}-q} 4i) + (\sum_{i=2q-1}^{3q-2} 2i) + (\sum_{i=4q-1}^{2q+2p-1} i) + \sum_{i=2q}^{3q-2} 2i)$$

$$+((\sum_{i=4q-3}^{6q-5} i) + ((\frac{p}{2}-q+1)(6q-3) + \sum_{i=0}^{\frac{p}{2}-q} 4i) + ((\frac{p}{2}-q+1)(6q-2) + \sum_{i=0}^{\frac{p}{2}-q} 4i) + (2\sum_{i=2q-1}^{3q-3} (2i+1)) + (6q-2) + (\sum_{i=4q-2}^{2q+2p-2} i))$$

Proof. The proof is similar to Lemma 2.4. ■

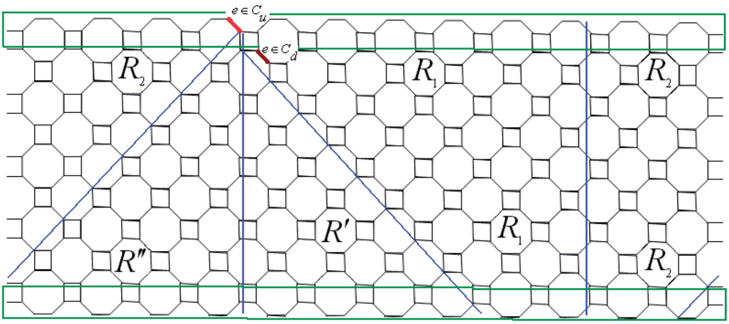

Fig. 7. Computing $W_{e0}\left(e_{C_d}, G\right)_1$ for $e \in C_d$ in T(12,5) where $q \prec \left[\frac{p}{2}\right] + 1$.

Lemma 2.15. For the set C_u , we have:

$$W_{e0}\left(C_u\right) = \frac{1}{2}\sum_{j=1}^{q} 2p\left(WCu_j\right) + pW_{e0}\left(e_{C_u}, G\right)_1$$

Proof. Suppose $e \in C_u$ be an edge on $j-th$ period. We divide the graph $G = T(p,q)$ into two sub-graphs G_1 and G_2 In this case, we have: $WCu_j = W_{e0}(e_{Cd}, G_1)_1 + W_{e0}(e_{Cu}, G_2)_1 - m_{cu}$ where m_{cu} is equal to the sum of distances between edges which located in the common region between graph G_1 and G_2, that is,

$$m_{cu} = 4 + 2\left((\sum_{i=1}^{2p-1} i) + (\sum_{i=2}^{2p} i) \right) + (\sum_{i=1}^{2p} i) + (2p) + (2p+1)$$

Since we have q periods, *2p* edges in each period and *2p* edges in lower row of $q-th$ period (that the sum of their distances is equal to $pW_{e0}\left(e_{C_u}, G\right)_1$), the desire result is obtain. ■

Lemma 2.16. For the set C_d, we have:

$$W_{e0}\left(C_d\right) = \frac{1}{2}\sum_{j=1}^{q-1} 2p\left(WCd_j\right) + pW_{e0}\left(e_{C_u}, G\right)_1$$

Proof. This is similar to the proof of Lemma 2.15, but we have $WCd_j = W_{e0}(e_{Cu}, G_1)_1 + W_{e0}(e_{Cd}, G_2)_1 - m_{cd}$ where m_{cd} is equal to the sum of distances between edges located in the common region between graph G_1 and G_2, that is,

$$m_{cd} = 4 + 2\left((\sum_{i=1}^{2p-1} i) + (\sum_{i=2}^{2p} i) \right) + (\sum_{i=1}^{2p} i) + (2p) + (2p+1)$$

Since the number of periods contain these edges are $q-1$ and there are $2p$ edges in lower row of last period, we obtain the desired results. ■

Lemma 2.17. Suppose $e \in D$. According to Fig.8, there are 5 regions for $e \in D$ in $T(p,q)$ that they satisfy the following relations:

$${}_AW_{e_0}\left(e, G\right)_1 = (2\sum_{i=1}^{2p} i) + (\sum_{i=1}^{p} 2i) + ((2\sum_{i=1}^{\frac{p}{2}-1}(4i+1)) + (2p+1))$$

$${}_{R_1}W_{e_0}\left(e, G\right)_1 = 2\left(\sum_{j=0}^{q-2}((\frac{p}{2}-j)(6j+3) + \sum_{i=0}^{\frac{p}{2}-j-1} 4i)\right) - (\sum_{j=0}^{q-2}(6j+3)) + (\sum_{j=0}^{q-2}\sum_{i=3j}^{p+j-1}(2i+4)) + (\sum_{k=0}^{2q-2}\sum_{i=3k+1}^{2p+k} i)$$

$$_{R_2}W_{e_0}(e,G)_1 = 2\left(\sum_{j=1}^{q-1}((\frac{p}{2}-j+1)(6j-4)+\sum_{i=0}^{\frac{p}{2}-j}4i)\right)-(\sum_{j=1}^{q-1}(6j-4))+(\sum_{j=1}^{q-1}\sum_{i=3j-1}^{p+j}(2i-1))+(\sum_{k=0}^{2q-2}\sum_{i=3k}^{2p+k} i)$$

$$_{R'}W_{e_0}(e,G)_1 = 2\left(\sum_{j=1}^{q-1}((q-j)(6j-2)+\sum_{i=0}^{q-j-1}4i)\right)+(\sum_{j=1}^{q-1}\sum_{i=3j}^{2q+j-1}(2i-1))+(\sum_{i=1}^{q-1}(6i+1))+(\sum_{k=1}^{2q-3}\sum_{i=3k+2}^{4q+j-3} i)$$

$$_{R''}W_{e_0}(e,G)_1 = 2\left(\sum_{j=1}^{q-1}((q-j)(6j)+\sum_{i=0}^{q-j-1}4i)\right)+(\sum_{j=1}^{q-2}\sum_{i=3j}^{2q+j-3}(2i+1))+(\sum_{i=1}^{q-2}(6i+3))+(\sum_{k=1}^{2q-3}\sum_{i=3k+1}^{4q+j-4} i)$$

Proof. Similar to the proof of Lemma 2.2 we obtain the desire result. ■

Lemma 2.18. Let $e \in D$ in Fig.8. Then

$$W_{e0}(e,G)_1 = \left({}_{A}W_{e0}(e,G)_1\right)+\left({}_{R_1}W_{e0}(e,G)_1\right)+\left({}_{R_2}W_{e0}(e,G)_1\right)+\left({}_{R'}W_{e0}(e,G)_1\right)+\left({}_{R''}W_{e0}(e,G)_1\right)$$

Proof. Straightforward. . ■

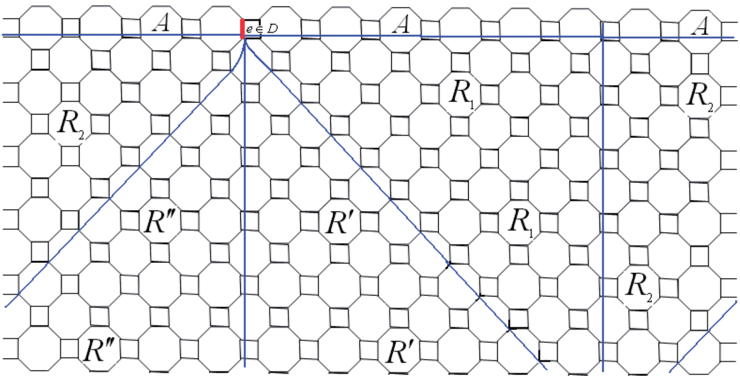

Fig. 8. The regions R', R'', R_1, R_2 and A in T(12,6) for $e \in D$ where $q \prec \left[\frac{p}{2}\right]+1$.

Lemma 2.19. For the set D, we have:

$$W_{e0}(D) = \frac{1}{2}\sum_{j=1}^{q} 2p\left(WD_j\right)$$

Proof. This is similar to the proof of Lemma 2.15, but we have $WD_j = W_{e0}(e_D, G_1)_1 + W_{e0}(e_D, G_2)_1 - m_D$ where m_D is equal to the sum of distances between edges which are located in the common region between graph G_1 and G_2, that is,

$$m_D = (4\sum_{i=1}^{2p} i) + (2 + \sum_{i=1}^{\frac{p}{2}-1}(8i+3)) + (\sum_{i=1}^{\frac{p}{2}}(8i+1))$$ ■

Lemma 2.20. Suppose $e \in E$ in Fig.9. Then

$W_{e0}(e_E, G)_1 = W_{e0}(e_D, G)_1 + (u_4) - (d_4)$ where $u_4 = (\sum_{i=1}^{\frac{p}{2}}(8i-1)) + (\sum_{i=1}^{\frac{p}{2}}(8i-3)) + (14 + 2\sum_{i=4}^{2p+1} i)$ and

$$d_4 = ((\sum_{i=4q-3}^{6q-7} i) + ((\frac{p}{2}-q+2)(6q-6) + \sum_{i=0}^{\frac{p}{2}-q+1} 4i) + ((\frac{p}{2}-q+1)(6q-5) + \sum_{i=0}^{\frac{p}{2}-q} 4i) + (\sum_{i=2q-1}^{3q-2}(2i-1)) + (\sum_{i=4q-2}^{2q+2p-2} i) + \sum_{i=2q}^{3q-2}(2i-1))$$
$$+ ((\sum_{i=4q-4}^{6q-8} i) + ((\frac{p}{2}-q+2)(6q-7) + \sum_{i=0}^{\frac{p}{2}-q+1} 4i) + ((\frac{p}{2}-q+1)(6q-6) + \sum_{i=0}^{\frac{p}{2}-q} 4i) + (2\sum_{i=2q-1}^{3q-3}(2i)) + (\sum_{i=4q-3}^{2q+2p-2} i))$$

Proof. This is similar to the proof of Lemma 2.4. ■

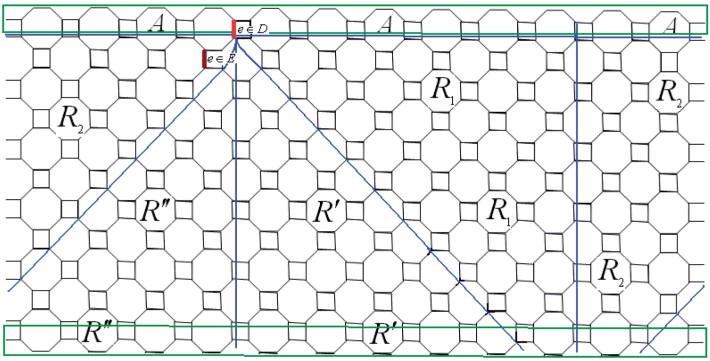

Fig. 9. Computing $W_{e0}(e_E, G)_1$ for $e \in E$ in T(12,6) where $q \prec \left[\frac{p}{2}\right] + 1$.

Lemma 2.21. For the set E , we have:

$$W_{e0}(E)=\frac{1}{2}\sum_{j=1}^{q-1}p\left(WE_j\right)$$

Proof. This is similar to the proof of Lemma 2.15, but we have $WE_j=W_{e0}(e_E,G_1)_1+W_{e0}(e_E,G_2)_1-m_E$ where m_E is equal to the sum of distances between edges which are located in the common region between graph G_1 and G_2, that is,

$$m_E=2\left((2\sum_{i=1}^{2p}i)+(14+2\sum_{i=4}^{2p+1}i)+(\sum_{i=1}^{\frac{p}{2}}(8i-1)+(\sum_{i=1}^{\frac{p}{2}}(8i-3))\right)+\left(2+(\sum_{i=1}^{\frac{p}{2}-1}(8i+3)\right)+\left((\sum_{i=1}^{\frac{p}{2}}(8i+1)\right).$$ ■

Based on the above results, we can state the following theorem:

Theorem 2.22. Let p be an even number and $q\prec\left[\frac{p}{2}\right]+1$. Then

$$W_{e0}(G)=W_{e0}(A_u)+W_{e0}(A_d)+W_{e0}(B_u)+W_{e0}(B_d)+W_{e0}(C_u)+W_{e0}(C_d)+W_{e0}(D)+W_{e0}(E)=$$
$$=p+34pq^2+12p^2q-10p^2-31pq-20p^2q^2+15pq^3-20p^3q+48p^2q^3+24pq^4+72p^3q^2+10p^3$$

■

(ii): p is odd.

