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Bidegreed trees with small index

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Abstract

Let $\mathcal{T}_\Delta^n$ be the class of trees on n vertices whose all vertices, other than pendant ones, are of degree Δ (bidegreed trees). In this paper we consider some problems related to the index (largest eigenvalue) of bidegreed trees focusing our attention on those trees with small index. In particular, we identify those trees from $\mathcal{T}_\Delta^n$ whose index is minimal.

1 Introduction

There are many papers in the literature where the largest eigenvalue (or the *index*) of the adjacency matrix of simple graphs is being considered (see [5] for more details). One special class of graphs, which attracts much attention among various researchers, especially in chemistry, consists of trees. Within the trees of fixed order, trees with maximum (or minimum) index are already identified; they are stars (resp. paths). Further efforts were focused on some special classes of trees (say, for trees of some fixed form, or trees with some prescribed invariants). Here we focus our attention on *bidegreed trees*, which are the trees whose vertices can have only two distinct degrees.

Let $\mathcal{T}_{\Delta}^{n}$ be the set of bidegreed trees on n vertices, whose all vertices other than pendant ones are of degree Δ. It is easy to prove that $\mathcal{T}_{\Delta}^{n}$ is non-empty only if $n \equiv 2 \mod (\Delta - 1)$; then each tree from $\mathcal{T}_{\Delta}^{n}$ has $\frac{n-2}{\Delta-1}$ vertices of degree Δ. In this paper we consider trees from $\mathcal{T}_{\Delta}^{n}$ whose index is small, or more precisely, less than $1 + \sqrt{\Delta - 1}$; in particular, we also identify those trees whose index is minimal.

For the basic notions and terminology on the spectral graph theory the readers are referred to [4]. To make the paper more self-contained, we will give here only a few basic facts. The spectrum of a graph G is the spectrum of its adjacency matrix (denoted by A_G). Recall, all eigenvalues of G are the roots of the characteristic polynomial of A_G (denoted by $\Phi_G(x) = \det(xI - A_G)$; note, all these eigenvalues are real since A_G is symmetric).

The largest eigenvalue of A_G is called the index (or *spectral radius*) of G, and is denoted by $\rho(= \rho(G))$. If G is connected, then its index is a simple eigenvalue; in addition, if G is a graph on n vertices (in particular a tree), there exists a positive eigenvector $\mathbf{x} = (x_1, \ldots, x_n)^T$ corresponding (such a vector is called the *principal eigenvector* if having the unit length). The following equation

$$\rho x_i = \sum_{i \sim j} x_j \quad (i = 1, 2, \ldots, n) \tag{1}$$

is called the *eigenvalue equation* for ρ corresponding to a vertex i of G; x_i is usually interpreted as the weight of the vertex i (with respect to $\mathbf{x}$), while $\sim$ denotes that the corresponding vertices are adjacent.

If $e = uv$ is a bridge of a graph G and if $G - e = G_1 \cup G_2$, where G_1 and G_2 are the components of G resulting from the deletion of e, we write $G = G_1 e G_2 = G_1 uv G_2$.

Recall, that any bidegreed tree T has only two kinds of vertices: vertices of degree Δ and vertices of degree 1, so we can visualize such trees just by looking at the vertices of degree Δ. We denote the graph induced by the vertices of degree Δ as the *skeleton* of T, and denote it by $Sk(T)$. In Fig. 1 we have depicted a bidegreed tree and its skeleton.

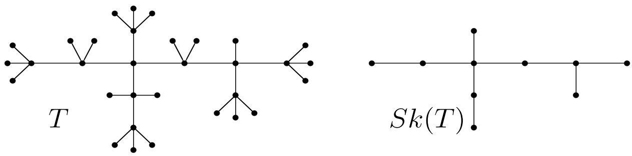

Fig.1: A bidegreed tree and its skeleton.

In the sequel, we will use to simplify our notation whenever there is no possibility of mis–interpretation (so the graphs in question, or polynomial variables, for the sake of brevity, will be omitted).

As already noted (see [4, 5]), trees with fixed order (number of vertices) having maximum or minimum index are identified. For trees with fixed order and maximum degree, the problem of identifying the trees with maximum index is resolved in [13]. It turns out that such trees have a star–like structure. For trees with minimum index in the same class, the expected structure (as noted in [1]) is path–like. Here we will resolve this problem for the bidegreed trees, which in particular will cover some chemical trees. More results on this topic, relevant to chemistry, can be found in [3, 7, 8, 9] and references therein.

