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THE ENERGY AND AN APPROXIMATION TO ESTRADA INDEX OF SOME TREES

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Abstract

The Bethe trees are unweighted rooted trees of k levels whose root vertex has degree d, the vertices from 2 to level $k-1$ have degree $d+1$ and the vertices at level k have degree 1. This paper gives an explicit formula for the energy and an approximation to the Estrada index of Bethe trees. We also obtain an explicit formula for the energy and an approximation for the Estrada index of double Bethe trees, that are unions of two Bethe trees with common root vertex.

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Introduction

Let G be an unweighted graph with n vertices. If we label the vertices of G by $1,\dots,n$, then the adjacency matrix of G, associated to this labelling is given by $A(G) = (a_{i,j})$ where $a_{i,j} = 1$ if the vertex i is connected, by an edge of G, to the vertex j and $a_{i,j} = 0$ otherwise. A permutation in the labelling of vertices generates similar adjacency matrices. These are symmetric, nonnegative matrices and their eigenvalues are known as the eigenvalues of the graph G. The concept of the energy of G was introduced by Ivan Gutman [1, 2] for approximating the total π-electron energy of a molecule. This concept, associated to the eigenvalues $\lambda_1 \le \cdots \le \lambda_n$ of G is defined as

$$E(G) = \sum_{j=1}^{n} |\lambda_j| \ .$$

Vladimir Nikiforov [3] defines the energy of a matrix M (square or not) as the sum of its singular values. We recall that if M is a symmetric matrix, then its singular values and the modulus of its eigenvalues coincide. Let M be a matrix with eigenvalues $\lambda_1(M) \le \cdots \le \lambda_k(M)$. Then its exponential matrix, $\exp(M)$ [4] is such that

$$E((\exp(M)) = \sum_{j=1}^{k} \exp(\lambda_j(M)) \ .$$

Ernesto Estrada [5] has introduced the molecular structure descriptor known as Estrada index. This concept, also associated to the eigenvalues of a graph G, is given by

$$EE(G) = \sum_{j=1}^{n} \exp(\lambda_j) \ .$$

Consequently, the Estrada index of a square matrix M, denoted by $EE(M)$, can be defined as

$$EE(M) := E(\exp(M)) \ .$$

Thus the Estrada index of a graph G is the Estrada index of its adjacency matrix.

The Bethe tree $B_{d,k}$ is an unweighted rooted tree of k levels whose root vertex has degree d, the vertices from 2 to level $k-1$ have degree $d+1$, and the vertices at level k have degree 1. Heilmann and Lieb [6] have obtained a decomposition of the matching polynomial of $B_{d,k}$. In trees the matching and characteristic polynomials coincide. In this paper we use this decomposition for obtaining the energy and an approximation to the Estrada index of $B_{d,k}$.

The double Bethe tree B_{d,k_1,k_2} is the union of B_{d,k_1} and B_{d,k_2} , which have only a common vertex, the root vertex.

Ivan Gutman [7] has determined decompositions of matching and characteristic polynomials for a class of compound graphs. Oscar Rojo [8] gives the explicit result for the compound graph B_{d,k_1,k_2} . With these results we obtain the energy and an approximation for the Estrada index of above specified trees. Ivan Gutman and Ante Graovac [9], approximate the Estrada index for the n-cycle C_n and the n-vertex path, P_n . They obtain $EE(P_n) \approx (n+1)\mathbf{I}_0 - \cosh(2)$, where

$$\mathbf{I}_0 = \sum_{\ell=0}^{\infty} \frac{1}{(\ell!)^2} = 2.27958530\ldots \ .$$

Our results are obtained by using similar technics.

