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# LARGE SETS OF NONCOSPECTRAL GRAPHS WITH EQUAL LAPLACIAN ENERGY[1]

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**Abstract**

Several alternative definitions to graph energy have appeared in literature recently, the first among them being the Laplacian energy, defined by Gutman and Zhou in [Linear Algebra Appl. 414 (2006), 29–37]. We show here that Laplacian energy apparently has small power of discrimination among threshold graphs, by showing that, for each $n$, there exists a set of $n$ mutually noncospectral connected threshold graphs with equal Laplacian energy with $O(\sqrt{n})$ vertices only. Nevertheless, situation becomes opposite when trees are considered, as it turns out that, up to 20 vertices, there exists no pair of noncospectral trees with equal Laplacian energies.

## 1 Introduction

Let $G = (V, E)$ be a finite, simple, undirected graph with vertices $V = \{1, 2, \ldots, n\}$ and $m = |E|$ edges. The degree of a vertex $u \in V$ will be denoted by $d_u$. Let $G$ have adjacency matrix $A$ with eigenvalues $\lambda_1 \geq \lambda_2 \geq \ldots \geq \lambda_n$, and Laplacian matrix $L = D - A$, where $D$ is the diagonal matrix of vertex degrees, with eigenvalues $\mu_1 \geq \mu_2 \geq \ldots \geq \mu_n = 0$.

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Additional details on the theory of graph spectra may be found in [1]. These eigenvalues obey the following well-known relations:

$$\sum_{i=1}^{n} \lambda_i = 0, \qquad \sum_{i=1}^{n} \mu_i = 2m. \tag{1}$$

The energy and the Laplacian energy of $G$ are now defined as follows

$$E = E(G) = \sum_{i=1}^{n} |\lambda_i|, \qquad LE = LE(G) = \sum_{i=1}^{n} \left| \mu_i - \frac{2m}{n} \right|. \tag{2}$$

The energy of a graph was defined by Gutman in [2] and it has a long known chemical applications; for details see the surveys [3, 4, 5]. Much work has appeared in literature, especially in the last decade. On the other hand, the Laplacian energy was defined in [6] only recently, with some further properties found in [7]-[10].

From (1) and (2) we can observe that both energies represent the absolute deviation of corresponding eigenvalues from their average value. Thus, we can introduce the following

**Definition 1** *The energy of a given matrix $M$, denoted as $M$-energy, is the absolute deviation of eigenvalues of $M$ from their average value.*

This way, the energy of a graph is its $A$-energy and the Laplacian energy of a graph is its $L$-energy. Other types of energy can be defined in the same way, the difference being only in the matrix under consideration: for example, the energy of a distance matrix is studied in [11, 12]. Among those found in literature, it is the Laplacian-like energy only, defined by Liu and Liu [13], that does not fit this setting (which, at the end, may happen to be to its advantage, as a number of extremal problems for Laplacian-like energy can be solved by considering the coefficients of characteristic polynomial of $L$ and finding transformations which are monotone on these coefficients [14, 15, 16]).

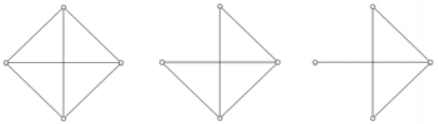
On the other hand, Nikiforov [17] has recently introduced another concept of the energy of a complex matrix $M$ as the sum of the singular values of $M$, which made possible to determine the energy of random graphs.

A feasible use of energies, as numerical invariants, is to distinguish nonisomorphic graphs from each other. In that respect, for a given type of graph matrix $M$, graphs having equal $M$-energy will be called *$M$-equienergetic*. Of course, since $M$-energy is calculated from spectrum of $M$, $M$-cospectral graphs will trivially have the same $M$-energy. Thus,

**Definition 2** *For a given type of graph matrix $M$, two graphs $G$ and $H$ will be called* $M$-equienergetic *if they are not $M$-cospectral, yet have equal $M$-energies.*

A number of results on $A$-equienergetic graphs have appeared recently [18]-[27]. In principle, most of these results show that $A$-equienergetic graphs exist in various classes of graphs, and in some cases, sets of $n$ such graphs can be found (e.g., see [21]), although on very large number of vertices (of order $5^n$ in [21]).

