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LARGE SETS OF NONCOSPECTRAL GRAPHS WITH EQUAL LAPLACIAN ENERGY[1]

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Abstract

Several alternative definitions to graph energy have appeared in literature recently, the first among them being the Laplacian energy, defined by Gutman and Zhou in [Linear Algebra Appl. 414 (2006), 29–37]. We show here that Laplacian energy apparently has small power of discrimination among threshold graphs, by showing that, for each n, there exists a set of n mutually noncospectral connected threshold graphs with equal Laplacian energy with $O(\sqrt{n})$ vertices only. Nevertheless, situation becomes opposite when trees are considered, as it turns out that, up to 20 vertices, there exists no pair of noncospectral trees with equal Laplacian energies.

1 Introduction

Let $G = (V, E)$ be a finite, simple, undirected graph with vertices $V = \{1, 2, \ldots, n\}$ and $m = |E|$ edges. The degree of a vertex $u \in V$ will be denoted by d_u. Let G have adjacency matrix A with eigenvalues $\lambda_1 \geq \lambda_2 \geq \ldots \geq \lambda_n$, and Laplacian matrix $L = D - A$, where D is the diagonal matrix of vertex degrees, with eigenvalues $\mu_1 \geq \mu_2 \geq \ldots \geq \mu_n = 0$.

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Additional details on the theory of graph spectra may be found in [1]. These eigenvalues obey the following well-known relations:

$$\sum_{i=1}^{n} \lambda_i = 0, \qquad \sum_{i=1}^{n} \mu_i = 2m. \tag{1}$$

The energy and the Laplacian energy of G are now defined as follows

$$E = E(G) = \sum_{i=1}^{n} |\lambda_i|, \qquad LE = LE(G) = \sum_{i=1}^{n} \left| \mu_i - \frac{2m}{n} \right|. \tag{2}$$

The energy of a graph was defined by Gutman in [2] and it has a long known chemical applications; for details see the surveys [3, 4, 5]. Much work has appeared in literature, especially in the last decade. On the other hand, the Laplacian energy was defined in [6] only recently, with some further properties found in [7]-[10].

From (1) and (2) we can observe that both energies represent the absolute deviation of corresponding eigenvalues from their average value. Thus, we can introduce the following

Definition 1 *The energy of a given matrix M, denoted as M-energy, is the absolute deviation of eigenvalues of M from their average value.*

This way, the energy of a graph is its A-energy and the Laplacian energy of a graph is its L-energy. Other types of energy can be defined in the same way, the difference being only in the matrix under consideration: for example, the energy of a distance matrix is studied in [11, 12]. Among those found in literature, it is the Laplacian-like energy only, defined by Liu and Liu [13], that does not fit this setting (which, at the end, may happen to be to its advantage, as a number of extremal problems for Laplacian-like energy can be solved by considering the coefficients of characteristic polynomial of L and finding transformations which are monotone on these coefficients [14, 15, 16]).

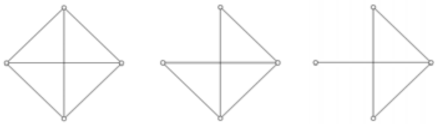
On the other hand, Nikiforov [17] has recently introduced another concept of the energy of a complex matrix M as the sum of the singular values of M, which made possible to determine the energy of random graphs.

A feasible use of energies, as numerical invariants, is to distinguish nonisomorphic graphs from each other. In that respect, for a given type of graph matrix M, graphs having equal M-energy will be called *M-equienergetic.* Of course, since M-energy is calculated from spectrum of M, M-cospectral graphs will trivially have the same M-energy. Thus,

Definition 2 *For a given type of graph matrix M, two graphs G and H will be called* M-equienergetic *if they are not M-cospectral, yet have equal M-energies.*

A number of results on A-equienergetic graphs have appeared recently [18]-[27]. In principle, most of these results show that A-equienergetic graphs exist in various classes of graphs, and in some cases, sets of n such graphs can be found (e.g., see [21]), although on very large number of vertices (of order 5^n in [21]).

When it comes to Laplacian energy, it appears that it might not be well suited to distinguish among nonisomorphic graphs, as there exists a triplet of L-equienergetic graphs on four vertices already:



L-spectrum:	$[4,4,4,0]$	$[4,4,2,0]$	$[4,3,1,0]$
L-energy:	6	6	6

Our main task here is to show that the above example is not a coincidence. In particular, we show that for any $n \in \mathbf{N}$ there exists a set of n L-equienergetic threshold graphs on $O(\sqrt{n})$ vertices only. It turns out that these graphs have equal number of edges as well. We find them in the class of threshold graphs.

2 Threshold graphs

Threshold graphs are a simple class of graphs, which due to their wide applicability, keeps reappearing under various names. A good survey on the properties of threshold graphs is [28].