This paper has four sections. In the first, we continue recalling some results from Complex Variable Theory in addition to Dirichlet's kernel. In the second section, the main results are

Theorem 1. *For the matrix M_r given by Eq. (9),*

$$E(M_r) = 2a\left(\frac{\sin\left(\left\lfloor \frac{r}{2}\right\rfloor + \frac{1}{2}\right)\frac{\pi}{r+1}}{\sin\frac{\pi}{2(r+1)}} - 1\right) \ .$$

In particular. for the path P_n ,

$$E(P_n) = 2\left(\frac{\sin\left(\left\lfloor \frac{n}{2}\right\rfloor + \frac{1}{2}\right)\frac{\pi}{n+1}}{\sin\frac{\pi}{2(n+1)}} - 1\right) \ .$$

Moreover

$$EE(M_r) \approx (r+1)\sum_{\ell=0}^{\infty} \frac{a^{2\ell}}{(\ell!)^2} - \cosh(2a) \ .$$

We observe that in particular for P_n

$$EE(P_n) \approx (n+1)\sum_{\ell=0}^{\infty} \frac{1}{(\ell!)^2} - \cosh(2) \, .$$

which has been proven in [9].

In the third section we obtain:

Theorem 2. *For Bethe tree $B_{d,k}$*

$$E(B_{d,k}) = E(S_k) + \sum_{j=1}^{k-1} d^{k-j-1} (d-1) E(S_j)$$

where for all $j = 1, \ldots, k$, the energy $E(S_j)$ is given by Eq. (23).

Moreover

$$EE(B_{d,k}) = EE(S_k) + \sum_{j=1}^{k-1} d^{k-j-1} (d-1) EE(S_j)$$

where for $j = 1, \ldots, k$, the Estrada index $EE(S_j)$ is approximated in Eq. (24).

Finally in the fourth section we prove:

Theorem 3. *For the tree B_{d,k_1,k_2}*

$$E(B_{d,k_1,k_2}) = E(X) + \sum_{j=1}^{k_1-1} d^{k_1-j-1} (d-1) E(S_{1,j}) + \sum_{j=1}^{k_2-1} d^{k_2-j-1} (d-1) E(S_{2,j})$$

where for $i = 1, 2$ and $j = 1, \ldots, k_i - 1$, the energies $E(S_{i,j})$ and $E(X)$ are given by Eqs. (31) and (33), respectively. Moreover,

$$\begin{aligned} EE(B_{d,k_1,k_2}) &= EE(X) + \sum_{j=1}^{k_1-1} d^{k_1-j-1} (d-1) EE(S_{1,j}) \\ &+ \sum_{j=1}^{k_2-1} d^{k_2-j-1} (d-1) EE(S_{2,j}) \end{aligned}$$

where for $i = 1, 2$ and $j = 1, \ldots, k_i - 1$, the Estrada indices $EE(S_{i,j})$ and $EE(X)$ are given by Eqs. (32) and (34), respectively.

Let $\mathbf{C}$ be the complex number set. From now on we use the concept of complex analytical function [see [10]].

Lemma 4. *(Laurent Series Expansion) [10] Let $b \in \mathbf{C}$. Let g be analytic in the annulus $\{z \in \mathbf{C} : R_1 < |z - b| < R_2\}$. Then*

$$g(z) = \sum_{n=-\infty}^{n=\infty} b_n (z-b)^n$$

where for $R_1 < r_1 < r_2 < R_2$ the convergence is absolute and uniform over the closure of annulus $\{z \in \mathbf{C} : r_1 < |z-b| < r_2\}$ and also the coefficients b_n are given by

$$b_n = \frac{1}{2\pi i} \int\limits_{\gamma} \frac{g(z)}{(z-b)^{n+1}}\, dz \tag{1}$$

where γ is the circle $|z-b| = r$ for any r, $R_1 < r < R_2$. Moreover this series is unique.

Lemma 5. *Let $a > 0$ and $R > 1$. Let*

$$g(z) = \exp\left[a\left(z+z^{-1}\right)\right] + \exp\left[-a\left(z+z^{-1}\right)\right] .$$

Then g is analytic in the punctured disk $\{z \in \mathbf{C} : 0 < |z| < R\}$. Moreover, if

$$g(z) = \sum_{n=-\infty}^{n=\infty} b_n\, z^n$$

is its Laurent expansion in the above disk, then

$$b_0 = \frac{1}{2\pi i} \int\limits_{\gamma} \frac{g\left(z\right)}{z} dz = 2 \sum_{\ell=0}^{\infty} \frac{a^{2\ell}}{\left(\ell!\right)^2} \tag{2}$$

where γ is the circle $|z| = 1$.