When it comes to Laplacian energy, it appears that it might not be well suited to distinguish among nonisomorphic graphs, as there exists a triplet of $L$-equienergetic graphs on four vertices already:



| | | | |
|---|---|---|---|
| $L$-spectrum: | $[4, 4, 4, 0]$ | $[4, 4, 2, 0]$ | $[4, 3, 1, 0]$ |
| $L$-energy: | 6 | 6 | 6 |

Our main task here is to show that the above example is not a coincidence. In particular, we show that for any $n \in \mathbf{N}$ there exists a set of $n$ $L$-equienergetic threshold graphs on $O(\sqrt{n})$ vertices only. It turns out that these graphs have equal number of edges as well. We find them in the class of threshold graphs.

## 2 Threshold graphs

Threshold graphs are a simple class of graphs, which due to their wide applicability, keeps reappearing under various names. A good survey on the properties of threshold graphs is [28].

Basically, a threshold graph is obtained in a recursive process, where one starts with an isolated vertex and at each step either a new isolated vertex is added or a new vertex adjacent to all previous vertices is added. This construction process can be encoded with a sequence of 0s and 1s, where 0 represents addition of an isolated vertex, while 1 represents addition of a vertex adjacent to all previous vertices. Thus, an $n$-vertex threshold graph can be encoded with a sequence of $n-1$ symbols. For our purposes, we will extend this encoding with an arbitrary initial element (0 or 1) that will correspond to the starting isolated vertex. Thus, in our case, an $n$-vertex threshold graph will be encoded with a sequence of $n$ symbols, where symbol at position $k$ describes the nature of vertex $k$. It is immediate to see from this encoding that two threshold graphs are isomorphic if and only if they have the same encoding sequence (without initial element).

Suppose that $G$ is a threshold graph with encoding sequence $a_1 a_2 \dots a_n \in \{0,1\}^n$, and let $d_1 \geq d_2 \geq \dots \geq d_n$ be its degree sequence. The degree sequence can be represented via its Ferrers diagram, as shown by example in Fig. 1.

A particularly nice property of a class of threshold graphs is that

**Lemma 1 ([28])** *The Laplacian spectrum of a threshold graph $G$ is the conjugate of its degree sequence.*

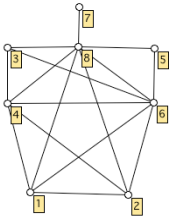
Thus, Laplacian spectrum of $G$ consists of eigenvalues

$$\mu_i = |\{j : d_j \geq i\}|, \qquad i = 1, 2, \dots, n. \tag{3}$$

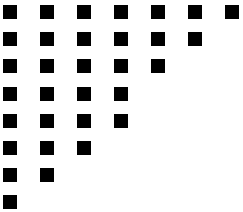
Let us now study the effects of two particular operations on the encoding sequence of a threshold graph:

**Operation A**. *Changing 01 with 10 in encoding sequence.*

Suppose $a_i = 0$ and $a_{i+1} = 1$ and form a new threshold graph $G'$ encoded by a sequence $a_1 \dots a_{i-1} 10 a_{i+2} \dots a_n$. Then $G'$ is obtained by removing edge $\{i, i+1\}$ from $G$, so that the degrees of $i$ and $i+1$ decrease by one, while the degrees of all other vertices remain




Ferrers diagram



Encoding sequence 01010101

$L$-spectrum $[8, 7, 6, 5, 3, 2, 1, 0]$

Figure 1: An example of a threshold graph.

intact. Then from (3) we see that, in the Laplacian spectrum of $G'$, the eigenvalues $\mu_{d'}$ and $\mu_{d''}$ decrease by one, while other eigenvalues remain intact.

