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SMALL AND NOT SO SMALL EQUIENERGETIC GRAPHS[1]

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Abstract

We show that, contrary to the claims of several authors that there exists no equienergetic graphs on seven vertices, there exists 21 sets of noncospectral, equienergetic graphs on seven vertices. We also give the number and cardinality of sets of equienergetic connected graphs on eight, nine and ten vertices, and equienergetic trees and chemical trees on up to 22 vertices. Finally, we construct two new infinite families of equienergetic graphs.

1 Introduction

Let G be a finite, simple and undirected graph with the adjacency matrix A and eigenvalues $\lambda_1 \geq \lambda_2 \geq \ldots \geq \lambda_n$. We denote the *characteristic polynomial* of the adjacency matrix of graph G by $P_G(\lambda)$. Two nonisomorphic graphs are *cospectral* if they have the same spectrum.

The *energy* of the graph G is a sum $E(G) = \sum_{i=1}^{n} |\lambda_i|$. Two graphs are *equienergetic* if they are not cospectral and have same energy. A set of graphs is said to be *equiset* if all

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graphs in the set have the same energy and there are at least two non-cospectral graphs in the set.

In ([8]) authors claim that there are no equienergetic graphs on seven and nine vertices. A pair of equienergetic graphs on nine vertices has been found in ([11]), but still, both ([11, 12]) remain to position that there are no equienergetic graphs on seven vertices.

We show here that equienergetic graphs on seven vertices do exist, and in no less than 21 equisets. We also give cardinalities of equisets of connected graphs on eight, nine and ten vertices, as well as numbers and cardinalities of equisets of trees and chemical trees on up to 22 vertices, thus extending the results obtained in [1].

In addition, in Section 4 we describe two new families of pairs of equienergetic graphs on $2n$ vertices, for $n \geq 3$.

2 Equienergetic graphs on seven vertices

There exist 853 nonisomorphic connected graphs on seven vertices. High precision computing (66+ decimals) of *Mathematica* has been used to determine all sets of equienergetic graphs among them. There exist 21 equisets on seven vertices, 16 of them are pairs, four of them are triplets, and one equiset has cardinality six. Graphs from all equisets are shown in Fig. 1, while their spectra and energies, or just characteristic polynomials where we could not find a closed form, are given in Table 1.

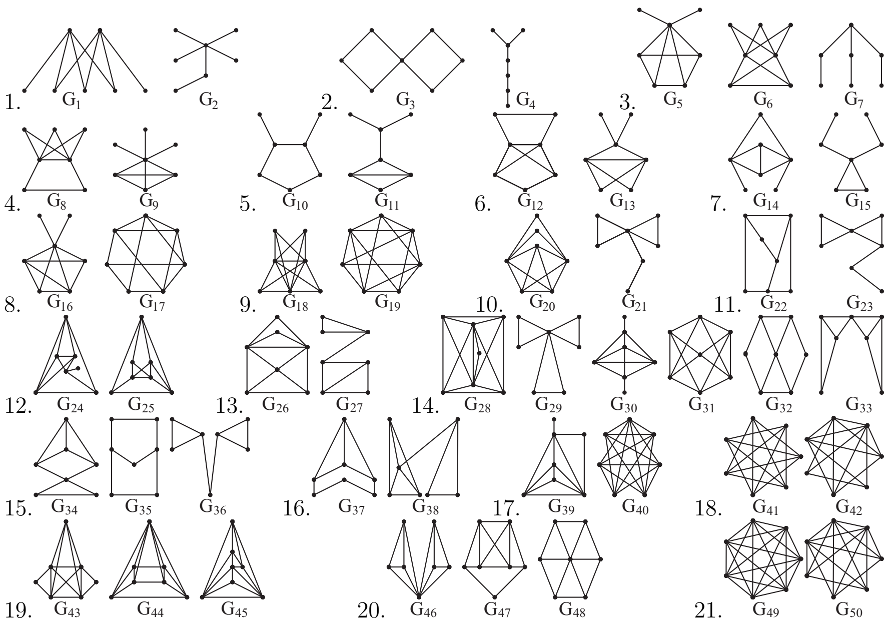

Figure 1: Seven-vertex connected graphs belonging to 21 equisets.