The organization of the paper is as follows. In Section 2 we give some necessary facts to be used later. In Section 3 we give some bounds (lower and upper) on the index of trees in $\mathcal{T}_\Delta^n$ having minimum index, and explain some features of the upper bound. In Section 4 we prove our main results which enables us to resolve the conjecture from [1]. Finally, in Section 5 we provide some concluding remarks, and give two conjectures relevant to extension of Theorem 2.4.

2 Preliminaries

In this section we give some more definitions and precise the notation. Some graphs of particular form will be introduced. Finally, we provide some basic theoretical tools to be used further on.

The following formulas (also known as "Schwenk formulas", see [11]) will be widely used in the sequel proofs.

Theorem 2.1. *Let G be a (simple) graph. Denote by $\mathcal{C}(v)$ ($\mathcal{C}(e)$) the set of all cycles in G containing a vertex v (resp. an edge $e = uv$). Then we have:*

(i) $\Phi_G(x) = x\Phi_{G-v}(x) - \sum_{w\sim v} \Phi_{G-v-w}(x) - 2 \sum_{C\,\in\,\mathcal{C}(v)} \Phi_{G-V(C)}(x)$

(ii) $\Phi_G(x) = \Phi_{G-e}(x) - \Phi_{G-v-u}(x) - 2 \sum_{C\,\in\,\mathcal{C}(e)} \Phi_{G-V(C)}(x).$

We assume that $\Phi_G(x) = 1$ if G is the empty graph (i.e. with no vertices).

Remark 2.2. *Theorem 2.1(ii) reduces to a simpler formula under the condition that the edge $e = uv$ is a bridge. Then if $G - e = G_1 \cup G_2$, where $u \in G_1$ and $v \in G_2$, Theorem 2.1(ii) reduces to:*

$$\Phi_{G_1\,uv\,G_2} = \Phi_{G_1}\Phi_{G_2} - \Phi_{G_1-u}\Phi_{G_2-v}. \tag{2}$$

The next result (see [12], or [6] p. 56) concerns the *splitting* of a vertex of a graph. Let v be a vertex of a graph G (of degree at least 2), and let $N(v) = \{w \mid w \sim v\}$. We say that G' is obtained from G by the splitting of v if G' is formed from $G - v$ by adding two new vertices v_1 and v_2, and edges v_1w_1 ($w_1 \in N_1$), v_2w_2 ($w_2 \in N_2$), where $N_1 \cup N_2$ is a non–trivial bipartition of $N(v)$.

Theorem 2.3. *If G' is obtained from G by splitting a vertex, then $\rho(G') < \rho(G)$.*

The next result on graph perturbations was obtained in [10] (see also [6]).

Theorem 2.4. *Let $G(l,m)$ $(l,m \geq 0)$ be a graph obtained from a non–trivial connected rooted graph G by adding to its root r two hanging paths of length l and m. If $l \geq m \geq 1$ then*

$$\rho(G(l,m)) > \rho(G(l+1,m-1)).$$

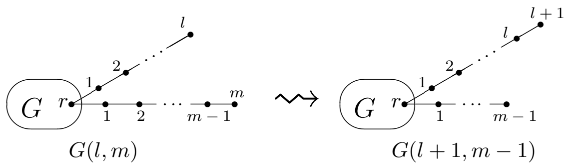

Fig. 2: The perturbation described in Theorem 2.4.

In order to prove our main result (Theorem 4.8) we consider a perturbation analogous to the above one, where G is a bidegreed tree from $\mathcal{T}^n_\Delta$; now the hanging paths are replaced by hanging *caterpillars* (i.e. trees whose removal of pendant vertices make them paths) – for more details see Corollary 4.9.

Now we introduce a special type of caterpillar, in fact the only bidegreed caterpillar in $\mathcal{T}^n_\Delta$. We denote such a caterpillar as T_k (see Fig. 3):

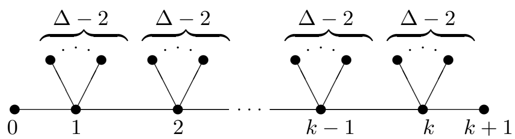

Fig. 3: The tree T_k.