Basically, a threshold graph is obtained in a recursive process, where one starts with an isolated vertex and at each step either a new isolated vertex is added or a new vertex adjacent to all previous vertices is added. This construction process can be encoded with a sequence of 0s and 1s, where 0 represents addition of an isolated vertex, while 1 represents addition of a vertex adjacent to all previous vertices. Thus, an n-vertex threshold graph can be encoded with a sequence of $n-1$ symbols. For our purposes, we will extend this encoding with an arbitrary initial element (0 or 1) that will correspond to the starting isolated vertex. Thus, in our case, an n-vertex threshold graph will be encoded with a sequence of n symbols, where symbol at position k describes the nature of vertex k. It is immediate to see from this encoding that two threshold graphs are isomorphic if and only if they have the same encoding sequence (without initial element).

Suppose that G is a threshold graph with encoding sequence $a_1a_2\dots a_n \in \{0,1\}^n$, and let $d_1 \geq d_2 \geq \dots \geq d_n$ be its degree sequence. The degree sequence can be represented via its Ferrers diagram, as shown by example in Fig. 1.

A particularly nice property of a class of threshold graphs is that

Lemma 1 ([28]) *The Laplacian spectrum of a threshold graph G is the conjugate of its degree sequence.*

Thus, Laplacian spectrum of G consists of eigenvalues

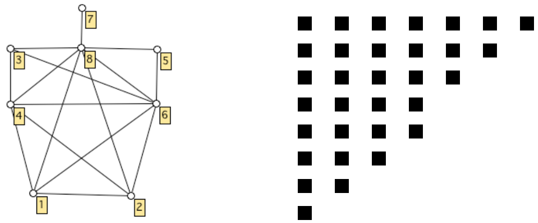
$$\mu_i = |\{j : d_j \geq i\}|, \qquad i = 1,2,\dots,n. \tag{3}$$

Let us now study the effects of two particular operations on the encoding sequence of a threshold graph:

Operation A. *Changing 01 with 10 in encoding sequence.*

Suppose $a_i = 0$ and $a_{i+1} = 1$ and form a new threshold graph G' encoded by a sequence $a_1 \dots a_{i-1}10a_{i+2}\dots a_n$. Then G' is obtained by removing edge $\{i, i+1\}$ from G, so that the degrees of i and $i+1$ decrease by one, while the degrees of all other vertices remain

Ferrers diagram



Encoding sequence 01010101 L-spectrum $[8, 7, 6, 5, 3, 2, 1, 0]$

Figure 1: An example of a threshold graph.

intact. Then from (3) we see that, in the Laplacian spectrum of G', the eigenvalues $\mu_{d'}$ and $\mu_{d''}$ decrease by one, while other eigenvalues remain intact.

Operation B. *Changing 10 with 01 in encoding sequence.*

Suppose $a_i = 1$ and $a_{i+1} = 0$ and form a new threshold graph G'' encoded by a sequence $a_1 \dots a_{i-1} 01 a_{i+2} \dots a_n$. Then G'' is obtained by adding edge $\{i, i+1\}$ to G, so that the degrees of i and $i+1$ increase by one, while the degrees of all other vertices remain intact. Then from (3) we see that, in the Laplacian spectrum of G'', the eigenvalues $\mu_{d'+1}$ and $\mu_{d''+1}$ increase by one, while other eigenvalues remain intact.

3 A particular threshold graph and its mates

If we apply just one of operations A and B to a threshold graph G, then its number of edges changes, and so, the term $2m/n$ in the formula for L-energy changes as well. In such case, if we wish to know how does L-energy change, we first need to know how many eigenvalues of L fall into each of intervals $[0, 2(m-1)/n]$, $[2(m-1)/n, 2m/n]$ and $[2m/n, +\infty]$. However, we can work around this problem if we simultaneously apply both operations to G, as then the number of edges remain the same.

The threshold graph G_k, $k \geq 3$, that will be used to generate a large set of L-equienergetic graphs, has encoding sequence

$$\underbrace{010101\dots010101}_{k \text{ times}}. \tag{4}$$

For example, G_5 is the first graph shown in Fig. 2.

From its encoding sequence, it is easy to see that G_k has $2k$ vertices with degrees, given in the same order as vertices,

$$k, k, k-1, k+1, k-2, k+2, \dots, 1, 2k-1,$$

where the vertex at position $2i-1$, corresponding to 0 in (4), has degree $k-i+1$, while the vertex at position $2i$, corresponding to 1 in (4), has degree $k+i-1$, $i = 1, 2, \dots, k$.

From (3), its Laplacian spectrum is

$$[2k, 2k-1, \ldots, k+2, k+1, k-1, k-2, \ldots, 1, 0].$$

The average degree of G_k is exactly k, and thus, its Laplacian energy is equal to $k(k+1)$.