This case is almost similar to the case (i), but there are some differences which have been mentioned as follows:

Lemma 2.23. Suppose $e\in A_u$, then there are two regions R *and* R', such that

$${}_RW_{e_0}(e,G)_1=2\left(\sum_{j=0}^{q-2}((\frac{p-1}{2}-j+1)(6j+2)+\sum_{i=0}^{\frac{p-1}{2}-j}4i)\right)-(\sum_{j=0}^{q-2}(6j+2))+(\sum_{j=0}^{q-2}\sum_{i=3j}^{p+j-2}(2i+3))$$
$$+((\frac{p-1}{2}-q+2)(6q-4)+\sum_{i=0}^{\frac{p-1}{2}-q+1}4i))+((\frac{p-1}{2}-q+1)(6q-3)+\sum_{i=0}^{\frac{p-1}{2}-q}4i))+(\sum_{k=0}^{2q-1}\sum_{i=3k}^{2p+k}i)$$

$${}_{R'}W_{e_0}(e,G)_1=2\left(\sum_{j=1}^{q-1}((q-j)(6j)+\sum_{i=0}^{q-j-1}4i)\right)+(\sum_{j=1}^{q-1}\sum_{i=3j}^{2q+j-1}(2i+1))+(\sum_{j=1}^{q}(6j-3))+(\sum_{k=1}^{2q-2}\sum_{i=3k+1}^{4q+j-2}i).$$ ■

Lemma 2.24. Suppose $e \in A_u$. Then

$$W_{e0}(e,G)_1 = 2\left(\left({}_RW_{e0}(e,G)_1\right) + \left({}_{R'}W_{e0}(e,G)_1\right)\right) - (t_{11}) + (t_{21}) \text{ where } t_{11} = \sum_{i=2p}^{2p+2q-1} i \text{ and}$$

$$t_{21} = \left(\sum_{i=1}^{q-1}(8i+1)\right) + 4q .$$

■

Lemma 2.25. Suppose $e \in A_d$. Then

$$W_{e0}\left(e_{A_d},G\right)_1 = W_{e0}\left(e_{A_u},G\right)_1 + (u_1) - (d_1) \quad \text{where} \quad u_1 = 2\left(1 + \sum_{i=1}^{\frac{p-1}{2}}(8i+1)\right) + \left(2 + (2\sum_{i=2}^{2p} i) + (2p+1)\right) \quad \text{and}$$

$$d_1 = (4q) + 2\left(\begin{array}{l} (\sum_{i=4q-2}^{6q-5} i) + ((\frac{p-1}{2} - q + 2)(6q-4) + \sum_{i=0}^{\frac{p-1}{2}-q+1} 4i) + ((\frac{p-1}{2} - q + 1)(6q-3) + \sum_{i=0}^{\frac{p-1}{2}-q} 4i) \\ +(\sum_{i=2q-1}^{3q-2}(2i+1)) + (\sum_{i=4q-1}^{2q+2p-2} i) + (\sum_{i=2q}^{3q-2}(2i+1)) \end{array}\right) + (2q+2p-1)$$

■

Lemma 2.26. Suppose $e \in B_u$. Then there are two regions R *and* R', such that

$$_RW_{e_0}(e,G)_1 = 2\left(\sum_{j=0}^{q-2}((\frac{p-1}{2} - j + 1)(6j) + \sum_{i=0}^{\frac{p-1}{2}-j} 4i)\right) - (\sum_{j=0}^{q-2} 6j) + (\sum_{j=0}^{q-2}\sum_{i=3j}^{p+j-1}(2i+1))$$

$$+((\frac{p-1}{2} - q + 2)(6q-6) + \sum_{i=0}^{\frac{p-1}{2}-q+1} 4i)) + ((\frac{p-1}{2} - q + 2)(6q-5) + \sum_{i=0}^{\frac{p-1}{2}-q+1} 4i)) + (1 + \sum_{k=0}^{2q-1}\sum_{i=3k-2}^{2p+k-1} i)$$

$$_{R'}W_{e_0}(e,G)_1 = 2\left(\sum_{j=1}^{q-1}((q-j)(6j+1) + \sum_{i=0}^{q-j-1} 4i)\right) + (\sum_{j=1}^{q-2}\sum_{i=3j}^{2q+j-3}(2i+2)) + (\sum_{j=1}^{q-1}(6j-2)) + (\sum_{k=1}^{2q-3}\sum_{i=3k+2}^{4q+j-3} i)$$

■

Lemma 2.27. Suppose $e \in B_u$. Then

$$W_{e0}(e,G)_1 = 2\left(\left({}_{R}W_{e0}(e,G)_1\right)+\left({}_{R'}W_{e0}(e,G)_1\right)\right)+(t_{12})+(t_{22})$$

where $t_{12}=\sum_{i=2p}^{2p+2q-1} i$ and $t_{22}=2+(\sum_{i=1}^{q-1}(8i+3))$. ■

Lemma 2.28. Suppose $e\in B_d$. Then

$$W_{e0}\left(e_{B_d},G\right)_1 = W_{e0}\left(e_{B_u},G\right)_1 + (u_2)-(d_2) \text{ where } u_2=\left(2\sum_{i=1}^{\frac{p-1}{2}}(8i-3)+(2p)\right)+\left(4+2\left(3+\sum_{i=3}^{2p} i\right)+(2p+1)\right)$$

and

$$d_2=(4q-2)+2\left(\begin{array}{l}(\sum_{i=4q-3}^{6q-7} i)+((\frac{p-1}{2}-q+2)(6q-6)+\sum_{i=0}^{\frac{p-1}{2}-q+1}4i)+((\frac{p-1}{2}-q+2)(6q-5)+\sum_{i=0}^{\frac{p-1}{2}-q+1}4i)+\\(2\sum_{i=2q-1}^{3q-3}(2i+1))+(\sum_{i=4q-2}^{2q+2p-2} i))\end{array}\right)+(2q+2p-1)$$

■

Remark 2.29. In Lemmas 2.8, 2.9, 2.10, and 2.11 we substituted only m_i's by m_i''s where $1\le i\le 4$ that have been denoted as follows:

$$m_1' = 4+2\left((\sum_{i=1}^{2p-1} i)+(\sum_{i=3}^{2p} i)+3+(\sum_{i=1}^{\frac{p-1}{2}}(8i-3))+(2p)\right)+(2p)+(2p+1)$$

$$m_2' = 2+2\left((\sum_{i=1}^{2p-1} i)+(\sum_{i=2}^{2p} i)+(1+\sum_{i=1}^{\frac{p-1}{2}}(8i+1))\right)+(2p)+(2p+1)$$

$$m_3' = 2+2\left((\sum_{i=1}^{2p-1} i)+(\sum_{i=2}^{2p} i)+(1+\sum_{i=1}^{\frac{p-1}{2}}(8i+1))\right)+(2p)+(2p+1)$$

$$m_4' = 4+2\left((\sum_{i=1}^{2p-1} i)+(\sum_{i=3}^{2p} i)+(3+(\sum_{i=1}^{\frac{p-1}{2}}(8i-3))+2p)\right)+(2p)+(2p+1)$$

Lemma 2.30. Let $e \in C_u$. Then there are 4 regions for $e \in C_u$ in $T(p,q)$ that satisfy the following relations:

$$_{R_1}W_{e_0}(e,G)_1 = 2\left(\sum_{j=0}^{q-2}((\frac{p-1}{2}-j+1)(6j+1)+\sum_{i=0}^{\frac{p-1}{2}-j}4i)\right)-(2+\sum_{j=1}^{q-2}(6j+1))+(\sum_{j=0}^{q-2}\sum_{i=3j}^{p+j-1}(2i+2))$$

$$+((\frac{p-1}{2}-q+2)(6q-5)+\sum_{i=0}^{\frac{p-1}{2}-q+1}4i))+((\frac{p-1}{2}-q+2)(6q-4)+\sum_{i=0}^{\frac{p-1}{2}-q+1}4i))+(1+\sum_{k=0}^{2q-1}\sum_{i=3k-1}^{2p+k}i)$$

$$_{R_2}W_{e_0}(e,G)_1 = 2\left(\sum_{j=1}^{q-1}((\frac{p-1}{2}-j+2)(6j-3)+\sum_{i=0}^{\frac{p-1}{2}-j+1}4i)\right)-(\sum_{j=1}^{q-1}(6j-3))+(\sum_{j=1}^{q-1}\sum_{i=3j-1}^{p+j-1}2i)-\left(\sum_{j=1}^{q-1}((6j-3)+(4(\frac{p-1}{2}-j+1))\right)$$

$$+((\frac{p-1}{2}-q+1)(6q-3)+\sum_{i=0}^{\frac{p-1}{2}-q}4i))+((\frac{p-1}{2}-q+1)(6q-2)+\sum_{i=0}^{\frac{p-1}{2}-q}4i))+(\sum_{i=1}^{2q-1}(3i+1))+(\sum_{k=1}^{2q}\sum_{i=3k-2}^{2p+k-2}i)$$

$$_{R'}W_{e_0}(e,G)_1 = 2\left(\sum_{j=1}^{q-1}((q-j)(6j-1)+\sum_{i=0}^{q-j-1}4i)\right)+(\sum_{j=1}^{q-1}\sum_{i=3j}^{2q+j-1}2i)+(\sum_{j=1}^{q-1}(6j+2))+(\sum_{k=1}^{2q-3}\sum_{i=3k+3}^{4q+j-2}i)$$

$$_{R''}W_{e_0}(e,G)_1 = 2\left(\sum_{j=0}^{q-1}((q-j)(6j+3)+\sum_{i=0}^{q-j-1}4i)\right)+(\sum_{j=1}^{q-1}\sum_{i=3j}^{2q+j-1}(2i-2))-(\sum_{j=0}^{q-1}(6j+3))+(\sum_{k=1}^{2q-2}\sum_{i=3k+1}^{4q+j-2}i)$$

■

Lemma 2.31. Suppose $e \in C_d$. Then

$W_{e0}\left(e_{C_d},G\right)_1 = W_{e0}\left(e_{C_u},G\right)_1 + (u_3) - (d_3)$ where $u_3 = \left(\sum_{i=1}^{2p} i\right) + \left(4 + (2\sum_{i=2}^{2p} i) + (2p+1)\right)$ and

$$d_3 = ((\sum_{i=4q-2}^{6q-6} i) + ((\frac{p-1}{2}-q+2)(6q-5)+\sum_{i=0}^{\frac{p-1}{2}-q+1}4i) + ((\frac{p-1}{2}-q+2)(6q-4)+\sum_{i=0}^{\frac{p-1}{2}-q+1}4i) + (\sum_{i=2q-1}^{3q-2}2i) + (\sum_{i=4q-1}^{2q+2p-1}i) + \sum_{i=2q}^{3q-2}2i)$$

$$+((\sum_{i=4q-3}^{6q-5} i) + ((\frac{p-1}{2}-q+1)(6q-3)+\sum_{i=0}^{\frac{p-1}{2}-q}4i) + ((\frac{p-1}{2}-q+1)(6q-2)+\sum_{i=0}^{\frac{p-1}{2}-q}4i) + (2\sum_{i=2q-1}^{3q-3}(2i+1)) + (6q-2) + (\sum_{i=4q-2}^{2q+2p-2} i))$$

■

Remark 2.32. For Lemmas 2.15 and 2.16 we substituted only m_i's by m_i''s that have been denoted as follows:

$$m_{cu}' = 4+2\left((\sum_{i=1}^{2p-1} i)+(\sum_{i=2}^{2p} i)\right)+(\sum_{i=1}^{2p} i)+(2p)+(2p+1)$$

$$m_{cd}' = 4+2\left((\sum_{i=1}^{2p-1} i)+(\sum_{i=2}^{2p} i)\right)+(\sum_{i=1}^{2p} i)+(2p)+(2p+1)$$

