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A Note on Wiener Index

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Abstract

The Wiener index of a connected graph is equal to the sum of distances between all pairs of its vertices. In this paper, we find a lower bound for the Wiener index in terms of graph invariants.

1 Introduction

The Wiener index of a graph was introduced by H.Wiener([1]). He observed a relationship between the boiling point of paraffin and the Wiener index. For a vertex v of a connected graph G, let $d_i(v)$ be the number of vertices at distance i from v. Let $dds(v) = (d_0(v), d_1(v), \ldots\ldots, d_{n-1}(v))$. It is called the distance degree sequence of a vertex v. This paper gives a lower bound of W(G) in terms of graph invariants by considering the distance degree of vertices.

2 Results

2.1 Definitions

Given a connected simple graph $G = (V, E)$, $(\mid V \mid = n, \mid E \mid = m)$. The distance $d(v, w)$ between two vertices v and w is the minimum length of the paths connecting them.

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For an integer i and a vertex v, let $N_i(v) = \{w \in V \mid d(v,w) = i\}$, $d_i(v) =| N_i(v) |$, $(d_0(v) := 1)$. Then, $dds(v) = (d_0(v), d_1(v), d_2(v), \ldots, d_{n-1}(v))$ is called the distance degree sequence of a vertex v. We let $\Delta_i = \max_v\{d_i(v)\}$, and $\delta_i = \min_v\{d_i(v)\}$ $e(v) = \max_w\{d(v,w)\}$. The radius and the diameter of G ($r = r(G)$ and $d = d(G)$) are defined as follows, $r = r(G) = \min_v\{e(v)\}$, $d = d(G) = \max_v\{e(v)\}$, respectively. Write D_i for the number of unordered pairs of vetices whose distance is i.

A nondecreasing sequence S: $a_1, a_2, \ldots, a_n$ of nonnegative integers is called an eccentric sequence if there exists a connected graph G whose vertices can be labelled $v_1, v_2, \ldots, v_n$ so that $e(v_i) = a_i$ for all i. The vertex connectivity of a connected graph G, denoted by $\kappa(G)$, is the minimum number of vertices whose removal can either disconnect G or reduce it to a 1-vertex graph. A graph G is k-connected if G is connected and $k \le \kappa(G)$. Finally, the Wiener index of G, denoted by $W(G)$, is defined as follows.

$$W(G) = \sum_{v,w \in V} d(v,w). \qquad (v \text{ and } w \text{ are unordered})$$

Observation 1. For any vertex $v \in V$,

$$\sum_{i=0}^{e(v)} d_i(v) = n .$$

Observation 2.

$$\sum_{v} d_i(v) = 2D_i .$$

Observation 3.

$$W(G) = \sum_{i=1}^{d} iD_i .$$

2.2 Lemmas

Lemma 1.

$$n\delta_i \le 2D_i \le n\Delta_i \qquad (1 \le i \le d)$$

Proof. This follows from, $2D_i = \sum_v d_i(v)$ (see Observation 2).

Lemma 2. ([8]) Let G be k-connected. Then, for any vertex $v \in V$,

$$k \le d_i(v). \qquad (1 \le i \le e(v)-1)$$

Lemma 3. ([9])

Suppose a nondecreasing sequence $a_1,\ a_2,\ .\ .\ .\ ,\ a_n$ $(n \ge 2)$ is eccentric. Let h is any integer with $a_1 < h \le a_n$, then $a_i = a_{i+1} = h$ for some i $(2 \le i \le n-1)$.

2.3 Result

Theorem. Let G be k-connected. Then the Wiener index of G is at least

$$\frac{nr}{2}\left(\frac{k(r-1)}{2}+1\right)+\frac{d-r}{2}\left(2d+(d-r-1)\left(dk-1-\frac{k(2d-2r-1)}{3}\right)\right) \ .$$

Proof. From Lemma 3, there are at least two vertices $v_1,\ v_2$ such that $e(v_1) = e(v_2) = d$. And from Lemma 2, we have $d_i(v_1) \ge k,\ d_i(v_2) \ge k, \quad (1 \le i \le d-1)$. Also, from Lemma 3, there are at least two vertices $v_3,\ v_4$ (distinct from $v_1,\ v_2$) such that $e(v_3) = e(v_4) = d-1$, and we have $d_i(v_3) \ge k,\ d_i(v_4) \ge k \quad (1 \le i \le (d-1)-1)$. And so on. Thus from Observation 2, we have

$$D_d \ge 1, \quad D_{d-1} \ge 1+k, \quad D_{d-2} \ge 1+2k,\ .\ .\ .\ .$$
$$.\ .\ .\ .\ ,\ D_{r+1} = D_{d-(d-r-1)} \ge 1+(d-r-1)k.$$

Therefore

$$\begin{aligned}\sum_{i=r+1}^{d} i\,D_i &= (r+1)D_{r+1}+\cdots+dD_d \\ &\ge (r+1)(1+(d-r-1)k+\cdots+(d-1)(1+k)+d \\ &= \sum_{i=0}^{d-r-1}(d-i)(1+ik)\ .\end{aligned}$$

Thus

$$\begin{aligned}W(G) &= \sum_{i=1}^{d} iD_i = \sum_{i=1}^{r-1} iD_i + rD_r + \sum_{i=r+1}^{d} iD_i \\ &\ge \frac{nr}{2}(\frac{k(r-1)}{2}+1)+\sum_{i=r+1}^{d} iD_i \qquad (\text{since} \ \ D_r \ge \tfrac{n}{2}) \\ &= \frac{nr}{2}\left(\frac{k(r-1)}{2}+1\right) \\ &+ \frac{d-r}{2}\left(2d+(d-r-1)(dk-1-\frac{k(2d-2r-1)}{3})\right)\ .\end{aligned}$$

Remark: The lower bound of this Theorem is attained, if G is an even cycle.

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