Let us now apply operation **A** to pair 01 at positions $2i-1$ and $2i$, $1 \le i \le k$, and operation **B** to pair 10 at positions $2j$ and $2j+1$, $1 \le j \le k-1$, in the encoding sequence (4) (assuming that $\{2i-1, 2i\} \cap \{2j, 2j+1\} = \emptyset$). Then the degrees of vertices $2i-1$ and $2i$ decrease by one, while those of vertices $2j$ and $2j+1$ increase by one, so that the new degree sequence becomes

$$\underbrace{k, k, \ldots, k-i+2, k+i-2}_{\text{vertices } 1 \text{ to } 2i-2}, \quad \mathbf{k-i, k+i-2}, \quad \underbrace{k-i, k+i, \ldots, k-j+1}_{\text{vertices } 2i+1 \text{ to } 2j-1},$$
$$\mathbf{k+j, k-j+1}, \quad \underbrace{k+j, \ldots, 1, 2k-1}_{\text{vertices } 2j+2 \text{ to } 2k}.$$

These operations decrease or increase by one corresponding Laplacian eigenvalues of G_k, so that their new values are

$$\mu'_{k-i+1} = k+i-1, \tag{5}$$
$$\mu'_{k-j+1} = k+j+1, \tag{6}$$
$$\mu'_{k+i-1} = k-i, \tag{7}$$
$$\mu'_{k+j} = k-j+1, \tag{8}$$

while the rest of the L-spectrum remains intact. Thus, two L-eigenvalues that were larger than k increase and decrease by one, respectively, and two L-eigenvalues that were smaller than k increase and decrease by one, respectively. Since the average degree remains k in a newly obtained graph, we see that the L-energy does not change, and so, G_k and a newly obtained graph are L-equienergetic.

Next, notice that operation **A** can be applied to G_k in $k-1$ ways (we do not apply it to last pair 01 in the encoding sequence, as it results in a disconnected graph). If **A** is applied to the first pair 01, then operation **B** can be applied in $k-2$ ways, while in other cases, **B** can be applied in $k-3$ ways. Thus, we can apply them simultaneously in k^2-4k+4 ways. Since all these threshold graphs have distinct encoding sequences, no two of them are isomorphic. Moreover, it is easy to see from (5)-(8) that no two of them may be L-cospectral as well. Thus, we have just shown

Theorem 1 *For each $k \ge 3$, there exists a set of k^2-4k+5 L-equienergetic graphs on $2k$ vertices.*

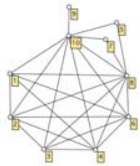
The construction of such set is exemplified for $k=5$ in Fig. 2.

4 Concluding remarks

We see from Theorem 1 that there exist arbitrariy large sets of L-equienergetic graphs, with a relatively small number of vertices, and moreover with the same number of edges, all of which show that L-energy is not well suited to distinguish threshold graphs from each other. This conclusion may extend to connected graphs as well, as it turns out that there exists

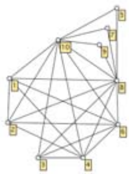
Sequence 0101010101
[10, 9, 8, 7, 6, 4, 3, 2, 1, 0]

Sequence 0011011001
[10, 8, 8, 7, 7, 5, 3, 1, 1, 0]



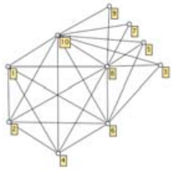
Sequence 0011100101
[10, 9, 7, 7, 7, 5, 2, 2, 1, 0]

Sequence 0100111001
[10, 8, 8, 8, 6, 4, 4, 1, 1, 0]

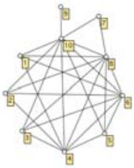


Sequence 0101100011
[10, 10, 7, 7, 6, 4, 2, 2, 2, 0]

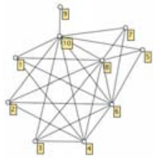
Sequence 0110001101
[10, 9, 9, 6, 6, 3, 3, 3, 1, 0]



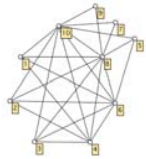
Sequence 0110010011
[10, 10, 8, 6, 6, 3, 3, 2, 2, 0]



Sequence 1000110101
[10, 9, 8, 8, 4, 4, 4, 2, 1, 0]



Sequence 1001001101
[10, 9, 9, 7, 4, 4, 3, 3, 1, 0]



Sequence 1001010011
[10, 10, 8, 7, 4, 4, 3, 2, 2, 0]

Figure 2: $k^2 - 4k + 5 = 10$ L-equienergetic graphs on $2k = 10$ vertices, with encoding sequence and Laplacian spectrum below each graph.

- 297 pairs of noncospectral L-equienergetic graphs among 853 connected graphs on seven vertices,
- 13044 pairs among 11117 connected graphs on eight vertices (implying, in fact, existence of a large number of L-equienergetic sets containing at least four graphs each), and
- 39304 pairs among 261080 connected graphs on nine vertices.

Nevertheless, L-energy may be fit for business in more restricted graph classes. For example, our computer search employing first a Java-based program that puts trees in hash map with L-energy as keys, calculated using Colt library (available from `http://acs.lbl.gov/~hoschek/colt/`), and then checking the findings with Wolfram's Mathematica, revealed that

there is not a single pair of L-equienergetic trees up to 20 vertices!

There is a large number of trees whose L-energy differs by less than 10^{-11}, but no two of them are really L-equienergetic. This is in sharp contrast with the fact that there are already 120 pairs of equienergetic trees on 20 vertices (which was checked by the same programs, except that A-energy was calculated instead of L-energy).

Of course, it is hard to believe that there exists no pair of L-equienergetic trees at all, and they probably do exist on a larger number of vertices. Thus, we close this paper by leaving the following

Open problem. *Find a pair of L-equienergetic trees.*

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