Proof. Let $0 < r < R$ and let γ be the circle $|z| = 1$. If we use $\exp(z) = \sum\limits_{n=1}^{\infty} z^n/n!$, then the exponential expansion in $\mathbf{C}$ and the Newton expansion for $\left(z+z^{-1}\right)^{2\ell}$ for each term of $g(z)$ yield the following expansion in the punctured disk $\{z \in \mathbf{C} : 0 < |z| < R\}$

$$\sum_{\ell=0}^{\infty} \sum_{j=0}^{2\ell} \frac{2a^{2\ell}}{(2\ell!)} \frac{(2\ell)!}{(2\ell-j)!j!}\, z^{2\ell-2j} . \tag{3}$$

We observe that Eq. (3) is a Laurent series expansion of g. As a consequence of the above Lemma, we have its uniqueness, and by searching the constant term we see that Eq. (2) is implied by Eq. (1). ■

Corollary 6. Let $J = \int\limits_0^{\frac{\pi}{2}} \left[\exp\left(2a\cos x\right) + \exp\left(-2a\cos x\right)\right]\, dx$. Then

$$J = \pi \sum_{\ell=0}^{\infty} \frac{a^{2\ell}}{\left(\ell!\right)^2} . \tag{4}$$

Proof. Let γ be the circle $|z| = 1$. Then $\gamma(x) = e^{ix}$, $x \in [0, 2\pi]$ is a parametrization of γ. Hence

$$\frac{1}{2\pi i}\int_{\gamma}\frac{g(z)}{z}dz = \frac{1}{2\pi}\int_{0}^{2\pi}\left[\exp(2a\cos x) + \exp(-2a\cos x)\right]\, dx \;. \tag{5}$$

For an appropriate change of variable,

$$\int_{0}^{2\pi}\left[\exp(2a\cos x) + \exp(-2a\cos x)\right]\, dx = 4\int_{0}^{\frac{\pi}{2}}\left[\exp(2a\cos x) + \exp(-2a\cos x)\right]\, dx \;.$$

Replacing this in Eq. (5) and by using Eq. (2), the result follows.

Let $L > 0$, concerning the convergence of Fourier series of a $2L-$periodic function we find the function

$$D(L, n, x) = \frac{1}{L}\left(\frac{1}{2} + \sum_{k=1}^{n}\cos\frac{k\pi x}{L}\right) \tag{6}$$

known as Dirichlet's kernel. ■

Lemma 7. *For* $x \neq 0, \pm 2L, \pm 4L, \ldots$,

$$D(L, n, x) = \frac{1}{2L}\frac{\sin(n+\frac{1}{2})\frac{\pi x}{L}}{\sin\frac{\pi x}{2L}} \;.$$

Proof. This proof can be found in [11]. ■

¿From this Lemma and by taking L as a positive integer and $x = 1$ in Eq. (6) we obtain

$$\sum_{k=1}^{n} 2\,\cos\frac{k\pi}{L} = \frac{\sin\left(n+\dfrac{1}{2}\right)\dfrac{\pi}{L}}{\sin\dfrac{\pi}{2L}} - 1 \;. \tag{7}$$

2. The energy and an approximation for the Estrada index of certain matrices

Let $a > 0$. Consider the tridiagonal symmetric $r \times r$ matrices

$$F_r = \begin{bmatrix} 0 & 1 & & & \\ 1 & 0 & \ddots & & \\ & \ddots & \ddots & \ddots & \\ & & \ddots & 0 & 1 \\ & & & 1 & 0 \end{bmatrix} \tag{8}$$

and

$$M_r = a\, F_r \ . \tag{9}$$

Observe that F_n is the adjacency matrix of the path P_n. It is well known that

$$\lambda_\ell\,(M_r) = 2a\,\cos\frac{(r+1-\ell)\pi}{r+1}\ ,\ j = 1,\ldots,r \tag{10}$$

and that these eigenvalues satisfy the relations

$$\lambda_\ell(M_r) = -\lambda_{r+1-\ell}(M_r)\ ,\ \ell = 1,\ldots,r\ .$$

Moreover, if r is an odd number, then $\lambda_{(r+1)/2}(M_r) = 0$.