**Operation B**. *Changing 10 with 01 in encoding sequence.*

Suppose $a_i = 1$ and $a_{i+1} = 0$ and form a new threshold graph $G''$ encoded by a sequence $a_1 \dots a_{i-1}01a_{i+2} \dots a_n$. Then $G''$ is obtained by adding edge $\{i, i+1\}$ to $G$, so that the degrees of $i$ and $i+1$ increase by one, while the degrees of all other vertices remain intact. Then from (3) we see that, in the Laplacian spectrum of $G''$, the eigenvalues $\mu_{d'+1}$ and $\mu_{d''+1}$ increase by one, while other eigenvalues remain intact.

# 3 A particular threshold graph and its mates

If we apply just one of operations $A$ and $B$ to a threshold graph $G$, then its number of edges changes, and so, the term $2m/n$ in the formula for $L$-energy changes as well. In such case, if we wish to know how does $L$-energy change, we first need to know how many eigenvalues of $L$ fall into each of intervals $[0, 2(m-1)/n]$, $[2(m-1)/n, 2m/n]$ and $[2m/n, +\infty]$. However, we can work around this problem if we simultaneously apply both operations to $G$, as then the number of edges remain the same.

The threshold graph $G_k$, $k \geq 3$, that will be used to generate a large set of $L$-equienergetic graphs, has encoding sequence

$$\underbrace{010101\dots010101}_{k \text{ times}}. \tag{4}$$

For example, $G_5$ is the first graph shown in Fig. 2.

From its encoding sequence, it is easy to see that $G_k$ has $2k$ vertices with degrees, given in the same order as vertices,

$$k, k, k-1, k+1, k-2, k+2, \dots, 1, 2k-1,$$

where the vertex at position $2i-1$, corresponding to 0 in (4), has degree $k-i+1$, while the vertex at position $2i$, corresponding to 1 in (4), has degree $k+i-1$, $i = 1, 2, \dots, k$.

From (3), its Laplacian spectrum is

$$[2k, 2k-1, \ldots, k+2, k+1, k-1, k-2, \ldots, 1, 0].$$

The average degree of $G_k$ is exactly $k$, and thus, its Laplacian energy is equal to $k(k+1)$.

Let us now apply operation **A** to pair 01 at positions $2i-1$ and $2i$, $1 \leq i \leq k$, and operation **B** to pair 10 at positions $2j$ and $2j+1$, $1 \leq j \leq k-1$, in the encoding sequence (4) (assuming that $\{2i-1, 2i\} \cap \{2j, 2j+1\} = \emptyset$). Then the degrees of vertices $2i-1$ and $2i$ decrease by one, while those of vertices $2j$ and $2j+1$ increase by one, so that the new degree sequence becomes

$$\underbrace{k, k, \ldots, k-i+2, k+i-2}_{\text{vertices 1 to } 2i-2}, \quad \mathbf{k-i, k+i-2}, \quad \underbrace{k-i, k+i, \ldots, k-j+1}_{\text{vertices } 2i+1 \text{ to } 2j-1},$$
$$\mathbf{k+j, k-j+1}, \quad \underbrace{k+j, \ldots, 1, 2k-1}_{\text{vertices } 2j+2 \text{ to } 2k}.$$

These operations decrease or increase by one corresponding Laplacian eigenvalues of $G_k$, so that their new values are

$$\mu'_{k-i+1} = k+i-1, \tag{5}$$
$$\mu'_{k-j+1} = k+j+1, \tag{6}$$
$$\mu'_{k+i-1} = k-i, \tag{7}$$
$$\mu'_{k+j} = k-j+1, \tag{8}$$

while the rest of the $L$-spectrum remains intact. Thus, two $L$-eigenvalues that were larger than $k$ increase and decrease by one, respectively, and two $L$-eigenvalues that were smaller than $k$ increase and decrease by one, respectively. Since the average degree remains $k$ in a newly obtained graph, we see that the $L$-energy does not change, and so, $G_k$ and a newly obtained graph are $L$-equienergetic.

