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ON DISTANCE ENERGY OF GRAPHS

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Abstract

The D-eigenvalues of a graph G are the eigenvalues of its distance matrix D, and the D-energy $E_D(G)$ is the sum of the absolute values of its D-eigenvalues. Two graphs are said to be D-equienergetic if they have the same D-energy. In this note we obtain bounds for the distance spectral radius and D-energy of graphs of diameter 2. Pairs of equiregular D-equienergetic graphs of diameter 2, on $p = 3t + 1$ vertices are also constructed.

INTRODUCTION

Let G be a connected graph with vertex set $V(G) = \{v_1, v_2, \ldots, v_p\}$ and size (= number of edges) q. The distance matrix or D-matrix, D, of G is defined as

$D = [d_{ij}]$, where d_{ij} is the distance between the vertices v_i and v_j in G. The eigenvalues $\mu_1, \mu_2, \ldots, \mu_p$ of the D-matrix of G are said to be the D-eigenvalues of G and to form the D-spectrum of G, denoted by $spec_D(G)$.

Since the D-matrix of G is symmetric, all of its eigenvalues are real and can be ordered as $\mu_1 \geq \mu_2 \geq \cdots \geq \mu_p$. Two graphs G and H are said to be D-cospectral if $spec_D(G) = spec_D(H)$. The D-energy $E_D(G)$ of G is defined as

$$E_D(G) = \sum_{i=1}^{p} |\mu_i| \ . \tag{1}$$

Eq. (1) is put forward in full analogy to the definition of the (ordinary) graph energy E, namely

$$E(G) = \sum_{i=1}^{p} |\lambda_i| \tag{2}$$

where $\lambda_1, \lambda_2, \ldots, \lambda_p$ are the eigenvalues of the adjacency matrix of G. For basic facts on graph energy E see the book [11]; for the most recent research on E see [10,12,14–16,25,28,29,31,32].

Two graphs with the same D-energy are called D-equienergetic. We are, of course, interested in D-equienergetic graphs only if these are not D-cospectral.

The characteristic polynomial of the D-matrix and the corresponding spectrum were considered in [6–9,13,30]. The D-energy seems to be defined here for the first time.

In this paper we are concerned with the D-spectra and D-energies of graphs of diameter 2. Moore and Moser showed [3] that almost all graphs are of diameter two. Thus a discussion of graphs of small diameter pertains to almost all graphs.

This paper is organized as follows. In the next section we establish the distance spectrum of some graphs of diameter 2 and 3. In the following section a lower bound for the largest eigenvalue of D, and bounds for the D-energy are obtained. In the last section some pairs of equiregular D-equienergetic graphs of diameter 2 are constructed.

All graphs considered in this paper are simple. Our spectral graph theoretic terminology follows that of the book [4].

We shall need the following lemmas.

Lemma 1 [4]. Let G be a graph with an adjacency matrix A and $spec(G) = \{\lambda_1, \lambda_2, \ldots, \lambda_p\}$. Then $\det A = \prod_{i=1}^{p} \lambda_i$. In addition, for any polynomial $P(x)$, $P(\lambda)$ is an eigenvalue of $P(A)$ and hence $\det P(A) = \prod_{i=1}^{p} P(\lambda_i)$.

Lemma 2 [5]. Let

$$A = \begin{bmatrix} A_0 & A_1 \\ A_1 & A_0 \end{bmatrix}$$

be a symmetric 2×2 block matrix. Then the spectrum of A is the union of the spectra of $A_0 + A_1$ and $A_0 - A_1$.

Lemma 3 [4]. Let M, N, P, Q be matrices, and let M be invertible. Let

$$S = \begin{bmatrix} M & N \\ P & Q \end{bmatrix} .$$

Then $\det S = \det M \cdot \det [Q - PM^{-1}N]$. If M and P commute, then $\det S = \det[MQ - PN]$.

Lemma 4 [4]. Let G be an r-regular connected graph, $r \geq 3$, with $spec(G) = \{r, \lambda_2, \ldots, \lambda_p\}$. Then

$$spec(L(G)) = \begin{pmatrix} 2r-2 & \lambda_2 + r - 2 & \cdots & \lambda_p + r - 2 & -2 \\ 1 & 1 & \cdots & 1 & p(r-2)/2 \end{pmatrix} .$$

Lemma 5 [4]. Let G be an r-regular connected graph on p vertices with an adjacency matrix A, and let $r, \lambda_2, \ldots, \lambda_m$ be its distinct eigenvalues. Let J be the all-one square matrix of order p. Then there exists a polynomial $P(x)$ such that $P(A) = J$, and

$$P(x) = p \, \frac{(x - \lambda_2)(x - \lambda_3) \cdots (x - \lambda_m)}{(r - \lambda_2)(r - \lambda_3) \cdots (r - \lambda_m)}$$

so that $P(r) = p$ and $P(\lambda_i) = 0$, for all $\lambda_i \neq r$.