We denote by m_r or by $\lfloor r/2 \rfloor$ the greatest integer less than or equal to $r/2$.

Let $E(M_r)$ and $EE(M_r)$ denote, respectively, the energy and the Estrada index of the matrix M_r. Therefore by Eq. (10),

$$E(M_r) = \sum_{\ell=1}^{r}\left|2a\cos\frac{\ell\pi}{r+1}\right|\ . \tag{11}$$

If

$$\upsilon\,(r) = \sum_{\ell=1}^{m_r}\left[\exp\left(2a\cos\frac{\pi\ell}{r+1}\right) + \exp\left(-2a\cos\frac{\pi\ell}{r+1}\right)\right] \tag{12}$$

then it is clear that

$$EE\,(M_r) = \begin{cases} \upsilon\,(r) & \text{if } r \text{ is an even number,} \\ 1+\upsilon\,(r) & \text{if } r \text{ is an odd number} \end{cases} \tag{13}$$

Theorem 8. *For the matrix M_r in Eq. (9),*

$$E(M_r) = 2a\left(\frac{\sin\left(m_r+\frac{1}{2}\right)\frac{\pi}{r+1}}{\sin\frac{\pi}{2(r+1)}} - 1\right)\ .$$

In particular, for the path P_n ,

$$E(P_n) = 2\left(\frac{\sin\left(\left\lfloor \frac{n}{2}\right\rfloor + \frac{1}{2}\right)\frac{\pi}{n+1}}{\sin\frac{\pi}{2(n+1)}} - 1\right) .$$

Proof. aking into account Eq. (10) for the eigenvalues, we can rewrite Eq. (11) as

$$E(M_r) = 2a\sum_{\ell=1}^{m_r} 2\cos\frac{\ell\pi}{r+1} . \tag{14}$$

Finally we observe that the sum in Eq. (14) can be obtained from Eq. (7) when we exchange n by m_r and L by $r+1$. Thus the result is proved. ■

Lemma 9. *Let $\upsilon(r)$ defined in Eq. (12). Then*

$$\upsilon(r) \approx \begin{cases} (r+1)\sum_{\ell=0}^{\infty}\frac{a^{2\ell}}{(\ell!)^2} - \cosh(2a) & \text{if } r \text{ is aneven number,} \\ \\ (r+1)\sum_{\ell=0}^{\infty}\frac{a^{2\ell}}{(\ell!)^2} - \cosh(2a) - 1 & \text{if } r \text{ is an odd number.} \end{cases}$$

Proof. Let $r = 2p$ and J be the integral defined in Corollary 6. Let

$$P = \left\{0, \frac{\pi}{r+1}, \frac{2\pi}{r+1}, \cdots, \frac{p\pi}{r+1}\right\}$$

be a partition of the interval $I = \left[0, \frac{\pi}{2}\right]$. In addition, we consider the function $f(x) = \exp(2a\cos x) + \exp(-2a\cos x)$. Thus

$$f(0) = 2\cosh(2a) \qquad \text{and} \qquad f(\pi/2) = 2 .$$

For f defined on I let consider the following Riemann sums

$$R(P, f) = \sum_{\ell=0}^{p}\frac{\pi}{r+1} f\left(\frac{\pi\ell}{r+1}\right) \tag{15}$$

and

$$R_1(P, f) = \sum_{\ell=1}^{p}\frac{\pi}{r+1} f\left(\frac{\pi\ell}{r+1}\right) = R(P, f) - \frac{\pi}{r+1} f(0) . \tag{16}$$