Next, notice that operation **A** can be applied to $G_k$ in $k-1$ ways (we do not apply it to last pair 01 in the encoding sequence, as it results in a disconnected graph). If **A** is applied to the first pair 01, then operation **B** can be applied in $k-2$ ways, while in other cases, **B** can be applied in $k-3$ ways. Thus, we can apply them simultaneously in $k^2-4k+4$ ways. Since all these threshold graphs have distinct encoding sequences, no two of them are isomorphic. Moreover, it is easy to see from (5)-(8) that no two of them may be $L$-cospectral as well. Thus, we have just shown

**Theorem 1** *For each $k \geq 3$, there exists a set of $k^2-4k+5$ $L$-equienergetic graphs on $2k$ vertices.*

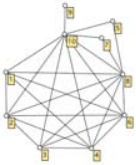
The construction of such set is exemplified for $k=5$ in Fig. 2.

# 4 Concluding remarks

We see from Theorem 1 that there exist arbitrariy large sets of $L$-equienergetic graphs, with a relatively small number of vertices, and moreover with the same number of edges, all of which show that $L$-energy is not well suited to distinguish threshold graphs from each other. This conclusion may extend to connected graphs as well, as it turns out that there exists

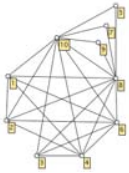
Sequence 0101010101
$[10, 9, 8, 7, 6, 4, 3, 2, 1, 0]$

Sequence 0011011001
$[10, 8, 8, 7, 7, 5, 3, 1, 1, 0]$



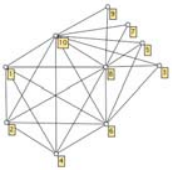
Sequence 0011100101
$[10, 9, 7, 7, 7, 5, 2, 2, 1, 0]$

Sequence 0100111001
$[10, 8, 8, 8, 6, 4, 4, 1, 1, 0]$



Sequence 0101100011
$[10, 10, 7, 7, 6, 4, 2, 2, 2, 0]$

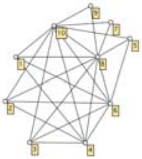
Sequence 0110001101
$[10, 9, 9, 6, 6, 3, 3, 3, 1, 0]$



Sequence 0110010011
$[10, 10, 8, 6, 6, 3, 3, 2, 2, 0]$

Sequence 1000110101
$[10, 9, 8, 8, 4, 4, 4, 2, 1, 0]$

Sequence 1001001101
$[10, 9, 9, 7, 4, 4, 3, 3, 1, 0]$



Sequence 1001010011
$[10, 10, 8, 7, 4, 4, 3, 2, 2, 0]$

Figure 2: $k^2 - 4k + 5 = 10$ $L$-equienergetic graphs on $2k = 10$ vertices, with encoding sequence and Laplacian spectrum below each graph.

- 297 pairs of noncospectral $L$-equienergetic graphs among 853 connected graphs on seven vertices,
- 13044 pairs among 11117 connected graphs on eight vertices (implying, in fact, existence of a large number of $L$-equienergetic sets containing at least four graphs each), and
- 39304 pairs among 261080 connected graphs on nine vertices.

Nevertheless, $L$-energy may be fit for business in more restricted graph classes. For example, our computer search employing first a Java-based program that puts trees in hash map with $L$-energy as keys, calculated using Colt library (available from `http://acs.lbl.gov/~hoschek/colt/`), and then checking the findings with Wolfram's Mathematica, revealed that

*there is not a single pair of $L$-equienergetic trees up to 20 vertices!*

There is a large number of trees whose $L$-energy differs by less than $10^{-11}$, but no two of them are really $L$-equienergetic. This is in sharp contrast with the fact that there are already 120 pairs of equienergetic trees on 20 vertices (which was checked by the same programs, except that $A$-energy was calculated instead of $L$-energy).

Of course, it is hard to believe that there exists no pair of $L$-equienergetic trees at all, and they probably do exist on a larger number of vertices. Thus, we close this paper by leaving the following

**Open problem.** *Find a pair of $L$-equienergetic trees.*

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