We will also need some particular graphs that will be considered later. In the next figures we introduce this notation in addition to the usual one:

- all vertices, if not told otherwise, are depicted as small black vertices;
- a co–clique of order $\Delta - 4$ will be depicted as a big white vertex;
- a co–clique of order $\Delta - 2$ will be depicted as a big black vertex;
- an edge between a vertex r (small vertex) and a co–clique means that r is adjacent to all vertices of the co–clique.

Under the above notation we can depict T_k as follows:

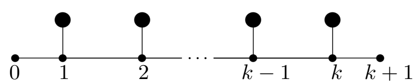

Fig. 4: The tree T_k under the new notation.

Let P_i^k be the graph obtained from T_k by deleting two hanging vertices at the vertex i. In Fig. 5 we depict the graph P_i^k:

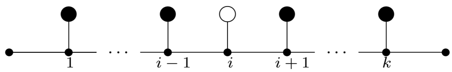

Fig. 5: The graph P_i^k.

Finally let us consider the graph $P_{i,i+1}^k$ as the graph obtained from T_k by deleting two hanging vertices at vertices i and $i+1$, i.e.:

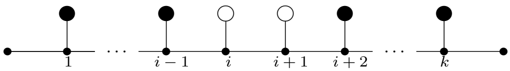

Fig. 6: The graph $P_{i,i+1}^k$.

3 $1+\sqrt{\Delta-1}$ as an upper bound

Here we first give a (lower and upper) bound on the index of trees having the minimum index (in $\mathcal{T}_\Delta^n$). We will next justify the role of our upper bound ($= 1+\sqrt{\Delta-1}$) in classifying the trees within $\mathcal{T}_\Delta^n$ according to the values of their index.

According to computational data, it was conjectured in [1], that the tree having the minimum index in $\mathcal{T}_\Delta^n$ is of the form as in Fig. 3.

To verify the above conjecture, we prove:

Theorem 3.1. *Let $\hat{T}$ be a tree with minimum index in $\mathcal{T}_\Delta^n$ and diameter ≥ 3. Then*

$$\frac{1}{2}+\sqrt{\Delta-\frac{3}{4}} \leq \rho(\hat{T}) < 1+\sqrt{\Delta-1},$$

where the equality on the left holds only for diameter equal 3.

Proof Let $\rho=\rho(\hat{T})$. First we consider the left inequality. The graph $T=T_2$ with diameter 3 (see Fig. 3) is an induced subgraph for any tree in $\mathcal{T}_\Delta^n$. Let $\mu=\rho(T)$. From (1) we easily get that

$$\mu=\frac{1}{2}+\sqrt{\Delta-\frac{3}{4}}$$

as the largest root of the following equation

$$\mu^2-\mu-(\Delta-1)=0.$$

Since $\mu \leq \rho$ (by the Interlacing Theorem, see for example [4], p. 19) we have that the left inequality holds (with equality only if the diameter is equal to 3).

Secondly, we consider the right inequality. Let $T' = T_k \in \mathcal{T}^n_\Delta$, and set $\lambda = \rho(T')$. Let U' be the unicyclic graph obtained from T' by identifying two vertices at maximum distance (so the leftmost and the rightmost vertices in Fig. 3), and by adding $\Delta - 2$ pendent edges at this new vertex. Then the index of U' goes up (by Theorem 2.3 and Interlacing Theorem). On the other hand (see also [2]) the index of U' (for any k) is the greatest root of the equation

$$\lambda^2 - 2\lambda - \Delta + 2 = 0,$$

and the right inequality also holds. □

In the next two theorems we will show how the upper bound from Theorem 3.1 features in determining within $\mathcal{T}^n_\Delta$ those graphs with small index.

