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Zeroth-order general Randić index of trees with given order and distance conditions

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Abstract

The zeroth-order general Randić index of a graph $G = (V, E)$ is defined as

$$^0R_\alpha(G) = \sum_{v \in V} (d(v))^\alpha$$

where $d(v)$ is the degree of the vertex $v \in V$ and α is a pertinently chosen real number. In this paper, we consider the trees with order n, diameter d or radius r, and extremal zeroth-order general Randić index. We obtain the trees with first three largest zeroth-order general Randić index among all the trees with order n, diameter d or radius r.

1. Introduction

Let $G = (V, E)$ be a graph with vertex set V ($\mid V \mid= n$) and edge set E. The Zeroth-order Randić index defined by Kier and Hall [1] is $^0R = \sum_{v \in V} d(v)^{-1/2}$. Pavlović [2] gave a graph with the maximum value of ^0R. The general Randić index $R_\alpha(G)$ of G is defined as

$$R_\alpha(G) = \sum_{vw \in E} (d(v)d(w))^\alpha$$

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where α is an arbitrary real number. For $\alpha = -1/2$, the general Randić index $R_\alpha(G)$ reduces to the Randić index $R_{-1/2}(G)$ ([3]). Li and Zheng [4] defined the zeroth-order general Randić index ${}^0R_\alpha(G)$ of a graph G as ${}^0R_\alpha(G) = \sum_{v \in V}(d(v))^\alpha$ for any real number α. Hu, Li, Shi, Xu ([5]) characterized the simple connected graphs with extremal zeroth-order general Randić index. In [6] the molecular (n, m)-graphs with the largest and smallest zeroth-order general Randić indices were characterized. In [7] all trees with the first three largest and smallest zeroth-order general Randić indices were determined when $\alpha \in \{k, -k, -1/k\}$, where $k \geq 2$ is an integer. Also, Zhang and Zhou [8] characterized the trees and unicyclic graphs of fixed maximum degree or fixed number of pendent vertices with extremal zeroth-order general Randić indices. For $\alpha > 1$ or $\alpha < 0$ (resp. $0 < \alpha < 1$), we characterize the trees with first three largest (resp. smallest) zeroth-order general Randić index among all the trees of order n and diameter d or radius r using the similar argument given in [8].

2. Results.

Denote by $D(G) = [d_1, d_2, \cdots, d_n]$ the degree sequence of the graph G, where d_i is the degrees of G such that $d_1 \geq d_2 \geq \cdots \geq d_n$. Given a connected graph G. For a vertex v, then the eccentricity $e(v)$ of v is defined as $e(v) = \max_{w \in V}\{d(v, w)\}$ where $d(v, w)$ is the minimum length (number of edges) of the paths connecting them. The radius $r = r(G)$ and the diameter $d = d(G)$ of G are defined as $r = \min_{v \in V}\{e(v)\}$, $d = \max_{v \in V}\{e(v)\}$, respectively. If a tree T has diameter d, then T has a path P of length d. If there is a graph G, such that $d_i \geq d_j \geq 2$, let G' be the graph obtained from G by replacing the pair (d_i, d_j) by $(d_i + 1, d_j - 1)$.

Lemma 1([6, 8]). For the two graph G and G', specified above, we have

$$(i)\,{}^0R_\alpha(G) < {}^0R_\alpha(G') \quad (\alpha > 1 \text{ or } \alpha < 0)$$

$$(ii)\,{}^0R_\alpha(G) > {}^0R_\alpha(G') \quad (0 < \alpha < 1).$$

Theorem 2. Let T be a tree with $n \geq 3$, $n-1 \geq d \geq 2$ and $\alpha > 1$ or $\alpha < 0$ (resp. $0 < \alpha < 1$).

(1) ${}^0R_\alpha(T)$ attains the largest (resp. smallest) value if and only if $D(T) = [n-d+1, 2^{d-2}, 1^{n-d+1}]$.

(2) For $n-3 \geq d \geq 3$, ${}^0R_\alpha(T)$ attains the second largest (resp. smallest) value if and only if $D(T) = [n-d, 3, 2^{d-3}, 1^{n-d+1}]$.

(3) For $d = n-4$, ${}^0R_\alpha(T)$ attains the third largest (resp. smallest) value if and only if $D(T) = [3^3, 2^{n-8}, 1^5]$, and for $n-5 \geq d \geq 3$, ${}^0R_\alpha(T)$ attains the third largest (resp. smallest) value if and only if $D(T) = [n-d-1, 4, 2^{d-3}, 1^{n-d+1}]$.

Proof.
The statement is trivial for $d = 2$. Suppose that $d \geq 3$. Fix a path P of length d on T, and $v \in V$ is a vertex with maximum degree on P. If $D(T) \neq [n-d+1, 2^{d-2}, 1^{n-d-1}]$, then there is at least one vertex w satisfying (a): $w \notin P$ and $d(w) \geq 2$, or (b): $w \neq v$, $w \in P$ and $d(w) \geq 3$. Consider the first case (a). We chose w so that w is adjacent to exactly $d(w)-1$ pendent vertices $\{w_1, w_2, \ldots, w_{d(w)-1}\}$. Let $T_w = T - ww_1 - \cdots - ww_{d(w)-1} + vw_1 + \cdots + vw_{d(w)-1}$.