G_1	$Sp(G_1) = [-\sqrt{10}, 0^5, \sqrt{10}]$	$E = 2\sqrt{10}$
G_2	$Sp(G_2) = [-\sqrt{3+\sqrt{5}}, -\sqrt{3-\sqrt{5}}, 0^3, \sqrt{3-\sqrt{5}}, \sqrt{3+\sqrt{5}}]$	
G_3	$Sp(G_3) = [-\sqrt{6}, -\sqrt{2}, 0^3, \sqrt{2}, \sqrt{6}]$	$E = 2\sqrt{2} + 2\sqrt{6}$
G_4	$Sp(G_4) = [-\sqrt{2+\sqrt{3}}, -\sqrt{2}, -\sqrt{2-\sqrt{3}}, 0, \sqrt{2-\sqrt{3}}, \sqrt{2}, \sqrt{2+\sqrt{3}}]$	
G_5	$Sp(G_5) = [-2^2, 0^3, 2-\sqrt{2}, 2+\sqrt{2}]$	$E = 8$
G_6	$Sp(G_6) = [-3, -1, 0^3, 2-\sqrt{2}, 2+\sqrt{2}]$	
G_7	$Sp(G_7) = [-2, -1^2, 0, 1^2, 2]$	
G_8	$\phi(G_8, \lambda) = -\lambda^3(2+\lambda)\left(6 - 6\lambda - 2\lambda^2 + \lambda^3\right)$	
G_9	$\phi(G_9, \lambda) = -\lambda^2(1+\lambda)^2\left(6 - 6\lambda - 2\lambda^2 + \lambda^3\right)$	
G_{10}	$\phi(G_{10}, \lambda) = -(-1+\lambda)^2\lambda(2+\lambda)\left(-2 - 4\lambda + \lambda^3\right)$	
G_{11}	$\phi(G_{11}, \lambda) = -\lambda^2(1+\lambda)(2+\lambda)\left(5 - \lambda - 3\lambda^2 + \lambda^3\right)$	
G_{12}	$\phi(G_{12}, \lambda) = -\lambda^3(2+\lambda)\left(8 - 6\lambda - 2\lambda^2 + \lambda^3\right)$	
G_{13}	$\phi(G_{13}, \lambda) = -\lambda^2(1+\lambda)^2\left(8 - 6\lambda - 2\lambda^2 + \lambda^3\right)$	
G_{14}	$Sp(G_{14}) = [-1-\sqrt{2}, -1^2, 0, -1+\sqrt{2}, 1, 3]$	$E = 6 + 2\sqrt{2}$
G_{15}	$Sp(G_{15}) = [-2, -1^2, 1-\sqrt{2}, 1^2, 1+\sqrt{2}]$	
G_{16}	$\phi(G_{16}, \lambda) = -\lambda^2(1+\lambda)(2+\lambda)\left(4 - 4\lambda - 3\lambda^2 + \lambda^3\right)$	
G_{17}	$\phi(G_{17}, \lambda) = -\lambda^3(3+\lambda)\left(4 - 4\lambda - 3\lambda^2 + \lambda^3\right)$	
G_{18}	$Sp(G_{18}) = [1-\sqrt{13}, -1^2, 0^3, 1+\sqrt{13}]$	$E = 2 + 2\sqrt{13}$
G_{19}	$Sp(G_{19}) = [1-\sqrt{13}, -2, 0^4, 1+\sqrt{13}]$	
G_{20}	$Sp(G_{20}) = [-1-\sqrt{3}, -1^2, 0^2, -1+\sqrt{3}, 4]$	$E = 6 + 2\sqrt{3}$
G_{21}	$Sp(G_{21}) = [-2, -1^2, 1-\sqrt{3}, 1^2, 1+\sqrt{3}]$	
G_{22}	$\phi(G_{22}, \lambda) = -(-1+\lambda)\left(-1 + 2\lambda + \lambda^2\right)\left(2 + \lambda - 5\lambda^2 - \lambda^3 + \lambda^4\right)$	
G_{23}	$\phi(G_{23}, \lambda) = -(1+\lambda)\left(-2 + \lambda^2\right)\left(2 + \lambda - 5\lambda^2 - \lambda^3 + \lambda^4\right)$	
G_{24}	$\phi(G_{24}, \lambda) = -(-1+\lambda)(1+\lambda)^2\left(2 - 2\lambda - 10\lambda^2 - \lambda^3 + \lambda^4\right)$	