The first theorem gives some strong structural restrictions which could be better visualized in the skeletons of trees from $\mathcal{T}^n_\Delta$. For this aim we define:

$$\mathcal{S}^n_d = \{T : T \in \mathcal{T}^n_\Delta,\ \Delta(Sk(T)) = d\},$$

where $2 \leq d \leq \Delta$. So the graphs from $\mathcal{T}^n_\Delta$ are classified according to the maximum degree in the skeleton. Clearly, $\mathcal{S}^n_2 = \{T_k\}$ (see Fig. 3). We also have

$$\mathcal{T}^n_\Delta = \bigcup_{d=2}^{\Delta} \mathcal{S}^n_d.$$

Theorem 3.2. *If $T \in \mathcal{S}^n_d$ where $d \geq 5$, then $\rho(T) \geq 1 + \sqrt{\Delta - 1}$.*

Proof. Let $T \in \mathcal{T}^n_\Delta$ be a tree having a vertex, say v, of degree Δ adjacent to p vertices of degree 1 and q vertices of degree Δ (note $p + q = \Delta$), then the graph depicted below is an induced subgraph of T (note, a label close to any oval denotes the number of vertices inside it).

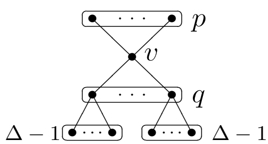

Fig. 7: An induced subgraph of T.

In what follows we make use of the divisor technique (see [6], p. 38). It is easy to see that the vertices in each level belong to a cell of an equitable partition (of the above tree) defined as follows: the first cell consists only of the vertex v; the second one consists of the q vertices in the oval below v; the third one consists of the p vertices in the oval above v; all other vertices belong to the fourth cell. It is easy to see that the adjacency matrix of the corresponding divisor is equal to:

$$D = \begin{pmatrix} 0 & q & p & 0 \\ 1 & 0 & 0 & \Delta - 1 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{pmatrix}$$

The characteristic polynomial of D is equal to:

$$\Phi_D(x) = x^4 - x^2(2\Delta - 1) + p(\Delta - 1).$$

As well–known, the largest eigenvalue of D coincides with the index of T (see, for example, [6], p. 38). Consequently, if $\Phi_D(1+\sqrt{\Delta - 1}) \leq 0$ then $\rho(T) \geq 1+\sqrt{\Delta - 1}$. By simple calculations we next get that $\Phi_D(1 + \sqrt{\Delta - 1}) \leq 0$ if

$$p \leq \Delta - 4 - \frac{2}{\sqrt{\Delta - 1}}.$$

Therefore, it follows that $q \geq 5$, as requited.

This completes the proof. □

The next theorem deals with the limit point for the index of trees from Fig. 3. It turns out that $1 + \sqrt{\Delta - 1}$ is the best possible upper bound for the minimum index of bidegreed trees with maximum degree Δ (since it is also the limit point).

Theorem 3.3. $\lim_{k\to\infty} \rho(T_k) = 1 + \sqrt{\Delta - 1}$.

Proof Let $\mathbf{x} = (x_0, x_1, \ldots, x_k, x_{k+1}, \ldots, x_n)^T$ be an eigenvector corresponding to $\rho(T_k)$ (see Fig. 3). Let $\rho = \rho(T_k)$ for a fixed k. Observe first that each pendant vertex at h ($h = 1, \ldots, k$) has a weight (with respect to $\mathbf{x}$) equal to $\frac{x_h}{\rho}$ (by (1)). Further application of the eigenvalue equations gives:

$$\rho x_h = x_{h-1} + (\Delta - 2)\frac{x_h}{\rho} + x_{h+1} \qquad (h = 1, 2, \ldots, k).$$

Taking that

$$\rho - \frac{\Delta - 2}{\rho} = 2\cos(t), \tag{3}$$

where $0 < t < \frac{\pi}{2}$ (note, $\rho < 1 + \sqrt{\Delta - 1}$), we get

$$x_h = \frac{\cos(\frac{k+1}{2} - h)t}{\cos(\frac{k+1}{2})t} \qquad (h = 0, \ldots, k + 1);$$

here we have taken that $x_0 = x_{k+1} = 1$ (due to an appropriate scaling and also symmetry of T_k).

For a fixed k, we have (by (3)) that

$$\rho(T_k) = \cos(t_k) + \sqrt{\Delta - 1 - \sin^2(t_k)}.$$

Clearly, $\rho(T_k)$ increases with k, while t_k decreases. So if t_k tends to 0 as k tends to infinity, we are done. Otherwise, assume that $\lim_{k\to\infty} t_k = \tau \neq 0$. Applying the eigenvalue equation at vertex 0 (or $k+1$) we have $\rho(T_k)x_0 = x_1$, and so we get

$$\rho(T_k) = \frac{\cos(\frac{k+1}{2} - 1)t_k}{\cos(\frac{k+1}{2})t_k},$$

or equivalently

$$\rho(T_k) = \cos(t_k) + \tan(\frac{k+1}{2}t_k)\sin(t_k).$$

The final contradiction follows from the above relation by putting that k tends to infinity (note, the part with tangent function does not converge).