Lemma 2.33. Let $e \in D$. Then there are 5 regions for $e \in D$ in $T(p,q)$ that satisfy the following relations:

$${}_{A}W_{e_0}(e,G)_1 = (2\sum_{i=1}^{2p} i)+(\sum_{i=1}^{p} 2i)+(2\sum_{i=1}^{\frac{p-1}{2}}(4i+1))$$

$${}_{R_1}W_{e_0}(e,G)_1 = 2\left(\sum_{j=0}^{q-2}((\frac{p-1}{2}-j+1)(6j+3)+\sum_{i=0}^{\frac{p}{2}-j} 4i)\right)-(\sum_{j=0}^{q-2}(6j+3))+(\sum_{j=0}^{q-2}\sum_{i=3j}^{p+j-2}(2i+4))+(\sum_{k=0}^{2q-2}\sum_{i=3k+1}^{2p+k} i)$$

$${}_{R_2}W_{e_0}(e,G)_1 = 2\left(\sum_{j=1}^{q-1}((\frac{p-1}{2}-j+2)(6j-4)+\sum_{i=0}^{\frac{p-1}{2}-j+1} 4i)\right)-(\sum_{j=1}^{q-1}(6j-4))+(\sum_{j=1}^{q-1}\sum_{i=3j-1}^{p+j}(2i-1))+(\sum_{k=0}^{2q-2}\sum_{i=3k}^{2p+k} i)$$

$${}_{R'}W_{e_0}(e,G)_1 = 2\left(\sum_{j=1}^{q-1}((q-j)(6j-2)+\sum_{i=0}^{q-j-1} 4i)\right)+(\sum_{j=1}^{q-1}\sum_{i=3j}^{2q+j-1}(2i-1))+(\sum_{i=1}^{q-1}(6i+1))+(\sum_{k=1}^{2q-3}\sum_{i=3k+2}^{4q+j-3} i)$$

$${}_{R''}W_{e_0}(e,G)_1 = 2\left(\sum_{j=1}^{q-1}((q-j)(6j)+\sum_{i=0}^{q-j-1} 4i)\right)+(\sum_{j=1}^{q-2}\sum_{i=3j}^{2q+j-3}(2i+1))+(\sum_{i=1}^{q-2}(6i+3))+(\sum_{k=1}^{2q-3}\sum_{i=3k+1}^{4q+j-4} i).$$ ■

Remark 2.34. In Lemma 2.19 we substituted only m_D by m_D' that have been denoted as follows:

$$m_D' = (4\sum_{i=1}^{2p} i)+(2+\sum_{i=1}^{\frac{p-1}{2}}(8i+3))+(\sum_{i=1}^{\frac{p-1}{2}}(8i+1))$$

Lemma 2.35. Suppose $e \in E$. Then

$W_{e0}(e_E,G)_1 = W_{e0}(e_D,G)_1+(u_4)-(d_4)$ where

$u_4 = (\sum_{i=1}^{\frac{p-1}{2}}(8i-1)+(2p+1))+(\sum_{i=1}^{\frac{p-1}{2}}(8i-3)+(2p))+(14+2\sum_{i=4}^{2p+1} i)$ and

$$d_4 = ((\sum_{i=4q-3}^{6q-7} i) + ((\frac{p-1}{2} - q + 2)(6q-6) + \sum_{i=0}^{\frac{p-1}{2}-q+1} 4i) + ((\frac{p-1}{2} - q + 2)(6q-5) + \sum_{i=0}^{\frac{p-1}{2}-q+1} 4i) + (\sum_{i=2q-1}^{3q-2} (2i-1)) + (\sum_{i=4q-2}^{2q+2p-2} i) + \sum_{i=2q}^{3q-2} (2i-1))$$
$$+((\sum_{i=4q-4}^{6q-8} i) + ((\frac{p-1}{2} - q + 2)(6q-7) + \sum_{i=0}^{\frac{p-1}{2}-q+1} 4i) + ((\frac{p-1}{2} - q + 2)(6q-6) + \sum_{i=0}^{\frac{p-1}{2}-q+1} 4i) + (2\sum_{i=2q-1}^{3q-3} (2i)) + (\sum_{i=4q-3}^{2q+2p-2} i))$$

■

Remark 2.36. In Lemma 2.21 we substituted only m_E by m_E' that have been denoted as follows:

$$m_E' = 2\left((2\sum_{i=1}^{2p} i) + (14 + 2\sum_{i=4}^{2p+1} i) + (\sum_{i=1}^{\frac{p-1}{2}} (8i-1) + (\sum_{i=1}^{\frac{p-1}{2}+1} (8i-3)) \right) + \left(2 + (\sum_{i=1}^{\frac{p-1}{2}} (8i+3) \right) + \left((\sum_{i=1}^{\frac{p-1}{2}} (8i+1) \right)$$

Based on the above results, we can state the following theorem:

Theorem 2.37. Let p be an odd number and $q \prec \left[\frac{p}{2}\right] + 1$. Then

$$W_{e0}(G) = 32pq^2 - \frac{46}{3}pq - 12p^3q + 24pq^4 + 10p^3 + 72p^3q^2 - 16p^2 + \frac{16}{3}pq^3 - 12p^2q^2 + 48p^2q^3 + 18p^2q$$

■

Case 2. $q = \left[\frac{p}{2}\right] + 1$

(i): p is even.

The proof of following Lemmas is the same of last Lemmas and therefore we deleted the proofs.

Lemma 2.38. Suppose $e \in A_u$, then there are two regions R *and* R', such that

$$_R W_{e_0}(e,G)_1 = 2\left(\sum_{j=0}^{q-2} ((\frac{p}{2} - j)(6j+2) + \sum_{i=0}^{\frac{p}{2}-j-1} 4i) \right) - (\sum_{j=0}^{q-2} (6j+2)) + (\sum_{j=0}^{q-2} \sum_{i=3j}^{p+j-1} (2i+3)) + (\sum_{k=0}^{p} \sum_{i=3k}^{2p+k} i)$$

$$_{R'} W_{e_0}(e,G)_1 = 2\left(\sum_{j=1}^{q-1} ((q-j)(6j) + \sum_{i=0}^{q-j-1} 4i) \right) + (\sum_{j=1}^{q-1} \sum_{i=3j}^{2q+j-1} (2i+1)) + (\sum_{j=1}^{q} (6j-3)) + (3p+3) + (\sum_{k=1}^{2q-2} \sum_{i=3k+1}^{4q+j-2} i)$$

■

Lemma 2.39. Suppose $e \in A_u$. Then

$W_{e0}(e,G)_1 = 2\left(\left({}_R W_{e0}(e,G)_1\right) + \left({}_{R'} W_{e0}(e,G)_1\right)\right) - (t_{11}) + (t_{21})$ where $t_{11} = (3p+3) + \sum_{i=2p}^{2p+2q-1} i$ and

$t_{21} = (\sum_{i=1}^{q-1}(8i+1)) + 4q$. ■

Lemma 2.40. Suppose $e \in A_d$. Then

$W_{e0}\left(e_{A_d}, G\right)_1 = W_{e0}\left(e_{A_u}, G\right)_1 + (u_1) - (d_1)$ where $u_1 = 2\left(1 + (\sum_{i=1}^{\frac{p}{2}-1}(8i+1)) + 2p\right) + \left(2 + (2\sum_{i=2}^{2p} i) + (2p+1)\right)$

and $d_1 = (4q) + 2\left((\sum_{i=4q-2}^{4q+p-3} i) + (\sum_{i=2q-1}^{3q-2}(2i+1)) + (\sum_{i=4q-1}^{4q+p-2} i) + (\sum_{i=2q}^{3q-3}(2i+1))\right) + (4q+2p-1)$. ■

Lemma 2.41. Suppose $e \in B_u$, then there are two regions R *and* R', such that

$${}_R W_{e_0}(e,G)_1 = 2\left(\sum_{j=0}^{q-2}((\frac{p}{2} - j + 1)(6j) + \sum_{i=0}^{\frac{p}{2}-j} 4i)\right) - (\sum_{j=0}^{q-2} 6j) + (\sum_{j=0}^{q-2}\sum_{i=3j}^{p+j-1}(2i+1)) + (1 + \sum_{k=0}^{p}\sum_{i=3k-2}^{2p+k-1} i)$$

$${}_{R'} W_{e_0}(e,G)_1 = 2\left(\sum_{j=1}^{q-1}((q-j)(6j+1) + \sum_{i=0}^{q-j-1} 4i)\right) + (\sum_{j=1}^{q-2}\sum_{i=3j}^{2q+j-3}(2i+2)) + (\sum_{j=1}^{q-1}(6j-2)) + (\sum_{k=1}^{2q-1}\sum_{i=3k+2}^{4q+j-3} i).$$ ■

Lemma 2.42. Suppose $e \in B_u$. Then

$$W_{e0}(e,G)_1 = 2\left(\left({}_R W_{e0}(e,G)_1\right) + \left({}_{R'} W_{e0}(e,G)_1\right)\right) + (t_{12}) + (t_{22})$$

where $t_{12} = \sum_{i=2p}^{2p+2q-1} i$ and $t_{22} = 2 + (\sum_{i=1}^{q-1}(8i+3))$. ■

Lemma 2.43. Suppose $e \in B_d$. Then

$$W_{e0}\left(e_{B_d},G\right)_1=W_{e0}\left(e_{B_u},G\right)_1+\left(u_2\right)-\left(d_2\right)$$ where $$u_2=\left(2\sum_{i=1}^{\frac{p}{2}}(8i-3)\right)+\left(4+2\left(3+\sum_{i=3}^{2p}i\right)+(2p+1)\right)$$ and

$$d_2=(4q-2)+2\left((\sum_{i=4q-3}^{4q+p-4}i)+(\sum_{i=2q}^{3q-3}(2i+1))+(\sum_{i=4q-2}^{4q+p-3}i))+(\sum_{i=2q}^{3q-2}(2i+1))\right)+(4q+p-3)\ .$$ ■

Lemma 2.44. For the set A_u, we have:

$$W_{e0}\left(A_u\right)=\frac{1}{2}\left(\underset{G_2:q=\left[\frac{p}{2}\right]+1}{pW_{e0}\left(e_{A_u},G_2\right)_1}+(\underset{G_1,G_2:q\le\left[\frac{p}{2}\right]}{\sum_{j=2}^{q-1}p\left(WAu_j\right)})+\underset{G_1:q=\left[\frac{p}{2}\right]+1}{pW_{e0}\left(e_{B_d},G_1\right)_1}+\underset{G:q=\left[\frac{p}{2}\right]+1}{pW_{e0}\left(e_{A_u},G\right)_1}\right)$$

■

Lemma 2.45. For the set A_d, we have:

$$W_{e0}\left(A_d\right)=\frac{1}{2}\left(\underset{G_2:q=\left[\frac{p}{2}\right]+1}{pW_{e0}\left(e_{A_d},G_2\right)_1}+(\underset{G_1,G_2:q\le\left[\frac{p}{2}\right]}{\sum_{j=2}^{q-1}p\left(WAd_j\right)})+\underset{G_1:q=\left[\frac{p}{2}\right]+1}{pW_{e0}\left(e_{B_u},G_1\right)_1}\right)$$

■

Lemma 2.46. For the set B_u, we have:

$$W_{e0}\left(B_u\right)=\frac{1}{2}\left(\underset{G_2:q=\left[\frac{p}{2}\right]+1}{pW_{e0}\left(e_{B_u},G_2\right)_1}+(\underset{G_1,G_2:q\le\left[\frac{p}{2}\right]}{\sum_{j=2}^{q-1}p\left(WBu_j\right)})+\underset{G_1:q=\left[\frac{p}{2}\right]+1}{pW_{e0}\left(e_{A_d},G_1\right)_1}+\underset{G:q=\left[\frac{p}{2}\right]+1}{pW_{e0}\left(e_{B_u},G\right)_1}\right)$$

■

Lemma 2.47. For the set B_d, we have:

$$W_{e0}\left(B_d\right)=\frac{1}{2}\left(\underset{G_2:q=\left[\frac{p}{2}\right]+1}{pW_{e0}\left(e_{B_d},G_2\right)_1}+\underset{G_1,G_2:q\le\left[\frac{p}{2}\right]}{(\sum_{j=2}^{q-1}p\left(WBd_j\right))}+\underset{G_1:q=\left[\frac{p}{2}\right]+1}{pW_{e0}\left(e_{A_u},G_1\right)_1}\right)$$