G_{25}	$\phi(G_{25}, \lambda) = -(-1+\lambda)\lambda(2+\lambda)\left(2 - 2\lambda - 10\lambda^2 - \lambda^3 + \lambda^4\right)$	
G_{26}	$\phi(G_{26}, \lambda) = -\lambda^2(2+\lambda)^2\left(4 - 4\lambda^2 + \lambda^3\right)$	
G_{27}	$\phi(G_{27}, \lambda) = -(-2+\lambda)(1+\lambda)^3\left(1 - 5\lambda - \lambda^2 + \lambda^3\right)$	
G_{28}	$Sp(G_{28}) = [-2^2, -1, 0^2, \frac{1}{2}\left(5-\sqrt{17}\right), \frac{1}{2}\left(5+\sqrt{17}\right)]$	$E = 10$
G_{29}	$Sp(G_{29}) = [-2, -1^3, 1^2, 3]$	
G_{30}	$Sp(G_{30}) = [-2, -1^3, 2-\sqrt{3}, 1, 2+\sqrt{3}]$	
G_{31}	$Sp(G_{31}) = [-3, -1^2, 0^2, 1, 4]$	
G_{32}	$Sp(G_{32}) = [-2^2, -1, 0, 1^2, 3]$	
G_{33}	$Sp(G_{33}) = [-2, -1^3, 0, 2, 3]$	
G_{34}	$\phi(G_{34}, \lambda) = -(-3+\lambda)\lambda(1+\lambda)^2\left(-2 - 4\lambda + \lambda^2 + \lambda^3\right)$	
G_{35}	$\phi(G_{35}, \lambda) = -(-1+\lambda)^2(1+\lambda)(2+\lambda)\left(2 - 4\lambda - \lambda^2 + \lambda^3\right)$	
G_{36}	$\phi(G_{36}, \lambda) = -(-2+\lambda)(1+\lambda)^3\left(2 - 4\lambda - \lambda^2 + \lambda^3\right)$	
G_{37}	$\phi(G_{22}, \lambda) = -(-1+\lambda)(1+\lambda)^3\left(8 - 5\lambda - 2\lambda^2 + \lambda^3\right)$	
G_{38}	$\phi(G_{23}, \lambda) = -(-2+\lambda)(1+\lambda)^3\left(-2 - 6\lambda - \lambda^2 + \lambda^3\right)$	
G_{39}	$\phi(G_{39}, \lambda) = -(-1+\lambda)(1+\lambda)(2+\lambda)\left(1 - 2\lambda - 7\lambda^2 - 2\lambda^3 + \lambda^4\right)$	
G_{40}	$\phi(G_{40}, \lambda) = -\lambda^3(2+\lambda)^2\left(-6 - 4\lambda + \lambda^2\right)$	
G_{41}	$\phi(G_{41}, \lambda) = -\lambda(1+\lambda)^2(2+\lambda)\left(8 - 4\lambda - 4\lambda^2 + \lambda^3\right)$	
G_{42}	$\phi(G_{42}, \lambda) = -(-4+\lambda)\left(-1 - \lambda + 2\lambda^2 + \lambda^3\right)^2$	
G_{43}	$Sp(G_{43}) = [-2, -1^2, 0, 1, \frac{1}{2}\left(3-\sqrt{33}\right), \frac{1}{2}\left(3+\sqrt{33}\right)]$	$E = 5 + \sqrt{33}$
G_{44}	$Sp(G_{44}) = [-2^2, 0^2, 1, \frac{1}{2}\left(3-\sqrt{33}\right), \frac{1}{2}\left(3+\sqrt{33}\right)]$	
G_{45}	$Sp(G_{45}) = [-2^2, 0^2, 1, \frac{1}{2}\left(3-\sqrt{33}\right), \frac{1}{2}\left(3+\sqrt{33}\right)]$	
G_{46}	$Sp(G_{46}) = [-1^4, 2, 1-\sqrt{7}, 1+\sqrt{7}]$	$E = 6 + 2\sqrt{7}$
G_{47}	$Sp(G_{47}) = [-2, -1^2, 1^2, 1-\sqrt{7}, 1+\sqrt{7}]$	
G_{48}	$Sp(G_{48}) = [-2, -1^2, 1^2, 1-\sqrt{7}, 1+\sqrt{7}]$	
G_{49}	$Sp(G_{49}) = [-1^6, 6]$	$E = 12$
G_{50}	$Sp(G_{50}) = [-2, -1^4, 1, 5]$	