This completes the proof. □

4 Main result

The main considerations of this section are motivated by Theorem 2.4. Here we will prove a partial analogue to this theorem (with hanging paths replaced by bidegreed caterpillars) as already stated in Section 2.

In addition (in the light of Theorem 3.2) we will assume that the trees under consideration belong to $\mathcal{S}_3^n$ or $\mathcal{S}_4^n$. The general case (for $\mathcal{S}_d^n$ with $d \geq 5$) can be considered similarly, but is technically more involved.

In what follows we will restrict ourselves to graphs from $\mathcal{S}_4^n$ (the same considerations for graphs from $\mathcal{S}_3^n$ can be carried analogously). Furthermore it is better to simplify the notation. Let $i' = i + 1$. Consider then, for any i, graphs G, $G' \in \mathcal{S}_4^n$ as depicted in Fig. 8, where $P = P_i^k$, $P' = P_{i'}^k$ (and in addition $P^* = P_{i,i'}^k$).

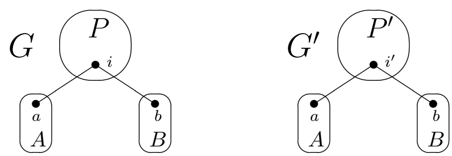

Fig. 8: The graphs G and G'.

Lemma 4.1. *Let $G, G' \in \mathcal{S}_4^n$ be the graphs depicted in Fig. 8. Then*

$$\Phi_{G'} - \Phi_G = \Phi_A\Phi_B(\Phi_{P'} - \Phi_P) - (\Phi_A\Phi_{B-b} + \Phi_{A-a}\Phi_B)(\Phi_{P'-i'} - \Phi_{P-i}).$$

Proof Applying (2) at edge ai we get:

$$\Phi_G = \Phi_A \Phi_{BbiP} - \Phi_{A-a} \Phi_{P-i} \Phi_B.$$

Another application of (2) at bi gives

$$= \Phi_A(\Phi_B \Phi_P - \Phi_{B-b} \Phi_{P-i}) - \Phi_{A-a} \Phi_B \Phi_{P-i}$$

$$= \Phi_A \Phi_B \Phi_P - (\Phi_A \Phi_{B-b} + \Phi_{A-a} \Phi_B) \Phi_{P-i}.$$

Similarly, for G', we get:

$$\Phi_{G'} = \Phi_A \Phi_B \Phi_{P'} - (\Phi_A \Phi_{B-b} + \Phi_{A-a} \Phi_B) \Phi_{P'-i'}$$

Now the assertion directly follows by subtracting the corresponding polynomials. □

In order to simplify the forthcoming proofs we introduce the graphs depicted in Fig. 9.

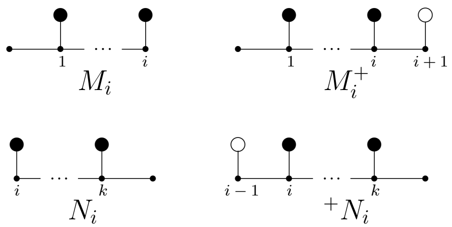

Fig. 9: Some graphs appearing in Lemmas 4.2 - 4.5.

Lemma 4.2. *Let M_i and N_i be the graphs depicted in Fig. 9. Then:*

$$\Phi_{P'} - \Phi_P = 2x^{3\Delta-9}(\Phi_{M_{i-1}} \Phi_{N_{i+3}} - \Phi_{M_{i-2}} \Phi_{N_{i+2}}).$$