■

Lemma 2.48. Let $e\in C_u$. Then there are 4 regions for $e\in C_u$ in $T(p,q)$ that satisfy the following relations:

$$_{R_1}W_{e_0}\left(e,G\right)_1=2\left(\sum_{j=0}^{q-2}((\frac{p}{2}-j+1)(6j+1)+\sum_{i=0}^{\frac{p}{2}-j}4i)\right)-(2+\sum_{j=0}^{q-2}(6j+1))+(\sum_{j=0}^{q-2}\sum_{i=3j}^{p+j-1}(2i+2))+(1+(\sum_{k=0}^{p}\sum_{i=3k-1}^{2p+k-1}i)+(\sum_{i=p}^{3q-3}2i))$$

$$_{R_2}W_{e_0}\left(e,G\right)_1=2\left(\sum_{j=1}^{q-1}((\frac{p}{2}-j+1)(6j-3)+\sum_{i=0}^{\frac{p}{2}-j}4i)\right)-(\sum_{j=1}^{q-1}(6j-3))+(\sum_{j=1}^{q-1}\sum_{i=3j-1}^{p+j-1}2i)+(\sum_{i=1}^{p-1}(3i+1))+(\sum_{k=1}^{p}\sum_{i=3k-2}^{2p+k-2}i)$$

$$_{R'}W_{e_0}\left(e,G\right)_1=2\left(\sum_{j=1}^{q-1}((q-j)(6j-1)+\sum_{i=0}^{q-j-1}4i)\right)+(\sum_{j=1}^{q-1}\sum_{i=3j}^{2q+j-1}2i)+(\sum_{j=1}^{q-1}(6j+2))+(\sum_{k=1}^{2q-3}\sum_{i=3k+3}^{4q+j-2}i)+(3p+1)$$

$$_{R''}W_{e_0}\left(e,G\right)_1=2\left(\sum_{j=0}^{q-2}((q-j)(6j+3)+\sum_{i=0}^{q-j-1}4i)\right)+(\sum_{j=1}^{q-1}\sum_{i=3j}^{2q+j-1}(2i-2))-(\sum_{j=0}^{q-2}(6j+3))+(\sum_{k=0}^{2q-3}\sum_{i=3k+1}^{4q+j-2}i)\,.$$

■

Lemma 2.49. Suppose $e\in C_d$. Then

$W_{e0}\left(e_{C_d},G\right)_1=W_{e0}\left(e_{C_u},G\right)_1+\left(u_3\right)-\left(d_3\right)$ where $u_3=\left(\sum_{i=1}^{2p}i\right)+\left(4+(2\sum_{i=2}^{2p}i)+(2p+1)\right)$ and

$$d_3=((\sum_{i=4q-2}^{4q+p-3}i)+(\sum_{i=2q-1}^{3q-3}2i)+(\sum_{i=4q-1}^{4q+p-2}i)+\sum_{i=2q}^{3q-2}2i)+((\sum_{i=4q-3}^{4q+p-4}i)+(2\sum_{i=2q-1}^{3q-3}(2i+1))+(\sum_{i=4q-2}^{4q+p-3}i))\,.$$ ■

Lemma 2.50. For the set C_u, we have:

$$W_{e0}\left(C_u\right)=\frac{1}{2}\left(\underset{G_2:q=\left[\frac{p}{2}\right]+1}{2pW_{e0}\left(e_{C_u},G_2\right)_1}+\left(\underset{G_1,G_2:q\le\left[\frac{p}{2}\right]}{\sum_{j=2}^{q-1}2p\left(WCu_j\right)}\right)+\underset{G_1:q=\left[\frac{p}{2}\right]+1}{2pW_{e0}\left(e_{C_d},G_1\right)_1}+\underset{G:q=\left[\frac{p}{2}\right]+1}{2pW_{e0}\left(e_{C_u},G\right)_1}\right)$$

■

Lemma 2.51. For the set C_d, we have:

$$W_{e0}\left(C_d\right)=\frac{1}{2}\left(\underset{G_2:q=\left[\frac{p}{2}\right]+1}{2pW_{e0}\left(e_{C_d},G_2\right)_1}+\left(\underset{G_1,G_2:q\le\left[\frac{p}{2}\right]}{\sum_{j=2}^{q-1}2p\left(WCd_j\right)}\right)+\underset{G_1:q=\left[\frac{p}{2}\right]+1}{2pW_{e0}\left(e_{C_u},G_1\right)_1}\right)$$

■

Lemma 2.52. Let $e\in D$. There are 5 regions for $e\in D$ in $T(p,q)$ that satisfy the following relations:

$${}_AW_{e_0}\left(e,G\right)_1=\left(2\sum_{i=1}^{2p}i\right)+\left(\sum_{i=1}^{p}2i\right)+\left(\left(2\sum_{i=1}^{\frac{p}{2}-1}(4i+1)\right)+(2p+1)\right)$$

$${}_{R_1}W_{e_0}\left(e,G\right)_1=2\left(\sum_{j=0}^{q-2}\left(\left(\frac{p}{2}-j\right)(6j+3)+\sum_{i=0}^{\frac{p}{2}-j-1}4i\right)\right)-\left(\sum_{j=0}^{q-2}(6j+3)\right)+\left(\sum_{j=0}^{q-2}\sum_{i=3j}^{p+j-1}(2i+4)\right)+\left(\sum_{k=0}^{2q-3}\sum_{i=3k+1}^{2p+k}i\right)$$

$${}_{R_2}W_{e_0}\left(e,G\right)_1=2\left(\sum_{j=1}^{q-1}\left(\left(\frac{p}{2}-j+1\right)(6j-4)+\sum_{i=0}^{\frac{p}{2}-j}4i\right)\right)-\left(\sum_{j=1}^{q-1}(6j-4)\right)+\left(\sum_{j=1}^{q-1}\sum_{i=3j-1}^{p+j}(2i-1)\right)+\left(\sum_{k=0}^{2q-2}\sum_{i=3k}^{2p+k}i\right)$$

$${}_{R'}W_{e_0}\left(e,G\right)_1=2\left(\sum_{j=1}^{q-1}\left((q-j)(6j-2)+\sum_{i=0}^{q-j-1}4i\right)\right)+\left(\sum_{j=1}^{q-1}\sum_{i=3j}^{2q+j-1}(2i-1)\right)+\left(\sum_{i=1}^{q-1}(6i+1)\right)+\left(\sum_{k=1}^{2q-3}\sum_{i=3k+2}^{4q+j-3}i\right)$$

$${}_{R''}W_{e_0}\left(e,G\right)_1=2\left(\sum_{j=1}^{q-1}\left((q-j)(6j)+\sum_{i=0}^{q-j-1}4i\right)\right)+\left(\sum_{j=1}^{q-2}\sum_{i=3j}^{2q+j-3}(2i+1)\right)+\left(\sum_{i=1}^{q-2}(6i+3)\right)+\left(\sum_{k=1}^{2q-3}\sum_{i=3k+1}^{4q+j-4}i\right).$$ ■

Lemma 2.53. For the set D, we have:

$$W_{e0}(D)=\frac{1}{2}\left(\begin{array}{c}\underset{G_2:q=\left[\frac{p}{2}\right]+1}{2pW_{e0}(e_D,G_2)_1}+(\underset{G_1,G_2:q\le\left[\frac{p}{2}\right]}{\sum_{j=2}^{q-1}2p(WD_j)})+\underset{G_1:q=\left[\frac{p}{2}\right]+1}{2pW_{e0}(e_D,G_1)_1}\end{array}\right)$$

■

Lemma 2.54. Suppose $e\in E$. Then

$W_{e0}(e_E,G)_1=W_{e0}(e_D,G)_1+(u_4)-(d_4)$ where $u_4=(\sum_{i=1}^{\frac{p}{2}}(8i-1))+(\sum_{i=1}^{\frac{p}{2}}(8i-3))+(14+2\sum_{i=4}^{2p+1}i)$ and

$d_4=((\sum_{i=4q-3}^{4q+p-4}i)+(\sum_{i=2q-1}^{3q-3}(2i-1))+(\sum_{i=4q-2}^{4q+p-3}i)+\sum_{i=2q}^{3q-2}(2i-1))+((\sum_{i=4q-4}^{4q+p-5}i)+(2\sum_{i=2q-1}^{3q-3}(2i))+(\sum_{i=4q-3}^{4q+p-4}i))$. ■

Lemma 2.55. For the set E, we have:

$$W_{e0}(E)=\frac{1}{2}\left(\begin{array}{c}\underset{G_2:q=\left[\frac{p}{2}\right]+1}{2pW_{e0}(e_E,G_2)_1}+(\underset{G_1,G_2:q\le\left[\frac{p}{2}\right]}{\sum_{j=2}^{q-1}2p(WE_j)})\end{array}\right)$$

■

By the above Lemmas, we have the following theorem:

Theorem 2.56. Let p be an even number and $q=\left[\frac{p}{2}\right]+1$. Then

$$W_{e0}(G)=\frac{465}{2}p+\frac{100}{3}p^4+72p^3q^2+48p^2q^3+24pq^4+\frac{680}{3}pq^3-162pq^2+20p^2q+\frac{373}{6}p^2+74p^3-100p^2q^2-\frac{1082}{3}pq-100p^3q$$

■

(ii): p is odd.

Lemma 2.57. Suppose $e\in A_u$, then there are two region R *and* R', such that

$$ {}_{R}W_{e_0}(e,G)_1 = 2\left(\sum_{j=0}^{q-1}\left(\left(\frac{p-1}{2}-j+1\right)(6j+2)+\sum_{i=0}^{\frac{p-1}{2}-j}4i\right)\right)-\left(\sum_{j=0}^{q-1}(6j+2)\right)+\left(\sum_{j=0}^{q-2}\sum_{i=3j}^{p+j-2}(2i+3)\right)+\left(\sum_{k=0}^{p}\sum_{i=3k}^{2p+k}i\right) $$

$$ {}_{R'}W_{e_0}(e,G)_1 = 2\left(\sum_{j=1}^{q-1}\left((q-j)(6j)+\sum_{i=0}^{q-j-1}4i\right)\right)+\left(\sum_{j=1}^{q-1}\sum_{i=3j}^{2q+j-1}(2i+1)\right)+\left(\sum_{j=1}^{q}(6j-3)\right)+\left(\sum_{k=1}^{2q-2}\sum_{i=3k+1}^{4q+j-2}i\right). \quad \blacksquare $$

Lemma 2.58. Suppose $e \in A_u$. Then

$W_{e0}(e,G)_1 = 2\left(\left({}_{R}W_{e0}(e,G)_1\right)+\left({}_{R'}W_{e0}(e,G)_1\right)\right)-(t_{11})+(t_{21})$ where $t_{11} = \sum_{i=2p}^{2p+2q-1} i$ and

$$ t_{21} = \left(\sum_{i=1}^{q-1}(8i+1)\right)+4q. \quad \blacksquare $$

Lemma 2.59. Suppose $e \in A_d$. Then

$W_{e0}\left(e_{A_d},G\right)_1 = W_{e0}\left(e_{A_u},G\right)_1+(u_1)-(d_1)$ where $u_1 = 2\left(1+\sum_{i=1}^{\frac{p-1}{2}}(8i+1)\right)+\left(2+\left(2\sum_{i=2}^{2p}i\right)+(2p+1)\right)$ and

$$ d_1 = (4q)+2\left(\left(\sum_{i=4q-2}^{4q+p-3}i\right)+\left(\sum_{i=2q-1}^{3q-3}(2i+1)\right)+\left(\sum_{i=4q-1}^{4q+p-2}i\right)+\left(\sum_{i=2q}^{3q-2}(2i+1)\right)\right)+(4q+p-2). \quad \blacksquare $$