It holds that $R(P, f) \approx J$ and $R_1(P, f) \approx J$. On the other hand, Eq. (12) implies

$$\upsilon(r) = \frac{r+1}{\pi}\left[\frac{1}{2} R_1(P, f) + \frac{1}{2} R_1(P, f)\right] \tag{17}$$

and therefore

$$\begin{aligned} v(r) &= \frac{r+1}{\pi}\left[\frac{1}{2}\left(R(P,f)-\frac{2\pi}{r+1}\cosh(2a)\right)+\frac{1}{2}\,R_1(P,f)\right] \\ &= \frac{r+1}{\pi}\left[\frac{1}{2}\left(R(P,f)+R_1(P,f)\right)-\frac{1}{2}\,\frac{2\pi}{r+1}\,\cosh(2a)\right] \\ &\approx \frac{r+1}{\pi}\,J-\cosh(2a)\ . \end{aligned}$$

Finally, we use Eq. (4) in order to obtain the result. Let $r = 2p+1$ and let

$$P=\left\{0,\frac{\pi}{r+1},\cdots,\frac{p\pi}{r+1},\frac{(p+1)\,\pi}{r+1}=\frac{\pi}{2}\right\}$$

be a partition of I. We consider the Riemann sums

$$R_p(P,f)=\sum_{\ell=1}^{p+1}\frac{\pi}{r+1}\,f\left(\frac{\pi\ell}{r+1}\right)$$

and also those defined in Eqs. (15) and (16). Clearly,

$$R_1(P,f)=R_p(P,f)-\frac{\pi}{r+1}\,f\left(\frac{\pi}{2}\right)\ . \tag{18}$$

By the relations in Eqs. (16), (17), and (18) we obtain:

$$\begin{aligned} v(r) &= \frac{r+1}{\pi}\left[\frac{1}{2}\left(R(P,f)-\frac{\pi}{r+1}\,f(0)\right)+\frac{1}{2}\left(R_p(P,f)-\frac{\pi}{r+1}\,f\left(\frac{\pi}{2}\right)\right)\right] \\ &= \frac{r+1}{\pi}\left[\frac{1}{2}\left(R(P,f)+R_p(P,f)\right)-\frac{1}{2}\,\frac{\pi}{r+1}\,f(0)-\frac{1}{2}\,\frac{\pi}{r+1}\,f\left(\frac{\pi}{2}\right)\right] \\ &\approx \frac{r+1}{\pi}\,J-\cosh(2a)-1\ . \end{aligned}$$

Finally we use Eq. (4) in order to obtain the result. ∎

Lemma 9 and the equality in Eq. (13) imply the following result.

Theorem 10. *For the matrix M_r in Eq. (9),*

$$EE\left(M_r\right)\approx(r+1)\sum_{\ell=0}^{\infty}\frac{a^{2\ell}}{(\ell!)^2}-\cosh\left(2a\right). \tag{19}$$

We observe that in particular for P_n,

$$EE(P_n) \approx (n+1) \sum_{\ell=0}^{\infty} \frac{1}{(\ell!)^2} - \cosh(2)$$

which has been proven in [9].

In view of the Riemann sums, formula (19) gives better accuracy for $r > 8$.

3. The energy and an approximation for the Estrada index of the Bethe tree $B_{d,k}$

In this section we obtain an approximation for the Estrada index of the Bethe tree $B_{d,k}$. In [6] a decomposition of the matching polynomial of $B_{d,k}$ has been given as an example. In trees the matching and characteristic polynomials coincide. We can rewrite this result as follows.

Theorem 11. *The characteristic polynomial of the Bethe tree $B_{d,k}$ has the following decomposition:*

$$\det(\lambda I - A(B_{d,k})) = P_k(\lambda) \prod_{j=1}^{k-1} P_j^{n_j}(\lambda) \tag{20}$$

where for all $j = 1, \ldots, k$, if F_j is as in Eq. (8), $S_j = \sqrt{d} F_j$,

$$P_j(\lambda) = \det(\lambda I - S_j) \tag{21}$$

and

$$n_j = d^{k-j-1}(d-1) \ . \tag{22}$$

As an immediate consequence, by taking $a = \sqrt{d}$ and $r = j$ in Theorems 8 and 10 we obtain the following expression for $E(S_j)$ and an approximation to $EE(S_j)$.