Table 1: Spectra and energies, or characteristic polynomials, of graphs in equisets.

3 Other equisets of graphs and trees

Tables 2, 3 and 4 contain number and cardinalities of equisets for connected graphs on five to ten vertices, trees and chemical trees on nine to 22 vertices, respectively. These graphs were generated by *nauty* ([14]). Table columns denote, respectively, numbers of vertices, equisets, and then numbers of equisets of cardinality two, three, ...

n	Equisets	2	3	4	5	6	≥ 6
5	1	1	-	-	-	-	-
6	4	3	1	-	-	-	-
7	21	16	4	-	-	1	-
8	132	87	23	11	5	-	7
9	984	619	236	68	28	9	24
10	14225	9092	3149	1059	448	206	271

Table 2: Number and cardinality of equisets of connected graphs on five to ten vertices

n	Equisets	2	3	4	5	6	≥ 6
9	1	1	-	-	-	-	-
13	1	1	-	-	-	-	-
14	1	1	-	-	-	-	-
15	4	4	-	-	-	-	-
16	7	6	1	-	-	-	-
17	3	2	1	-	-	-	-
18	15	12	3	-	-	-	-
19	30	24	5	1	-	-	-
20	89	65	19	4	1	-	-
21	205	148	40	10	6	-	1
22	780	614	118	36	7	5	-

Table 3: Equisets among trees on nine to 22 vertices.

n	Equisets	2	3	4	5	6	≥ 6
9	1	1	-	-	-	-	-
13	0	-	-	-	-	-	-
14	1	1	-	-	-	-	-
15	1	1	-	-	-	-	-
16	2	2	-	-	-	-	-
17	0	0	0	-	-	-	-
18	2	1	1	-	-	-	-
19	11	11	5	1	-	-	-
20	28	20	4	3	1	-	-
21	75	52	14	7	2	-	-
22	226	181	32	9	2	2	-

Table 4: Equisets among chemical trees on nine to 22 vertices.

4 Two new families of equienergetic graphs

Let $n \geq 3$ be an integer, and let K_n denote the complete graph on n vertices. In this section we construct two families of pairs of equienergetic graphs on $2n$ vertices.

Definition 1 *Let H_1 and H_2 be two copies of the complete graph K_n with vertex sets $V(H_1) = \{v_1, \dots, v_n\}$ and $V(H_2) = \{u_1, \dots, u_n\}$. Graph $\Gamma(n,p)$, $1 \leq p \leq n-1$, is a graph obtained from the union of graphs H_1 and H_2 by adding edges $\{v_i u_i | i \in \{1, \dots, p\}\}$.*

Example 2 *Graphs $\Gamma(5,1)$ and $\Gamma(5,4)$ are shown in Figure 2.*

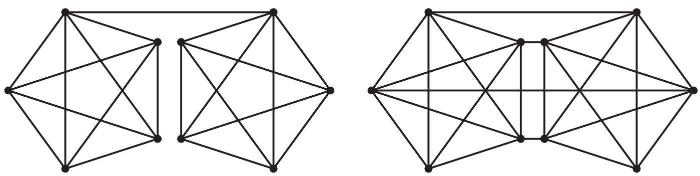

Figure 2: Graphs $\Gamma(5,1)$ and $\Gamma(5,4)$.