Lemma 2.60. Suppose $e \in B_u$, then there are two regions R *and* R', such that

$$ {}_{R}W_{e_0}(e,G)_1 = 2\left(\sum_{j=0}^{q-1}\left(\left(\frac{p-1}{2}-j+1\right)(6j)+\sum_{i=0}^{\frac{p-1}{2}-j}4i\right)\right)-\left(\sum_{j=0}^{q-1}6j\right)+\left(\sum_{j=0}^{q-1}\sum_{i=3j}^{p+j}(2i+1)\right)-(2p+2q-1)+\left(1+\sum_{k=1}^{p}\sum_{i=3k-2}^{2p+k-1}i\right) $$

$$ {}_{R'}W_{e_0}(e,G)_1 = 2\left(\sum_{j=1}^{q-1}\left((q-j)(6j+1)+\sum_{i=0}^{q-j-1}4i\right)\right)+\left(\sum_{j=1}^{q-2}\sum_{i=3j}^{2q+j-3}(2i+2)\right)+\left(\sum_{j=1}^{q-2}(6j-2)\right)+\left(\sum_{k=1}^{2q-3}\sum_{i=3k+2}^{4q+j-3}i\right). \quad \blacksquare $$

Lemma 2.61. Suppose $e \in B_u$. Then

$W_{e0}(e,G)_1 = 2\left(\left({}_{R}W_{e0}(e,G)_1\right)+\left({}_{R'}W_{e0}(e,G)_1\right)\right)+(t_{12})+(t_{22})$ where $t_{12} = \sum_{i=2p}^{2p+2q-1} i$ and

$t_{22} = 2 + (\sum_{i=1}^{q-1}(8i+3))$. ■

Lemma 2.62. Suppose $e \in B_d$. Then

$W_{e0}\left(e_{B_d},G\right)_1 = W_{e0}\left(e_{B_u},G\right)_1 + (u_2) - (d_2)$ where $u_2 = \left(2\sum_{i=1}^{\frac{p-1}{2}}(8i-3)+(2p)\right)+\left(4+2\left(3+\sum_{i=3}^{2p} i\right)+(2p+1)\right)$

and $d_2 = (4q-2) + 2\left((\sum_{i=4q-3}^{4q+p-4} i)+(2\sum_{i=2q-1}^{3q-3}(2i+1))+(\sum_{i=4q-2}^{4q+p-3} i)\right)+(4q+p-2)$. ■

Lemma 2.63. Let $e \in C_u$. Then there are 4 regions for $e \in C_u$ in $T(p,q)$ that satisfy the following relations:

$${}_{R_1}W_{e_0}(e,G)_1 = 2\left(\sum_{j=0}^{q-1}((\frac{p-1}{2}-j+1)(6j+1)+\sum_{i=0}^{\frac{p-1}{2}-j} 4i)\right)-(2+\sum_{j=1}^{q-1}(6j+1))+(\sum_{j=0}^{q-1}\sum_{i=3j}^{p+j-1}(2i+2))+(1+\sum_{k=0}^{p}\sum_{i=3k-1}^{2p+k} i)$$

$${}_{R_2}W_{e_0}(e,G)_1 = 2\left(\sum_{j=1}^{q-1}((\frac{p-1}{2}-j+2)(6j-3)+\sum_{i=0}^{\frac{p-1}{2}-j+1} 4i)\right)-(\sum_{j=1}^{q-1}(6j-3))+(\sum_{j=1}^{q-1}\sum_{i=3j-1}^{p+j-1} 2i)-\left(\sum_{j=1}^{q-1}((6j-3)+(4(\frac{p-1}{2}-j+1))\right)+(\sum_{i=1}^{p-1}(3i+1))+(\sum_{k=1}^{p}\sum_{i=3k-2}^{2p+k-2} i)$$

$${}_{R'}W_{e_0}(e,G)_1 = 2\left(\sum_{j=1}^{q-1}((q-j)(6j-1)+\sum_{i=0}^{q-j-1} 4i)\right)+(\sum_{j=1}^{q-1}\sum_{i=3j}^{2q+j-1} 2i)+(\sum_{j=1}^{q-1}(6j+2))+(\sum_{k=1}^{2q-3}\sum_{i=3k+3}^{4q+j-2} i)$$

$${}_{R''}W_{e_0}(e,G)_1 = 2\left(\sum_{j=0}^{q-1}((q-j)(6j+3)+\sum_{i=0}^{q-j-1} 4i)\right)+(\sum_{j=1}^{q-1}\sum_{i=3j}^{2q+j-1}(2i-2))-(\sum_{j=0}^{q-1}(6j+3))+(\sum_{k=0}^{2q-2}\sum_{i=3k+1}^{4q+j-2} i).$$ ■

Lemma 2.64. Suppose $e \in C_d$. Then

$W_{e0}\left(e_{C_d},G\right)_1 = W_{e0}\left(e_{C_u},G\right)_1 + (u_3) - (d_3)$ where $u_3 = \left(\sum_{i=1}^{2p} i\right) + \left(4 + (2\sum_{i=2}^{2p} i) + (2p+1)\right)$ and

$$d_3 = ((\sum_{i=4q-2}^{4q+p-3} i) + (\sum_{i=2q-1}^{3q-2} 2i) + (\sum_{i=4q-1}^{4q+p-2} i) + \sum_{i=2q}^{3q-2} 2i) + ((\sum_{i=4q-3}^{4q+p-4} i) + (\sum_{i=2q-1}^{3q-3} (2i+1)) + (\sum_{i=4q-2}^{4q+p-3} i) + (\sum_{i=2q-1}^{3q-2} (2i+1)) .\blacksquare$$

Lemma 2.65. Let $e \in D$. Then there are 5 regions for $e \in D$ in $T(p,q)$ that satisfy the following relations:

$${}_AW_{e_0}(e,G)_1 = (2\sum_{i=1}^{2p} i) + (\sum_{i=1}^{p} 2i) + (2\sum_{i=1}^{\frac{p-1}{2}} (4i+1))$$

$${}_{R_1}W_{e_0}(e,G)_1 = 2\left(\sum_{j=0}^{q-2} ((\frac{p-1}{2} - j + 1)(6j+3) + \sum_{i=0}^{\frac{p}{2}-j} 4i)\right) - (\sum_{j=0}^{q-2} (6j+3)) + (\sum_{j=0}^{q-2} \sum_{i=3j}^{p+j-2} (2i+4)) + (\sum_{k=0}^{2q-1} \sum_{i=3k+1}^{2p+k} i)$$

$${}_{R_2}W_{e_0}(e,G)_1 = 2\left(\sum_{j=1}^{q-1} ((\frac{p-1}{2} - j + 1)(6j-4) + \sum_{i=0}^{\frac{p-1}{2}-j} 4i)\right) - (\sum_{j=1}^{q-1} (6j-4)) + (\sum_{j=1}^{q-1} \sum_{i=3j-1}^{p+j} (2i-1)) + (\sum_{k=0}^{2q-2} \sum_{i=3k}^{2p+k} i)$$

$${}_{R'}W_{e_0}(e,G)_1 = 2\left(\sum_{j=1}^{q-1} ((q-j)(6j-2) + \sum_{i=0}^{q-j-1} 4i)\right) + (\sum_{j=1}^{q-1} \sum_{i=3j}^{2q+j-1} (2i-1)) + (\sum_{i=1}^{q-1} (6i+1)) + (\sum_{k=1}^{2q-3} \sum_{i=3k+2}^{4q+j-3} i)$$

$${}_{R''}W_{e_0}(e,G)_1 = 2\left(\sum_{j=1}^{q-1} ((q-j)(6j) + \sum_{i=0}^{q-j-1} 4i)\right) + (\sum_{j=1}^{q-2} \sum_{i=3j}^{2q+j-3} (2i+1)) + (\sum_{i=1}^{q-2} (6i+3)) + (\sum_{k=1}^{2q-3} \sum_{i=3k+1}^{4q+j-4} i). \qquad \blacksquare$$

Lemma 2.66. Suppose $e \in E$. Then

$W_{e0}(e_E,G)_1 = W_{e0}(e_D,G)_1 + (u_4) - (d_4)$ where

$$u_4 = (\sum_{i=1}^{\frac{p-1}{2}} (8i-1) + (2p+1)) + (\sum_{i=1}^{\frac{p-1}{2}} (8i-3) + (2p)) + (14 + 2\sum_{i=4}^{2p+1} i)$$ and

$$d_4 = ((\sum_{i=4q-3}^{4q+p-4} i) + (\sum_{i=2q-1}^{3q-2} (2i-1)) + (\sum_{i=4q-2}^{4q+p-3} i) + \sum_{i=2q}^{3q-2} (2i-1)) + ((\sum_{i=4q-4}^{4q+p-5} i) + (\sum_{i=2q-1}^{3q-3} (2i)) + (\sum_{i=4q-3}^{4q+p-4} i) + (\sum_{i=2q-1}^{3q-2} (2i))$$

$\blacksquare$

By the above Lemmas, we have the following theorem:

Theorem 2.67. Let p be an odd number and $q=\left[\frac{p}{2}\right]+1$. Then

$$W_{e0}(G)=-89p+\frac{656}{3}p\,q^{3}+60p^{2}q+\frac{967}{3}pq-498p\,q^{2}+\frac{207}{2}p^{3}+\frac{31}{6}p^{2}-92p^{3}q-92p^{2}q^{2}+48p^{2}q^{3}+24pq^{4}+72p^{3}q^{2}+\frac{100}{3}p^{4}$$

■

Case 3. $q\succ\left[\frac{p}{2}\right]+1$

(i): p is even.

Lemma 2.68. Suppose $e\in A_u$, then there are two regions R *and* R', such that

$${}_{R}W_{e_0}(e,G)_1=2\left(\sum_{j=0}^{\frac{p}{2}-1}((\frac{p}{2}-j)(6j+2)+\sum_{i=0}^{\frac{p}{2}-j-1}4i)\right)-(\sum_{j=0}^{\frac{p}{2}-1}(6j+2))+(\sum_{j=0}^{\frac{p}{2}-1}\sum_{i=3j}^{p+j-1}(2i+3))+(\sum_{k=0}^{p}\sum_{i=3k}^{2p+k}i)$$

$${}_{R'}W_{e_0}(e,G)_1=2\left(\sum_{j=1}^{\frac{p}{2}}((q-j)(6j)+\sum_{i=0}^{q-j-1}4i)\right)+(\sum_{j=1}^{\frac{p}{2}}\sum_{i=3j}^{2q+j-1}(2i+1))+(\sum_{j=1}^{\frac{p}{2}}(6j-3))+(\sum_{k=1}^{p}\sum_{i=3k+1}^{4q+j-2}i).$$

■

Lemma 2.69. Suppose $e\in A_u$. Then

$$W_{e0}(e,G)_1=2\left(\left({}_{R}W_{e0}(e,G)_1\right)+\left({}_{R'}W_{e0}(e,G)_1\right)\right)-(t_{11})+(t_{21})-(t_{31})$$ where $t_{11}=\sum_{i=2p}^{3p}i$,

$t_{21}=(\sum_{i=1}^{q-1}(8i+1))+4q$ and $t_{31}=\left((q-\frac{p}{2})(3p+3)+\sum_{i=0}^{q-\frac{p}{2}-1}4i\right)+\left((q-\frac{p}{2}-1)(3p+4)+\sum_{i=0}^{q-\frac{p}{2}-2}4i\right).$

■

Lemma 2.70. Suppose $e\in A_d$. Then

$$W_{e0}\left(e_{A_d},G\right)_1 = W_{e0}\left(e_{A_u},G\right)_1 + (u_1) - (d_1) \text{ where } u_1 = 2\left(1 + (\sum_{i=1}^{\frac{p}{2}-1}(8i+1)) + 2p\right) + \left(2 + (2\sum_{i=2}^{2p} i) + (2p+1)\right)$$

and $d_1 = (4q) + 2\left((\sum_{i=4q-2}^{4q+p-3} i) + (\sum_{i=2q-1}^{2q+\frac{p}{2}-2}(2i+1)) + (\sum_{i=4q-1}^{4q+p-2} i) + (\sum_{i=2q}^{2q+\frac{p}{2}-2}(2i+1))\right) + (4q+p-1)$. ■