Theorem 12. *For the matrices S_j, $j = 1, \ldots, k$, in Theorem 11,*

$$E(S_j) = 2d^{1/2} \left(\frac{\sin\left(m_j + \frac{1}{2}\right) \frac{\pi}{j+1}}{\sin \frac{\pi}{2(j+1)}} - 1 \right) \ . \tag{23}$$

Moreover,

$$EE(S_j) \approx (j+1) \sum_{\ell=0}^{\infty} \frac{d^{\ell}}{(\ell!)^2} - \cosh\left(2\sqrt{d}\right) \ . \tag{24}$$

¿From these facts and Eqs. (20), (21), and (22), we have:

Theorem 13. *For the Bethe tree* $B_{d,k}$ *,*

$$E\left(B_{d,k}\right) = E\left(S_k\right) + \sum_{j=1}^{k-1} d^{k-j-1}\left(d-1\right) E\left(S_j\right) \tag{25}$$

where for all $j = 1, \ldots, k$ *, the energy* $E(S_j)$ *is given by Eq. (23). Moreover,*

$$EE(B_{d,k}) = EE(S_k) + \sum_{j=1}^{k-1} d^{k-j-1}\left(d-1\right) EE(S_j) \tag{26}$$

where for $j = 1, \ldots, k$ *, the Estrada index* $EE(S_j)$ *is approximated in Eq. (24).*

Example 14. *Consider the binary tree* $B_{2,5}$ *given in Fig. 1.*

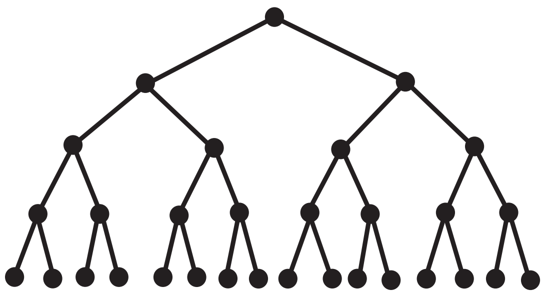

Fig. 1. *The Bethe tree* $B_{2,5}$ *.*

From Theorem 11 we obtain $E(B_{2,5}) = 33.366$ *and* $EE(B_{2,5}) = 72.2716$ *, and by Eq. (25),* $E(B_{2,5}) = 11.3137 + 8 + 6.3246 + 7.7279 = 33.366$.

On the other hand, let

$$s = s(2) = \sum_{\ell=0}^{10} \frac{2^{\ell}}{\left(\ell!\right)^2} = 4.2524$$

and $c = \cosh(2\sqrt{2}) = 8.4890$. Clearly $EE(S_1) = 1$ and $EE(S_2) = 2\cosh(\sqrt{2}) = 4.356\,4$. Then by an adaptation of Eq. (26),

$$EE(B_{2,5}) \approx 8 + 4\,(4.356\,4) + 2\,(8.5204) + 12.773 + 17.0254 = 72.265.$$

4. The energy and an approximation for the Estrada index of the tree B_{d,k_1,k_2}

In this section we obtain an approximation for the Estrada index of the double Bethe tree B_{d,k_1,k_2}. Let $A(B_{d,k_1,k_2})$ be the adjacency matrix of this tree. From [8], Theorem 4.a, we derive the following:

Theorem 15. *The characteristic polynomial of the double Bethe tree B_{d,k_1,k_2} has the following decomposition:*

$$\det\left(\lambda I - A\left(B_{d,k_1,k_2}\right)\right) = P\left(\lambda\right) \prod_{j=1}^{k_1-1} P_{1,j}^{n_{1,j}}\left(\lambda\right) \prod_{j=1}^{k_2-1} P_{2,j}^{n_{2,j}}\left(\lambda\right) \tag{27}$$

where for $i = 1, 2$ and $j = 1, \ldots, k_i$, if F_j is as in Eq. (8), then $S_{i,j} = \sqrt{d}\, F_j$ and $X = \sqrt{d}\, F_{k_1+k_2-1}$, as well as

$$P_{i,j}\left(\lambda\right) = \det\left(\lambda I - S_{i,j}\right) \tag{28}$$

$$P\left(\lambda\right) = \det\left(\lambda I - X\right) \tag{29}$$

$$n_{i,j} = d^{k_i-j-1}\left(d-1\right)\,. \tag{30}$$

¿From this result and by setting $a = \sqrt{d}$, $r = j$ and $r = k_1 + k_2 - 1$ into Theorems 8 and 10, we obtain below expressions $E(S_{i,j})$ and $E(X)$, in addition to the approximations for $EE(S_j)$ and $EE(X)$, respectively.