The next result is well known in matrix theory (see, e.g., [3]).

Lemma 3 *Let $S = \begin{pmatrix} A & B \\ C & D \end{pmatrix}$ where A and D are square matrices, and A is an invertible matrix. Then $\det(S) = \det(A)\det(D - CA^{-1}B)$. If A and C commute, then $\det(S) = \det(AD - BC)$.*

Theorem 4 *For $1 \leq p \leq n-1$, the characteristic polynomial of the adjacency matrix of $\Gamma(n,p)$ is equal to*

$$P_{\Gamma(n,p)}(\lambda) = \lambda^{p-1}(\lambda+1)^{2(n-p-1)}(\lambda+2)^{p-1}\left(\lambda^2 - (n-1)\lambda - p\right)\left(\lambda^2 - (n-3)\lambda - 2(n-1) - p\right). \tag{1}$$

Proof Let A and K be the adjacency matrices of the graphs $\Gamma(n,p)$ and K_n, respectively. Also, let us define C_n^p to be the square matrix of order n that has zeroes for all elements except for the first p diagonal elements, which are equal to 1. Then, $A = \begin{pmatrix} K & C_n^p \\ C_n^p & K \end{pmatrix}$, and

$$P_{\Gamma(n,p)}(\lambda) = \det(\lambda I_{2n} - A) = \begin{pmatrix} \lambda I_n - K & -C_n^p \\ -C_n^p & \lambda I_n - K \end{pmatrix}.$$

According to Lemma 3,

$$P_{\Gamma(n,p)}(\lambda) = \det(\lambda I_n - K)\det\left(\lambda I_n - K - C_n^p(\lambda I_n - K)^{-1}C_n^p\right). \tag{2}$$

Since $\lambda I_n - K$ has λ on the diagonal, and -1 in off-diagonal entries, we get

$$(\lambda I_n - K)^{-1} = \frac{1}{\det(\lambda I_n - K)} \begin{pmatrix} (\lambda+1)^{n-2}(\lambda - n + 2) & \cdots & (\lambda+1)^{n-2} \\ \vdots & \ddots & \vdots \\ (\lambda+1)^{n-2} & \cdots & (\lambda+1)^{n-2}(\lambda - n + 2) \end{pmatrix},$$

while the matrix $C_n^p(\lambda I_n - K)^{-1}C_n^p$ has the form

$$\begin{pmatrix} (\lambda I_n - K)^{-1}_{p\times p} & 0_{p\times(n-p)} \\ 0_{(n-p)\times p} & 0_{(n-p)\times(n-p)} \end{pmatrix},$$

where $(\lambda I_n - K)^{-1}_{p\times p}$ is a principal submatrix of order p of the matrix $(\lambda I_n - K)^{-1}$, and $0_{p\times q}$ is the zero matrix of order $p \times q$.

Replacing this and $\det(\lambda I_n - K) = P_{K_n}(\lambda) = (\lambda+1)^{n-1}(\lambda - (n-1))$ in (2), we arrive to (1) after somewhat cumbersome matrix computations. ■

Preceding theorem enables us to establish the spectrum of $\Gamma(n,p)$:

$$Sp(\Gamma(n,p)) = [-2^{(p-1)}, -1^{(2(n-p-1))}, 0^{(p-1)}, \frac{1}{2}(n-1\pm\sqrt{(n-1)^2+4p}), \frac{1}{2}(n-3\pm\sqrt{(n+1)^2-4p})],$$

where parenthesized exponents denote multiplicity. Multiplicities of eigenvalues obviously depend on p, so, for fixed n, no two graphs from the set $\{\Gamma(n,p) | p \in \{1,\ldots,n-1\}\}$ are cospectral.