Lemma 2.71. Suppose $e \in B_u$, then there are two regions R *and* R', such that

$$_R W_{e_0}(e,G)_1 = 2\left(\sum_{j=0}^{\frac{p}{2}}((\frac{p}{2} - j + 1)(6j) + \sum_{i=0}^{\frac{p}{2}-j} 4i)\right) - (\sum_{j=0}^{\frac{p}{2}} 6j) + (\sum_{j=0}^{\frac{p}{2}-1}\sum_{i=3j}^{p+j-1}(2i+1)) + (1 + \sum_{k=0}^{p}\sum_{i=3k-2}^{2p+k-1} i)$$

$$_{R'} W_{e_0}(e,G)_1 = 2\left(\sum_{j=1}^{\frac{p}{2}}((q-j)(6j+1) + \sum_{i=0}^{q-j-1} 4i)\right) + (\sum_{j=1}^{\frac{p}{2}}\sum_{i=3j}^{2q+j-3}(2i+2)) + (\sum_{j=1}^{\frac{p}{2}}(6j-2)) + (\sum_{k=1}^{p}\sum_{i=3k+2}^{4q+j-3} i).$$ ■

Lemma 2.72. Suppose $e \in B_u$. Then

$W_{e0}(e,G)_1 = 2\left(\left(_R W_{e0}(e,G)_1\right) + \left(_{R'} W_{e0}(e,G)_1\right)\right) + (t_{12}) - (t_{22}) + (t_{32})$ where

$$t_{12} = \sum_{i=2p}^{3p+1} i \ , \ t_{22} = \left((q - \frac{p}{2} - 1)(3p+4) + \sum_{i=0}^{q-\frac{p}{2}-2} 4i\right) + \left((q - \frac{p}{2} - 1)(3p+5) + \sum_{i=0}^{q-\frac{p}{2}-2} 4i\right)$$

and $t_{32} = 2 + (\sum_{i=1}^{q-1}(8i+3))$. ■

Lemma 2.73. Suppose $e \in B_d$. Then

$W_{e0}\left(e_{B_d},G\right)_1 = W_{e0}\left(e_{B_u},G\right)_1 + (u_2) - (d_2)$ where $u_2 = \left(2\sum_{i=1}^{\frac{p}{2}}(8i-3)\right) + \left(4+2\left(3+\sum_{i=3}^{2p} i\right)+(2p+1)\right)$ and

$$d_2 = (4q-2) + 2\left((\sum_{i=4q-3}^{4q+p-4} i) + (\sum_{i=2q}^{2q+\frac{p}{2}-2} (2i+1)) + (\sum_{i=4q-2}^{4q+p-3} i)) + (\sum_{i=2q}^{2q+\frac{p}{2}-1} (2i+1)) \right) + (4q+p-3)\ .$$

■

Lemma 2.74. For the set A_u, we have:

$$W_{e0}\left(A_u\right) = \frac{1}{2}\left((\sum_{\substack{j=1 \\ G_1:q\prec\left[\frac{p}{2}\right]+1}}^{\left[\frac{p}{2}\right]} p\left(W_j - m_1\right)) + p\,(W_{e0}\left(e_{B_d},G_1\right)_1 - m_1)_{G_1:q=\left[\frac{p}{2}\right]+1} + (\sum_{\substack{j=\left[\frac{p}{2}\right]+2 \\ G_1:q\succ\left[\frac{p}{2}\right]+1}}^{q} p\left(W_j - m_1\right)) \right)$$

$$+\frac{1}{2}\left((\sum_{\substack{j=1 \\ G_2:q\succ\left[\frac{p}{2}\right]+1}}^{q-\left[\frac{p}{2}\right]-1} p\left(W_j\right)) + p\,(W_{e0}\left(e_{A_u},G_2\right)_1)_{G_2:q=\left[\frac{p}{2}\right]+1} + (\sum_{\substack{j=q-\left[\frac{p}{2}\right]+1 \\ G_2:q\prec\left[\frac{p}{2}\right]+1}}^{q} p\left(W_j\right)) \right) + \frac{1}{2} p W_{e0}\left(e_{A_u},G\right)_{1\ G:q\succ\left[\frac{p}{2}\right]+1}$$

■

Lemma 2.75. For the set A_d, we have:

$$W_{e0}\left(A_d\right)=\frac{1}{2}\left(\begin{array}{l}(\sum\limits_{\substack{j=1\\ G_1:q\prec\left[\frac{p}{2}\right]+1}}^{\left[\frac{p}{2}\right]} p\left(W_j-m_2\right))+\underset{G_1:q=\left[\frac{p}{2}\right]+1}{p\,(W_{e0}\left(e_{B_u},G_1\right)_1-m_2)}+(\sum\limits_{\substack{j=\left[\frac{p}{2}\right]+2\\ G_1:q\succ\left[\frac{p}{2}\right]+1}}^{q-1} p\left(W_j-m_2\right))\end{array}\right)$$

$$+\frac{1}{2}\left(\begin{array}{l}(\sum\limits_{\substack{j=1\\ G_2:q\succ\left[\frac{p}{2}\right]+1}}^{q-\left[\frac{p}{2}\right]-1} p\left(W_j\right))+\underset{G_2:q=\left[\frac{p}{2}\right]+1}{p\,(W_{e0}\left(e_{A_d},G_2\right)_1)}+(\sum\limits_{\substack{j=q-\left[\frac{p}{2}\right]+1\\ G_2:q\prec\left[\frac{p}{2}\right]+1}}^{q-1} p\left(W_j\right))\end{array}\right)+\underset{G:q\succ\left[\frac{p}{2}\right]+1}{\frac{1}{2}\,p\,W_{e0}\left(e_{B_u},G\right)_1}$$

■

Lemma 2.76. For the set B_u, we have:

$$W_{e0}\left(B_u\right)=\frac{1}{2}\left(\begin{array}{l}(\sum\limits_{\substack{j=1\\ G_1:q\prec\left[\frac{p}{2}\right]+1}}^{\left[\frac{p}{2}\right]} p\left(W_j-m_3\right))+\underset{G_1:q=\left[\frac{p}{2}\right]+1}{p\,(W_{e0}\left(e_{A_d},G_1\right)_1-m_3)}+(\sum\limits_{\substack{j=\left[\frac{p}{2}\right]+2\\ G_1:q\succ\left[\frac{p}{2}\right]+1}}^{q} p\left(W_j-m_3\right))\end{array}\right)$$

$$+\frac{1}{2}\left(\begin{array}{l}(\sum\limits_{\substack{j=1\\ G_2:q\succ\left[\frac{p}{2}\right]+1}}^{q-\left[\frac{p}{2}\right]-1} p\left(W_j\right))+\underset{G_2:q=\left[\frac{p}{2}\right]+1}{p\,(W_{e0}\left(e_{B_u},G_2\right)_1)}+(\sum\limits_{\substack{j=q-\left[\frac{p}{2}\right]+1\\ G_2:q\prec\left[\frac{p}{2}\right]+1}}^{q} p\left(W_j\right))\end{array}\right)+\underset{G:q\succ\left[\frac{p}{2}\right]+1}{\frac{1}{2}\,p\,W_{e0}\left(e_{B_u},G\right)_1}$$

■

Lemma 2.77. For the set B_d, we have:

$$W_{e0}\left(B_d\right)=\frac{1}{2}\left(\Big(\sum_{\substack{j=1\\ G_1:q\prec\left[\frac{p}{2}\right]+1}}^{\left[\frac{p}{2}\right]} p\left(W_j-m_4\right)\Big)+p\,\Big(W_{e0}\left(e_{A_u},G_1\right)_1\Big|_{G_1:q=\left[\frac{p}{2}\right]+1}-m_4\Big)+\Big(\sum_{\substack{j=\left[\frac{p}{2}\right]+2\\ G_1:q\succ\left[\frac{p}{2}\right]+1}}^{q-1} p\left(W_j-m_4\right)\Big)\right)$$

$$+\frac{1}{2}\left(\Big(\sum_{\substack{j=1\\ G_2:q\succ\left[\frac{p}{2}\right]+1}}^{q-\left[\frac{p}{2}\right]-1} p\left(W_j\right)\Big)+p\,\Big(W_{e0}\left(e_{B_d},G_2\right)_1\Big|_{G_2:q=\left[\frac{p}{2}\right]+1}\Big)+\Big(\sum_{\substack{j=q-\left[\frac{p}{2}\right]+1\\ G_2:q\prec\left[\frac{p}{2}\right]+1}}^{q-1} p\left(W_j\right)\Big)\right)+\frac{1}{2}pW_{e0}\left(e_{A_u},G\right)_1\Big|_{G:q\succ\left[\frac{p}{2}\right]+1}$$

■

Lemma 2.78. Let $e\in C_u$. Then there are 4 regions for $e\in C_u$ in $T(p,q)$ that satisfy the following relations:

$${}_{R_1}W_{e_0}\left(e,G\right)_1=2\left(\sum_{j=0}^{\frac{p}{2}-1}\Big(\big(\frac{p}{2}-j+1\big)(6j+1)+\sum_{i=0}^{\frac{p}{2}-j}4i\Big)\right)-\Big(2+\sum_{j=0}^{\frac{p}{2}-1}(6j+1)\Big)+\Big(\sum_{j=0}^{\frac{p}{2}-1}\sum_{i=3j}^{p+j-1}(2i+2)\Big)+\Big(1+\Big(\sum_{k=0}^{p}\sum_{i=3k-1}^{2p+k-1}i\Big)+\Big(\sum_{i=p}^{\frac{3}{2}p}2i\Big)\Big)$$

$${}_{R_2}W_{e_0}\left(e,G\right)_1=2\left(\sum_{j=1}^{\frac{p}{2}}\Big(\big(\frac{p}{2}-j+1\big)(6j-3)+\sum_{i=0}^{\frac{p}{2}-j}4i\Big)\right)-\Big(\sum_{j=1}^{\frac{p}{2}}(6j-3)\Big)+\Big(\sum_{j=1}^{\frac{p}{2}}\sum_{i=3j-1}^{p+j-1}2i\Big)+\Big(\sum_{i=1}^{p-1}(3i+1)\Big)+\Big(\sum_{k=1}^{p}\sum_{i=3k-2}^{2p+k-2}i\Big)$$

$${}_{R'}W_{e_0}\left(e,G\right)_1=2\left(\sum_{j=1}^{\frac{p}{2}}\Big((q-j)(6j-1)+\sum_{i=0}^{q-j-1}4i\Big)\right)+\Big(\sum_{j=1}^{\frac{p}{2}}\sum_{i=3j}^{2q+j-1}2i\Big)+\Big(\sum_{j=1}^{\frac{p}{2}}(6j+2)\Big)+\Big(\sum_{k=1}^{p}\sum_{i=3k+3}^{4q+j-2}i\Big)+(3p+1)$$

$${}_{R''}W_{e_0}\left(e,G\right)_1=2\left(\sum_{j=0}^{\frac{p}{2}-1}\Big((q-j)(6j+3)+\sum_{i=0}^{q-j-1}4i\Big)\right)+\Big(\sum_{j=1}^{\frac{p}{2}}\sum_{i=3j}^{2q+j-1}(2i-2)\Big)-\Big(\sum_{j=0}^{\frac{p}{2}-1}(6j+3)\Big)+\Big(\sum_{k=0}^{p-1}\sum_{i=3k+1}^{4q+j-2}i\Big).$$ ■

Lemma 2.79. Suppose $e\in C_d$. Then

$W_{e0}\left(e_{C_d},G\right)_1=W_{e0}\left(e_{C_u},G\right)_1+\left(u_3\right)-\left(d_3\right)$ where $u_3=\left(\sum_{i=1}^{2p}i\right)+\left(4+\big(2\sum_{i=2}^{2p}i\big)+(2p+1)\right)$ and

$$d_3=((\sum_{i=4q-2}^{4q+p-3} i)+(\sum_{i=2q-1}^{2p+\frac{p}{2}-2} 2i)+(\sum_{i=4q-1}^{4q+p-2} i)+\sum_{i=2q}^{2p+\frac{p}{2}-1} 2i)+((\sum_{i=4q-3}^{4q+p-4} i)+(2\sum_{i=2q-1}^{2p+\frac{p}{2}-2}(2i+1))+(\sum_{i=4q-2}^{4q+p-3} i))\,. \qquad \blacksquare$$