Proof Consider the graphs P^*, M_i^+, ^+N_i as in Fig. 9. Applying repeatedly (2) we get

$$\Phi_{P'} - \Phi_P =$$

$$= (x^2 \Phi_{P^*} - 2x \Phi_{P^*-i}) - (x^2 \Phi_{P^*} - 2x \Phi_{P^*-i'})$$

$$= 2x(\Phi_{P^*-i'} - \Phi_{P^*-i}) = 2x^{\Delta-3}(\Phi_{M_{i-1}^+} \Phi_{N_{i+2}} - \Phi_{M_{i-1}} \Phi_{^+N_{i+2}})$$

$$= 2x^{3\Delta-9}(\Phi_{M_{i-1}} \Phi_{N_{i+3}} - \Phi_{M_{i-2}} \Phi_{N_{i+2}}),$$

as required. □

Lemma 4.3. *Under the above notations we have:*

$$\Phi_{P'-i'} - \Phi_{P-i} = x^{3\Delta-8}(\Phi_{M_{i-1}}\Phi_{N_{i+3}} - \Phi_{M_{i-2}}\Phi_{N_{i+2}}).$$

Proof Since

$$\Phi_{P'-i'} - \Phi_{P-i} = x^{\Delta-4}(\Phi_{M_i}\Phi_{N_{i+2}} - \Phi_{M_{i-1}}\Phi_{N_{i+1}}),$$

the proof follows by (2). □

Lemma 4.4. *Under the above notations we have:*

$$\Phi_{G'} - \Phi_G = -x^{3\Delta-9}(\Phi_A\Phi_\beta + \Phi_B\Phi_\alpha)(\Phi_{M_{i-1}}\Phi_{N_{i+3}} - \Phi_{M_{i-2}}\Phi_{N_{i+2}}),$$

where $\Phi_\alpha = \sum_{v\sim a}\Phi_{A-a-v}$ *and* $\Phi_\beta = \sum_{u\sim b}\Phi_{B-b-u}$.

Proof From the above lemmas we get:

$$\Phi_{G'}-\Phi_G = (2x^{3\Delta-9}\Phi_A\Phi_B - x^{3\Delta-8}(\Phi_A\Phi_{B-b}+\Phi_{A-a}\Phi_B))(\Phi_{M_{i-1}}\Phi_{N_{i+3}}-\Phi_{M_{i-2}}\Phi_{N_{i+2}})$$

$$= x^{3\Delta-9}(2\Phi_A\Phi_B - x(\Phi_A\Phi_{B-b} + \Phi_{A-a}\Phi_B))(\Phi_{M_{i-1}}\Phi_{N_{i+3}} - \Phi_{M_{i-2}}\Phi_{N_{i+2}})$$

$$= x^{3\Delta-9}(2\Phi_A\Phi_B - \Phi_A(\Phi_B + \Phi_\beta) - \Phi_B(\Phi_A + \Phi_\alpha))(\Phi_{M_{i-1}}\Phi_{N_{i+3}} - \Phi_{M_{i-2}}\Phi_{N_{i+2}})$$

$$= -x^{3\Delta-9}(\Phi_A\Phi_\beta + \Phi_B\Phi_\alpha)(\Phi_{M_{i-1}}\Phi_{N_{i+3}} - \Phi_{M_{i-2}}\Phi_{N_{i+2}}).$$

□

Lemma 4.5. *Let* $i' = i + 1 \le \frac{k+1}{2}$, *then for any* $x \ge \min\{\rho(G), \rho(G')\}$ *we have*

$$\Phi_{G'} - \Phi_G < 0.$$

Proof From Lemma 4.4

$$\Phi_{G'} - \Phi_G = -K^{(0)}(\Phi_{M_{i-1}}\Phi_{N_{i+3}} - \Phi_{M_{i-2}}\Phi_{N_{i+2}}),$$

where $K^{(0)}$ is positive for any $x \ge \min\{\rho(G), \rho(G')\}$. Applying repeatedly (2) we get:

$$\Phi_{G'} - \Phi_G = -K^{(0)}(\Phi_{M_{i-1}}\Phi_{N_{i+3}} - \Phi_{M_{i-2}}\Phi_{N_{i+2}})$$

$$= -K^{(1)}(\Phi_{M_{i-2}}\Phi_{N_{i+4}} - \Phi_{M_{i-3}}\Phi_{N_{i+3}})$$

$$\cdots$$

$$= -K^{(i-2)}(\Phi_{M_1}\Phi_{N_{2i+1}} - \Phi_{M_0}\Phi_{N_{2i}})$$

$$= -K^{(i-1)}(\Phi_{N_{2i+2}\cup(\Delta-1)K_1} - \Phi_{N_{2i+1}}).$$

Since $K^{(i-1)} > 0$ and since the graph $N_{2i+2} \cup (\Delta - 1)K_1$ is a proper spanning subgraph of N_{2i+1}, we have $\Phi_{N_{2i+2}\cup(\Delta-1)K_1} > \Phi_{N_{2i+1}}$, for all $x \geq \rho(N_{2i+1})$ (see [6], Lemma 6.2.1 p. 133). Then the assertion follows. $\square$