However, the energy of $\Gamma(n,p)$ is

$$E(\Gamma(n,p)) = 2n - 4 + \sqrt{(n-1)^2+4p} + \sqrt{(n+1)^2-4p},$$

so that

$$\begin{aligned} E(\Gamma(n,n-p)) &= 2n - 4 + \sqrt{(n-1)^2+4(n-p)} + \sqrt{(n+1)^2-4(n-p)} \\ &= 2n - 4 + \sqrt{(n+1)^2-4p} + \sqrt{(n-1)^2+4p)} \\ &= E(\Gamma(n,p)). \end{aligned}$$

Thus, we have proved

Theorem 5 *For $1 \le p < \lceil \frac{n}{2} \rceil$, $\Gamma(n,p)$ and $\Gamma(n,n-p)$ are equienergetic connected graphs.*

For $p = 0$, graphs $\Gamma(n,0)$ and $\Gamma(n,n)$ are also equienergetic with energy $E(\Gamma(n,0)) = E(\Gamma(n,n)) = 4(n-1)$, but note that $\Gamma(n,0)$ is disconnected graph.

Now we give another family of pairs of equienergetic graphs.

Definition 6 *Let H_1 and H_2 be two copies of the complete graph K_n with vertex sets $V(H_1) = \{v_1,\ldots,v_n\}$ and $V(H_2) = \{u_1,\ldots,u_n\}$. Graph $\Theta(n)$ is a graph obtained from the union of graphs H_1 and H_2 by joining each vertex v_i to vertices in $\{u_1,\ldots,u_n\} \setminus \{u_i\}$, $i = 1,\ldots,n$.*

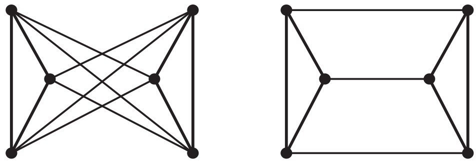

Figure 3: Graphs $\Theta(3)$ and $\Gamma(3,3)$.

Example 7 *Graphs* $\Theta(3)$ *and* $\Gamma(3,3)$ *are shown in Figure 3.*

Theorem 8 *For* $n \geq 3$*, graphs* $\Theta(n)$ *and* $\Gamma(n,n)$ *are equienergetic.*

Proof Let C_{2n} be an adjacency matrix of $\Theta(n)$. Then

$$\lambda I_{2n} - C_{2n} = \begin{pmatrix} A_n & B_n \\ B_n & A_n \end{pmatrix},$$

where

$$A_n = \begin{pmatrix} \lambda & -1 & -1 & \dots & -1 \\ -1 & \lambda & -1 & \dots & -1 \\ -1 & -1 & \lambda & \dots & -1 \\ & \vdots & & \ddots & \\ -1 & -1 & -1 & \dots & \lambda \end{pmatrix}, B_n = \begin{pmatrix} -1 & -1 & \dots & -1 & 0 \\ -1 & -1 & \dots & 0 & -1 \\ & \vdots & \ddots & \vdots & \\ -1 & 0 & \dots & -1 & -1 \\ 0 & -1 & \dots & -1 & -1 \end{pmatrix}.$$

Since matrices A_n and B_n commute, we can apply Lemma 3 to get

$$\det(C_{2n}) = \det(A_n^2) - \det(B_n^2).$$

Next,

$$A_n^2 - B_n^2 = \begin{pmatrix} \lambda^2 & -2\lambda & \dots & -2\lambda \\ -2\lambda & \lambda^2 & \dots & -2\lambda \\ & \vdots & \ddots & \vdots \\ -2\lambda & -2\lambda & \dots & \lambda^2 \end{pmatrix},$$

and we get

$$\det(C_{2n}) = \lambda^n(\lambda+2)^{n-1}(\lambda - 2(n-1)).$$

Thus, we get equality

$$E(\Theta(n)) = 4(n-1) = E(\Gamma(n,n)),$$

showing that $\Theta(n)$ and $\Gamma(n,n)$ are indeed equienergetic. ■

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