Lemma 2.80. For the set $e\in C_u$, we have:

$$W_{e0}\left(C_u\right)=\frac{1}{2}\left(\underset{G_1:q\prec\left[\frac{p}{2}\right]+1}{(\sum_{j=1}^{\left[\frac{p}{2}\right]} 2p\left(W_j-m_u\right))}+\underset{G_1:q=\left[\frac{p}{2}\right]+1}{2p\,(W_{e0}\left(e_{C_d},G_1\right)_1-m_u)}+\underset{G_1:q\succ\left[\frac{p}{2}\right]+1}{(\sum_{j=\left[\frac{p}{2}\right]+2}^{q} 2p\left(W_j-m_u\right))}\right)$$

$$+\frac{1}{2}\left(\underset{G_2:q\succ\left[\frac{p}{2}\right]+1}{(\sum_{j=1}^{q-\left[\frac{p}{2}\right]-1} 2p\left(W_j\right))}+\underset{G_2:q=\left[\frac{p}{2}\right]+1}{2p\,(W_{e0}\left(e_{C_u},G_2\right)_1)}+\underset{G_2:q\prec\left[\frac{p}{2}\right]+1}{(\sum_{j=q-\left[\frac{p}{2}\right]+1}^{q} 2p\left(W_j\right))}\right)+\underset{G:q\succ\left[\frac{p}{2}\right]+1}{\frac{1}{2}2pW_{e0}\left(e_{C_u},G\right)_1}$$

$\blacksquare$

Lemma 2.81. For the set C_d, we have:

$$W_{e0}\left(C_d\right)=\frac{1}{2}\left(\underset{G_1:q\prec\left[\frac{p}{2}\right]+1}{(\sum_{j=1}^{\left[\frac{p}{2}\right]} 2p\left(W_j-m_{cd}\right))}+\underset{G_1:q=\left[\frac{p}{2}\right]+1}{2p\,(W_{e0}\left(e_{C_u},G_1\right)_1-m_{cd})}+\underset{G_1:q\succ\left[\frac{p}{2}\right]+1}{(\sum_{j=\left[\frac{p}{2}\right]+2}^{q-1} 2p\left(W_j-m_{cd}\right))}\right)$$

$$+\frac{1}{2}\left(\underset{G_2:q\succ\left[\frac{p}{2}\right]+1}{(\sum_{j=1}^{q-\left[\frac{p}{2}\right]-1} 2p\left(W_j\right))}+\underset{G_2:q=\left[\frac{p}{2}\right]+1}{2p\,(W_{e0}\left(e_{C_d},G_2\right)_1)}+\underset{G_2:q\prec\left[\frac{p}{2}\right]+1}{(\sum_{j=q-\left[\frac{p}{2}\right]+1}^{q-1} 2p\left(W_j\right))}\right)+\underset{G:q\succ\left[\frac{p}{2}\right]+1}{\frac{1}{2}2pW_{e0}\left(e_{C_u},G\right)_1}$$

$\blacksquare$

Lemma 2.82. Let $e\in D$. There are 5 regions for $e\in D$ in $T(p,q)$ that satisfy the following relations:

$$ {}_{A}W_{e_0}(e,G)_1=(2\sum_{i=1}^{2p} i)+(\sum_{i=1}^{p}2i)+((2\sum_{i=1}^{\frac{p}{2}-1}(4i+1))+(2p+1)) $$

$$ {}_{R_1}W_{e_0}(e,G)_1=2\left(\sum_{j=0}^{\frac{p}{2}-1}((\frac{p}{2}-j)(6j+3)+\sum_{i=0}^{\frac{p}{2}-j-1}4i)\right)-(\sum_{j=0}^{\frac{p}{2}-1}(6j+3))+(\sum_{j=0}^{\frac{p}{2}-1}\sum_{i=3j}^{p+j-1}(2i+4))+(\sum_{k=0}^{p-1}\sum_{i=3k+1}^{2p+k}i) $$

$$ {}_{R_2}W_{e_0}(e,G)_1=2\left(\sum_{j=1}^{\frac{p}{2}}((\frac{p}{2}-j+1)(6j-4)+\sum_{i=0}^{\frac{p}{2}-j}4i)\right)-(\sum_{j=1}^{\frac{p}{2}}(6j-4))+(\sum_{j=1}^{\frac{p}{2}}\sum_{i=3j-1}^{p+j}(2i-1))+(\sum_{k=0}^{p}\sum_{i=3k}^{2p+k}i) $$

$$ {}_{R'}W_{e_0}(e,G)_1=2\left(\sum_{j=1}^{\frac{p}{2}}((q-j)(6j-2)+\sum_{i=0}^{q-j-1}4i)\right)+(\sum_{j=1}^{\frac{p}{2}}\sum_{i=3j}^{2q+j-1}(2i-1))+(\sum_{i=1}^{\frac{p}{2}}(6i+1))+(\sum_{k=1}^{p}\sum_{i=3k+2}^{4q+j-3}i) $$

$$ {}_{R''}W_{e_0}(e,G)_1=2\left(\sum_{j=1}^{\frac{p}{2}}((q-j)(6j)+\sum_{i=0}^{q-j-1}4i)\right)+(\sum_{j=1}^{\frac{p}{2}}\sum_{i=3j}^{2q+j-3}(2i+1))+(\sum_{i=1}^{\frac{p}{2}-1}(6i+3))+(\sum_{k=1}^{p}\sum_{i=3k+1}^{4q+j-4}i). $$ ■

Lemma 2.83. For the set D, we have:

$$ W_{e0}(D)=\frac{1}{2}\left((\sum_{\substack{j=1\\ G_1:q\prec\left[\frac{p}{2}\right]+1}}^{\left[\frac{p}{2}\right]}2p\left(W_j-m_D\right))+2p\,(W_{e0}(e_D,G_1)_1-m_D)_{G_1:q=\left[\frac{p}{2}\right]+1}+(\sum_{\substack{j=\left[\frac{p}{2}\right]+2\\ G_1:q\succ\left[\frac{p}{2}\right]+1}}^{q}2p\left(W_j-m_D\right))\right) $$

$$ +\frac{1}{2}\left((\sum_{\substack{j=1\\ G_2:q\succ\left[\frac{p}{2}\right]+1}}^{q-\left[\frac{p}{2}\right]-1}2p\left(W_j\right))+2p\,(W_{e0}(e_D,G_2)_1)_{G_2:q=\left[\frac{p}{2}\right]+1}+(\sum_{\substack{j=q-\left[\frac{p}{2}\right]+1\\ G_2:q\prec\left[\frac{p}{2}\right]+1}}^{q}2p\left(W_j\right))\right) $$

■

Lemma 2.84. Suppose $e\in E$. Then

$W_{e0}(e_E,G)_1 = W_{e0}(e_D,G)_1 + (u_4) - (d_4)$ where $u_4 = (\sum_{i=1}^{\frac{p}{2}}(8i-1)) + (\sum_{i=1}^{\frac{p}{2}}(8i-3)) + (14 + 2\sum_{i=4}^{2p+1} i)$ and

$$d_4 = ((\sum_{i=4q-3}^{4q+p-4} i) + (\sum_{i=2q-1}^{2q+\frac{p}{2}-2}(2i-1)) + (\sum_{i=4q-2}^{4q+p-3} i) + \sum_{i=2q}^{2q+\frac{p}{2}-1}(2i-1)) + ((\sum_{i=4q-4}^{4q+p-5} i) + (2\sum_{i=2q-1}^{2q+\frac{p}{2}-2}(2i)) + (\sum_{i=4q-3}^{4q+p-4} i)).$$ ■

Lemma 2.85. For the set E, we have:

$$W_{e0}(E) = \frac{1}{2}\left(\underset{G_1: q \prec \left[\frac{p}{2}\right]+1}{(\sum_{j=1}^{\left[\frac{p}{2}\right]} 2p(W_j - m_E))} + \underset{G_1: q = \left[\frac{p}{2}\right]+1}{2p(W_{e0}(e_E, G_1)_1 - m_E)} + \underset{G_1: q \succ \left[\frac{p}{2}\right]+1}{(\sum_{j=\left[\frac{p}{2}\right]+2}^{q-1} 2p(W_j - m_E))} \right)$$

$$+\frac{1}{2}\left(\underset{G_2: q \succ \left[\frac{p}{2}\right]+1}{(\sum_{j=1}^{q-\left[\frac{p}{2}\right]-1} 2p(W_j))} + \underset{G_2: q = \left[\frac{p}{2}\right]+1}{2p(W_{e0}(e_E, G_2)_1)} + \underset{G_2: q \prec \left[\frac{p}{2}\right]+1}{(\sum_{j=q-\left[\frac{p}{2}\right]+1}^{q-1} 2p(W_j))} \right)$$

By the above Lemmas, we have the following theorem:

Theorem 2.86. Let p be an even number and $q \succ \left[\frac{p}{2}\right]+1$. Then

$$W_{e0}(G) = \frac{341}{2}p + 96p^2q^3 + 68p^3q + \frac{187}{3}p^2 - \frac{675}{4}p^3 - \frac{3}{2}p^5 - 195pq + 36p^3q^2$$
$$-269p\,q^2 - 189p^2q + 268pq^3 + 57p^2q^2 + 12p^4q - \frac{1141}{12}p^4$$

■

(ii): p is odd.

Lemma 2.87. Suppose $e \in A_u$, then there are two regions R *and* R', such that

$$ {}_{R}W_{e_0}\left(e,G\right)_1 = 2\left(\sum_{j=0}^{\frac{p-1}{2}}\left(\left(\frac{p-1}{2}-j+1\right)(6j+2)+\sum_{i=0}^{\frac{p-1}{2}-j}4i\right)\right)-\left(\sum_{j=0}^{\frac{p-1}{2}}(6j+2)\right)+\left(\sum_{j=0}^{\frac{p-1}{2}-1}\sum_{i=3j}^{p+j-1}(2i+3)\right)+\left(\sum_{k=0}^{p}\sum_{i=3k}^{2p+k}i\right) $$

$$ {}_{R'}W_{e_0}\left(e,G\right)_1 = 2\left(\sum_{j=1}^{\frac{p-1}{2}}\left((q-j)(6j)+\sum_{i=0}^{q-j-1}4i\right)\right)+\left(\sum_{j=1}^{\frac{p-1}{2}}\sum_{i=3j}^{2q+j-1}(2i+1)\right)+\left(\sum_{j=1}^{\frac{p+1}{2}}(6j-3)\right)+\left(\sum_{k=1}^{p}\sum_{i=3k+1}^{4q+j-2}i\right). \qquad \blacksquare $$

Lemma 2.88. Suppose $e \in A_u$. Then

$W_{e0}\left(e,G\right)_1 = 2\left(\left({}_{R}W_{e0}\left(e,G\right)_1\right)+\left({}_{R'}W_{e0}\left(e,G\right)_1\right)\right)-\left(t_{11}\right)+\left(t_{21}\right)-\left(t_{31}\right)$ where $t_{11}=\sum_{i=2p}^{3p} i$,

$t_{21}=\left(\sum_{i=1}^{q-1}(8i+1)\right)+4q$ and $t_{31}=\left(\left(q-\frac{p-1}{2}-1\right)(3p+3)+\sum_{i=0}^{q-\frac{p-1}{2}-2}4i\right)+\left(\left(q-\frac{p-1}{2}-1\right)(3p+4)+\sum_{i=0}^{q-\frac{p-1}{2}-2}4i\right)$

$\blacksquare$

Lemma 2.89. Suppose $e \in A_d$. Then

$W_{e0}\left(e_{A_d},G\right)_1 = W_{e0}\left(e_{A_u},G\right)_1+\left(u_1\right)-\left(d_1\right)$ where $u_1 = 2\left(1+\sum_{i=1}^{\frac{p-1}{2}}(8i+1)\right)+\left(2+\left(2\sum_{i=2}^{2p}i\right)+(2p+1)\right)$ and

$$ d_1 = (4q)+2\left(\left(\sum_{i=4q-2}^{4q+p-3}i\right)+\left(\sum_{i=2q-1}^{2q+\frac{p-1}{2}-2}(2i+1)\right)+\left(\sum_{i=4q-1}^{4q+p-2}i\right)+\left(\sum_{i=2q}^{2q+\frac{p-1}{2}-1}(2i+1)\right)\right)+(4q+p-2). \qquad \blacksquare $$