Note that $\Phi_{G'} - \Phi_G < 0$ for any $x \geq \min\{\rho(G), \rho(G')\}$ implies that $\rho(G') > \rho(G)$.

Remark 4.6. *Assume now that the subgraph B (from Fig. 8) is reduced to a single vertex (say b). Now it is a matter of routine to check that the above proof can be adapted to hold for this case. Also, in order to cover the case with $\Delta = 3$ we can repeat the proofs of the above lemmas in almost the same way.*

Based on Lemmas 4.1 - 4.5 and Remark 4.6 we easily get:

Theorem 4.7. *Let G, G' be the graphs from $\mathcal{S}_3^n \cup \mathcal{S}_4^n$, with $i' = i + 1 \leq \frac{k+1}{2}$ (cf. Fig. 8). Then*

$$\rho(G') > \rho(G)$$

From the above theorem we deduce our main result:

Theorem 4.8. *T_k is the unique tree in $\mathcal{T}_\Delta^n$ with minimum index.*

Proof Let $T \in \mathcal{T}_\Delta^n$ be a tree with minimum index. By Theorem 3.2, T cannot belong to $\mathcal{S}_d^n$ if $d \geq 5$. It cannot belong to $\mathcal{S}_3^n$ or $\mathcal{S}_4^n$ due to Theorem 4.7. So it belongs to $\mathcal{S}_2^n$ $(= \{T_k\})$. $\square$

We can also deduce the following result:

Corollary 4.9. *Let $T \in \mathcal{T}_\Delta^n$ be a bidegreed tree. Let u be a vertex of degree 3 or 4 belonging to $Sk(T)$, having attached two hanging paths P_1 and P_2 of lengths l_1 and l_2 (with $l_2 \geq l_1 \geq 1$). Then the index of T decreases if the length of P_1 is decreased by one and the length of P_2 increases by one.*

5 Concluding remarks

As already noticed, Corollary 4.9 extends under some constraints Theorem 2.4. The authors of this paper are considering to extend this result to more general cases. A first conjecture that clearly follows from the above results is the following:

Conjecture 5.1. *Let $T \in \mathcal{T}_\Delta^n$ be a bidegreed tree, and let u be a vertex of $Sk(T)$ of degree greater or equal 5, having attached two hanging paths P_1 and P_2 of lengths l_1 and l_2 (with $l_2 \geq l_1 \geq 1$). Then the index of T decreases if the length of P_1 is decreased by one and the length of P_2 increases by one.*

In fact Conjecture 5.1 means that the same result of Theorem 4.7 holds for any graph (of that prescribed form) in $\mathcal{T}_\Delta^n$. Furthermore we expect that the same result can be obtained in a more general case. The following conjecture can be regarded as an extension to Theorem 2.4:

Conjecture 5.2. *Let G be a rooted graph having r as its root, with* $\deg(r) \geq \Delta - 2$. *Denote by $G_\Delta(l, m)$ (with $l, m \geq 0$), the graph obtained from G by adding at r two hanging caterpillars $T_l \in \mathcal{T}_\Delta^n$ and $T_m \in \mathcal{T}_\Delta^n$. If G is not the star $K_{1,\Delta-2}$ and $l \geq m \geq 1$ then*

$$\rho(G_\Delta(l, m)) > \rho(G_\Delta(l + 1, m - 1)).$$

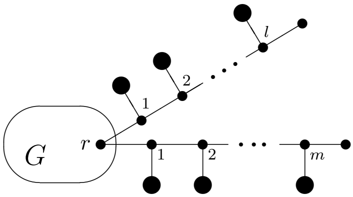

Fig. 10: The graph $G_\Delta(l, m)$.

Remark 5.3. *Note that if $\Delta = 2$ the above conjecture reduces to Theorem 2.4.*

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