Claim. ${}^0R_\alpha(T) < {}^0R_\alpha(T_w)$.

Proof.
By Lemma 1, it is easy to see that ${}^0R_\alpha(T) < {}^0R_\alpha(T_w)$ when $d(w) \leq d(v)$ in T. Suppose that $d(w) > d(v)$ in T, let $T_{v \leftrightarrow w}$ be the tree obtained from T by replacing the pair $(d(v), d(w))$ by $(d(w), d(v))$ by pendent vertices (of w) transformation. Then the degree of w is at most the degree of v in $T_{v \leftrightarrow w}$. Moreover, the tree T_w is obtained from $T_{v \leftrightarrow w}$ by the transformations (P2, line-5). Thus, by Lemma 1, we have ${}^0R_\alpha(T) < {}^0R_\alpha(T_w)$ when $d(w) > d(v)$ in T. ■

We repeat this operation while there remains a vertex satisfying (a). After that, we obtain a tree T' which has no vertex satisfying (a). Clearly, ${}^0R_\alpha(T) < {}^0R_\alpha(T')$. Next, suppose that there is at least one vertex w satisfying (b) for T' (instead for T). Let $N(w) = \{w_1, w_2, \cdots, w_{d(w)}\}$, where w_1, w_2 lies on the path P. Then we let

$T_i' = T' - ww_3 - \cdots - ww_{i+2} + vw_3 + \cdots + vw_{i+2}$ for $i = 1, 2, \ldots, d(w) - 2$. By Lemma 1, ${}^0R_\alpha(T') < {}^0R_\alpha(T_1') < \cdots < {}^0R_\alpha(T'_{d(w)-2})$. Repeating the operation above, we obtain a sequence $T', T_1', \ldots T_s'$ of trees with n vertices and diameter d, such that ${}^0R_\alpha(T') < {}^0R_\alpha(T_1') < \cdots < {}^0R_\alpha(T_s')$, and there is no pair of distinct vertices in T_s' with degree greater than or equal to 3. Obviously, $D(T_s') = [n - d + 1, 2^{d-2}, 1^{n-d-1}]$. This proves (1).

Suppose that $n - 3 \geq d \geq 3$. Since T_s' is obtained from T'_{s-1} by replacing some pair (d_i, d_j) by the pair (d_i+1, d_j-1) and $D(T_s') = [n-d+1, 2^{d-2}, 1^{n-d+1}]$, where $D(T'_{s-1}) = [d_1, d_2, \cdots, d_n]$ and $d_i \geq d_j \geq 3$, one can see that $D(T'_{s-1}) = [n - d, 3, 2^{d-3}, 1^{n-d+1}]$. This proves (2).

Similarly, $T'_{s-2} = (T'_{s-2})^1$ for $n-5 \geq d \geq 3$ or $(T'_{s-2})^2$ for $n-4 = d$, where $D((T'_{s-2})^1) = [n-d-1, 4, 2^{d-3}, 1^{n-d+1}]$ and $D((T'_{s-2})^2) = [n-d-1, 3^2, 2^{d-4}, 1^{n-d+1}] = [3^3, 2^{n-8}, 1^5]$. Since $(T'_{s-2})^1$ can be obtained from $(T'_{s-2})^2$ by replacing the pair $(3, 3)$ by $(4, 2)$, we have ${}^0R_\alpha((T'_{s-2})^2) < {}^0R_\alpha((T'_{s-2})^1)$. It follows that $T'_{s-2} = [3^3, 2^{n-8}, 1^5]$ for $d = n - 4$ and $T'_{s-2} = [n - d - 1, 4, 2^{d-3}, 1^{n-d+1}]$ for $n - 5 \geq d \geq 3$. This proves (3). ■

A vertex v is called central vertex if $e(v) = r$. It is well known that every tree has exactly one or two central vertices.

Note: A tree T has just one central vertex if and only if $d = 2r$. Hence, if T has two central vertices, then $d = 2r - 1$ (see, e.g [9], p37 EXERCISES 2.1 in [10]).

From Lemma 1 and Theorem 2, we chose a tree with $d = 2r - 1$ when given a radius $r \geq 2$. Thus we have a theorem as follows.

Theorem 3. Let T be a tree with $n \geq 3$, $\frac{n}{2} \geq r \geq 2$ and $\alpha > 1$ or $\alpha < 0$ (resp. $0 < \alpha < 1$).

(1) ${}^0R_\alpha(T)$ attains the largest (resp. smallest) value if and only if $D(T) = [n - 2r + 2, 2^{2r-3}, 1^{n-2r+2}]$.

(2) For $\frac{n-2}{2} \geq r \geq 2$, ${}^0R_\alpha(T)$ attains the second largest (resp. smallest) value if and only if $D(T) = [n - 2r + 1, 3, 2^{2r-4}, 1^{n-2r+2}]$.

(3) For $r = \frac{n-3}{2} \geq 3$, ${}^0R_\alpha(T)$ attains the third largest (resp. smallest) value if and

only if $D(T) = [3^3, 2^{n-8}, 1^5]$, and for $\frac{n-3}{2} > r \geq 2$, ${}^0R_\alpha(T)$ attains the third largest (resp. smallest) value if and only if $D(T) = [n-2r, 4, 2^{2r-4}, 1^{n-2r+2}]$.

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