Lemma 2.90. Suppose $e \in B_u$, then there are two regions R *and* R', such that

$$ {}_{R}W_{e_0}\left(e,G\right)_1 = 2\left(\sum_{j=0}^{\frac{p-1}{2}}\left(\left(\frac{p-1}{2}-j+1\right)(6j)+\sum_{i=0}^{\frac{p-1}{2}-j}4i\right)\right)-\left(\sum_{j=0}^{\frac{p-1}{2}}6j\right)+\left(\sum_{j=0}^{\frac{p-1}{2}}\sum_{i=3j}^{p+j-1}(2i+1)\right)+\left(1+\sum_{k=0}^{p}\sum_{i=3k-2}^{2p+k-1}i\right) $$

$$ {}_{R'}W_{e_0}\left(e,G\right)_1 = 2\left(\sum_{j=1}^{\frac{p-1}{2}}\left((q-j)(6j+1)+\sum_{i=0}^{q-j-1}4i\right)\right)+\left(\sum_{j=1}^{\frac{p-1}{2}}\sum_{i=3j}^{2q+j-3}(2i+2)\right)+\left(\sum_{j=1}^{\frac{p+1}{2}}(6j-2)\right)+\left(\sum_{k=1}^{p}\sum_{i=3k+2}^{4q+j-3}i\right). \qquad \blacksquare $$

Lemma 2.91. Suppose $e \in B_u$. Then

$$W_{e0}(e,G)_1 = 2\left(\left({}_RW_{e0}(e,G)_1\right)+\left({}_{R'}W_{e0}(e,G)_1\right)\right)+(t_{12})-(t_{22})+(t_{32})$$

where $t_{12} = \sum_{i=2p}^{3p+1} i$, $t_{22} = \left((q-\frac{p-1}{2}-1)(3p+4)+\sum_{i=0}^{q-\frac{p-1}{2}-2} 4i\right)+\left((q-\frac{p-1}{2}-2)(3p+5)+\sum_{i=0}^{q-\frac{p-1}{2}-3} 4i\right)$

and $t_{32} = 2+(\sum_{i=1}^{q-1}(8i+3))$. ■

Lemma 2.92. Suppose $e \in B_d$. Then

$W_{e0}\left(e_{B_d},G\right)_1 = W_{e0}\left(e_{B_u},G\right)_1 + (u_2)-(d_2)$ where $u_2 = \left(2\sum_{i=1}^{\frac{p-1}{2}}(8i-3)+(2p)\right)+\left(4+2\left(3+\sum_{i=3}^{2p} i\right)+(2p+1)\right)$

and $d_2 = (4q-2)+2\left((\sum_{i=4q-3}^{4q+p-4} i)+(\sum_{i=2q}^{2q+\frac{p-1}{2}-1}(2i+1))+(\sum_{i=4q-2}^{4q+p-3} i))+(\sum_{i=2q}^{2q+\frac{p-1}{2}-1}(2i+1))\right)+(4q+p-2)$. ■

Lemma 2.93. Let $e \in C_u$. Then there are 4 regions for $e \in C_u$ in $TUC_4C_8(p,q)$ that they satisfy the following relations:

$${}_{R_1}W_{e_0}(e,G)_1 = 2\left(\sum_{j=0}^{\frac{p-1}{2}}((\frac{p-1}{2}-j+1)(6j+1)+\sum_{i=0}^{\frac{p-1}{2}-j} 4i)\right)-(2+\sum_{j=0}^{\frac{p-1}{2}}(6j+1))+(\sum_{j=0}^{\frac{p-1}{2}}\sum_{i=3j}^{p+j-1}(2i+2))+(1+(\sum_{k=0}^{p}\sum_{i=3k-1}^{2p+k-1} i))$$

$${}_{R_2}W_{e_0}(e,G)_1 = 2\left(\sum_{j=1}^{\frac{p-1}{2}}((\frac{p-1}{2}-j+1)(6j-3)+\sum_{i=0}^{\frac{p-1}{2}-j} 4i)\right)-(\sum_{j=1}^{\frac{p-1}{2}}(6j-3))-(\sum_{j=1}^{\frac{p-1}{2}}((6j-3)+4(\frac{p-1}{2}-j+1))$$

$$+(\sum_{j=1}^{\frac{p-1}{2}}\sum_{i=3j-1}^{p+j-1} 2i)+(\sum_{i=1}^{p-1}(3i+1))+(\sum_{k=1}^{p}\sum_{i=3k-2}^{2p+k-2} i)$$

$$_{R'}W_{e_0}\left(e,G\right)_1=2\left(\sum_{j=1}^{\frac{p-1}{2}}((q-j)(6j-1)+\sum_{i=0}^{q-j-1}4i)\right)+(\sum_{j=1}^{\frac{p-1}{2}}\sum_{i=3j}^{2q+j-1}2i)+(\sum_{j=1}^{\frac{p-1}{2}}(6j+2))+\left((q-\frac{p-1}{2}-1)(3p+2)+\sum_{i=0}^{q-\frac{p-1}{2}-2}4i\right)$$

$$+\left((q-\frac{p-1}{2}-1)(3p+3)+\sum_{i=0}^{q-\frac{p-1}{2}-2}4i\right)+(\sum_{k=1}^{p}\sum_{i=3k+3}^{4q+j-2}i)+(3p+1)$$

$$_{R''}W_{e_0}\left(e,G\right)_1=2\left(\sum_{j=0}^{\frac{p-1}{2}-1}((q-j)(6j+3)+\sum_{i=0}^{q-j-1}4i)\right)+(\sum_{j=1}^{\frac{p-1}{2}}\sum_{i=3j}^{2q+j-1}(2i-2))-(\sum_{j=0}^{\frac{p-1}{2}-1}(6j+3))$$

$$+\left((q-\frac{p-1}{2})(3p)+\sum_{i=0}^{q-\frac{p-1}{2}-1}4i\right)+\left((q-\frac{p-1}{2}-1)(3p+1)+\sum_{i=0}^{q-\frac{p-1}{2}-2}4i\right)+(\sum_{k=0}^{p}\sum_{i=3k+1}^{4q+j-2}i)$$

■

Lemma 2.94. Suppose $e\in C_d$. Then

$W_{e0}\left(e_{C_d},G\right)_1=W_{e0}\left(e_{C_u},G\right)_1+\left(u_3\right)-\left(d_3\right)$ where $u_3=\left(\sum_{i=1}^{2p}i\right)+\left(4+(2\sum_{i=2}^{2p}i)+(2p+1)\right)$ and

$$d_3=((\sum_{i=4q-2}^{4q+p-3}i)+(\sum_{i=2q-1}^{2q+\frac{p-1}{2}-1}2i)+(\sum_{i=4q-1}^{4q+p-2}i)+\sum_{i=2q}^{2q+\frac{p-1}{2}-1}2i)+((\sum_{i=4q-3}^{4q+p-4}i)+(\sum_{i=2q-1}^{2q+\frac{p}{2}-2}(2i+1))+(\sum_{i=4q-2}^{4q+p-3}i)+(\sum_{i=2q-1}^{2q+\frac{p-1}{2}-1}(2i+1)))$$

■

Lemma 2.95. Let $e\in D$. Then there are 5 regions for $e\in D$ in $T(p,q)$ that satisfy the following relations:

$$_{A}W_{e_0}\left(e,G\right)_1=(2\sum_{i=1}^{2p}i)+(\sum_{i=1}^{p}2i)+(2\sum_{i=1}^{\frac{p-1}{2}-1}(4i+1))$$

$$_{R_1}W_{e_0}\left(e,G\right)_1=2\left(\sum_{j=0}^{\frac{p-1}{2}}((\frac{p-1}{2}-j+1)(6j+3)+\sum_{i=0}^{\frac{p-1}{2}-j}4i)\right)-(\sum_{j=0}^{\frac{p-1}{2}}(6j+3))+(\sum_{j=0}^{\frac{p-1}{2}-1}\sum_{i=3j}^{p+j-1}(2i+4))+(\sum_{k=0}^{p-1}\sum_{i=3k+1}^{2p+k}i)$$

$$_{R_2}W_{e_0}\left(e,G\right)_1=2\left(\sum_{j=1}^{\frac{p-1}{2}+1}((\frac{p-1}{2}-j+2)(6j-4)+\sum_{i=0}^{\frac{p-1}{2}-j+1}4i)\right)-(\sum_{j=1}^{\frac{p-1}{2}+1}(6j-4))+(\sum_{j=1}^{\frac{p-1}{2}}\sum_{i=3j-1}^{p+j-1}(2i-1))+(\sum_{k=0}^{p}\sum_{i=3k}^{2p+k}i)$$

$$ {}_{R'}W_{e_0}\left(e,G\right)_1 = 2\left(\sum_{j=1}^{\frac{p-1}{2}}((q-j)(6j-2)+\sum_{i=0}^{q-j-1}4i)\right)+(\sum_{j=1}^{\frac{p-1}{2}}\sum_{i=3j}^{2q+j-1}(2i-1))+(\sum_{i=1}^{\frac{p-1}{2}}(6i+1))+\left((q-\frac{p-1}{2}-1)(3p+1)+\sum_{i=0}^{q-\frac{p-1}{2}-2}4i\right) $$

$$ +\left((q-\frac{p-1}{2}-1)(3p+2)+\sum_{i=0}^{q-\frac{p-1}{2}-2}4i\right)+(\sum_{k=1}^{p}\sum_{i=3k+2}^{4q+j-3}i) $$

$$ {}_{R'}W_{e_0}\left(e,G\right)_1 = 2\left(\sum_{j=1}^{\frac{p-1}{2}}((q-j)(6j)+\sum_{i=0}^{q-j-1}4i)\right)+(\sum_{j=1}^{\frac{p-1}{2}}\sum_{i=3j}^{2q+j-3}(2i+1))+(\sum_{i=1}^{\frac{p-1}{2}}(6i+3))+\left((q-\frac{p-1}{2}-1)(3p+3)+\sum_{i=0}^{q-\frac{p-1}{2}-2}4i\right) $$

$$ +\left((q-\frac{p-1}{2}-2)(3p+4)+\sum_{i=0}^{q-\frac{p-1}{2}-3}4i\right)+(\sum_{k=1}^{p}\sum_{i=3k+1}^{4q+j-4}i) $$

■

Lemma 2.96. Suppose $e\in E$. Then

$W_{e0}\left(e_E,G\right)_1 = W_{e0}\left(e_D,G\right)_1 + \left(u_4\right) - \left(d_4\right)$ where

$$ u_4 = (\sum_{i=1}^{\frac{p-1}{2}}(8i-1)+(2p+1))+(\sum_{i=1}^{\frac{p-1}{2}}(8i-3)+(2p))+(14+2\sum_{i=4}^{2p+1}i) $$

and

$$ d_4 = ((\sum_{i=4q-3}^{4q+p-4}i)+(\sum_{i=2q-1}^{2q+\frac{p-1}{2}-1}(2i-1))+(\sum_{i=4q-2}^{4q+p-3}i)+\sum_{i=2q}^{2q+\frac{p-1}{2}-1}(2i-1))+((\sum_{i=4q-4}^{4q+p-5}i)+(\sum_{i=2q-1}^{2q+\frac{p-1}{2}-2}(2i))+(\sum_{i=4q-3}^{4q+p-4}i)+(\sum_{i=2q-1}^{2q+\frac{p-1}{2}-1}(2i)) $$

■

By the above results, we can state the following theorem:

Theorem 2.97. Let p be an odd number and $q \succ \left[\frac{p}{2}\right]+1$. Then

$$ W_{e0}\left(G\right) = -39p - \frac{381}{4}p^4 - \frac{3}{2}p^5 + \frac{820}{3}pq^3 + 96p^2q^3 + 61p^2q^2 + \frac{101}{2}p^2 - 178p^2q $$

$$ + \frac{962}{3}pq - 575p\,q^2 + \frac{115}{4}p^3 + 12p^4q + 70p^3q + 36p^3q^2 $$

■

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