Theorem 16. *For matrices $S_{i,j}$ $i = 1, 2$, $j = 1, \ldots, k_i - 1$, and the matrix X in Theorem 15,*

$$E(S_{i,j}) = 2\,d^{1/2}\left(\frac{\sin\left(\left\lfloor \frac{j}{2}\right\rfloor + \frac{1}{2}\right)\frac{\pi}{j+1}}{\sin\frac{\pi}{2(j+1)}} - 1\right) \tag{31}$$

and

$$EE(S_{i,j}) \approx (j+1) \sum_{\ell=0}^{\infty} \frac{d^\ell}{(\ell!)^2} - \cosh\left(2\sqrt{d}\right) \tag{32}$$

$$E\left(X\right) = 2\, d^{1/2} \left(\frac{\sin\left(\left\lfloor \frac{k_1+k_2-1}{2} \right\rfloor + \frac{1}{2}\right) \frac{\pi}{k_1+k_2}}{\sin \frac{\pi}{2(k_1+k_2)}} - 1 \right) \tag{33}$$

and

$$EE\left(X\right) \approx (k_1+k_2) \sum_{\ell=0}^{\infty} \frac{d^\ell}{(\ell!)^2} - \cosh\left(2\sqrt{d}\right) \ . \tag{34}$$

¿From these facts, and Eqs. (27), (28), (29), and (30), we obtain,

Theorem 17. *For the tree* B_{d,k_1,k_2} *,*

$$E(B_{d,k_1,k_2}) = E(X) + \sum_{j=1}^{k_1-1} d^{k_1-j-1}\,(d-1)\,E(S_{1,j}) + \sum_{j=1}^{k_2-1} d^{k_2-j-1}\,(d-1)\,E\left(S_{2,j}\right) \tag{35}$$

where for $i = 1, 2$ *and* $j = 1, \ldots, k_i - 1$ *, the energies* $E(S_{i,j})$ *and* $E(X)$ *are given by Eqs. (31) and (33), respectively. Moreover,*

$$\begin{aligned} EE(B_{d,k_1,k_2}) &= EE(X) + \sum_{j=1}^{k_1-1} d^{k_1-j-1}\,(d-1)\,EE(S_{1,j}) \\ &+ \sum_{j=1}^{k_2-1} d^{k_2-j-1}\,(d-1)\,EE(S_{2,j}) \end{aligned} \tag{36}$$

where for $i = 1, 2$ *and* $j = 1, \ldots, k_i - 1$ *, the Estrada indices* $EE(S_{i,j})$ *and* $EE(X)$ *are referred in Eqs. (32) and (34), respectively.*

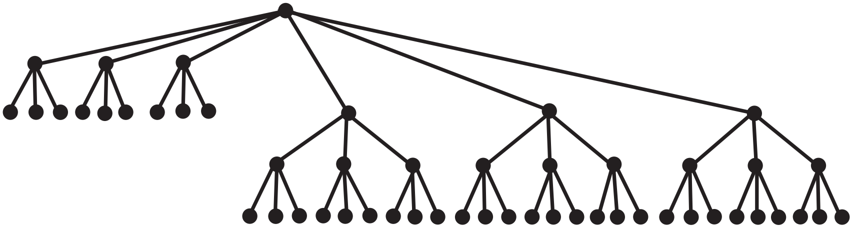

Fig. 2. The double Bethe tree $B_{3,3,4}$.

For the double Bethe tree $B_{3,3,4}$ given in Fig. 2, we obtain by means of a direct computation that its energy is equal to 49.6148 and its Estrada index is equal to 130.1 . The energy obtained with formula (35) is equal to 49.6142 . In the case of the Estrada, our adaptation of Eq. (36) gives $EE(B_{3,3,4}) \approx 130.0496$.

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