Lemma 6 [4,19]. For every $t \geq 3$, there exists a pair of non-cospectral cubic graphs on $2t$ vertices.

THE DISTANCE SPECTRUM OF SOME GRAPHS

In this section we calculate the distance spectrum of some graphs of diameter 2 or 3. The distance energy of some particular graphs are also obtained.

Graphs of diameter 2

Let G be a graph of diameter 2, A its adjacency matrix, and $\overline{A}$ the adjacency matrix of its complement $\overline{G}$. Then $d(u,v)=1$ if u adj v in G, and $d(u,v)=2$ if u adj v in $\overline{G}$. Thus the distance matrix of G is $A+2\overline{A}$.

Lemma 7. Let G be a (p,q)-graph of diameter 2, and let its D-eigenvalues be $\mu_1,\mu_2,\ldots,\mu_p$. Then

$$\sum_{i=1}^{p}\mu_i^2 = 2(2\,p^2-2p-3q)\ .$$

Proof. In the distance matrix D of G there are $2q$ elements equal to unity, and $p(p-1)-2q$ elements equal to two. Therefore,

$$\begin{aligned}\sum_{i=1}^{p}\mu_i^2 &= \sum_{i=1}^{p}(D^2)_{ii}=\sum_{i=1}^{p}\sum_{j=1}^{p}d_{ij}\,d_{ji}=\sum_{i=1}^{p}\sum_{j=1}^{p}(d_{ij})^2\\ &= (2q)\cdot 1^2+(p^2-p-2q)\cdot 2^2\end{aligned}$$

and the lemma follows. □

Theorem 1. Let G be an r-regular graph of diameter 2, and let its (ordinary) spectrum be $spec(G)=\{r,\lambda_2,\ldots,\lambda_p\}$. Then the D-spectrum of G is $spec_D(G)=\{2p-r-2,-(\lambda_2+2),\ldots,-(\lambda_p+2)\}$.

Proof. The theorem follows from the fact that the D-matrix of G is $A+2\overline{A}$ and from Lemma 5. □

Examples.

$$spec_D(K_{n,n})=\begin{pmatrix}3n-2 & n-2 & -2\\ 1 & 1 & 2n-2\end{pmatrix}$$

$$spec_D(CP(n))=\begin{pmatrix}2n & -2 & 0\\ 1 & n & n-1\end{pmatrix}$$

where $CP(n)$ denotes the $(2n)$-vertex regular graph of degree $2n-2$ (obtained by deleting n independent edges from the complete graph K_{2n}), sometimes referred to as the "cocktail party graph".

The graph product $G \times K_2$

Theorem 2. Let G be an r-regular graph of diameter 1 or 2 with an adjacency matrix A and $spec(G) = \{\lambda_1, \lambda_2, \ldots, \lambda_p\}$. Then $H = G \times K_2$ is $(r+1)$-regular and of diameter 2 or 3 with

$$spec_D(H) = \begin{pmatrix} 5p - 2(r+2) & -2\,(\lambda_i + 2) & -p & 0 \\ 1 & 1 & 1 & p-1 \end{pmatrix}, \quad i = 2, \ldots, p\,.$$

Proof. Since G is of diameter 1 or 2, its distance matrix is $A + 2\overline{A}$. Then the distance matrix of H is of the form

$$\begin{bmatrix} A + 2\overline{A} & A + 2\overline{A} + J \\ A + 2\overline{A} + J & A + 2\overline{A} \end{bmatrix}.$$

The theorem then follows by Lemma 2. $\square$

The wheel graph $W_{1,p}$ is defined as the join of p-vertex cycle C_p and K_1 [4].

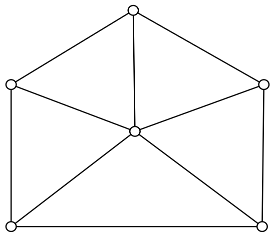

Figure 1: $W_{1,5} = C_5 \nabla K_1$

Theorem 3. The distance energy of the wheel graph is given by $E_D(W_{1,p}) = 2\left(p - 2 + \sqrt{p^2 - 3p + 4}\right)$.

Proof. Let A be an adjacency matrix of C_p with $spec(C_p) = \{2, \lambda_2, \lambda_3, \ldots, \lambda_p\}$. Then the distance matrix of the wheel graph can be written as

$$\begin{bmatrix} A + 2\overline{A} & J_{p\times 1} \\ J_{1\times p} & 0 \end{bmatrix}.$$

By Lemma 3,

$$spec_D\left(W_{1,p}\right) = \begin{pmatrix} p - 2 \pm \sqrt{p^2 - 3p + 4} & -\left(\lambda_i + 2\right) \\ 1 & 1 \end{pmatrix}, \quad i = 2, \ldots, p\,.$$

Since $\lambda_i + 2 > 0$ for all $i = 2, \ldots, p$, the theorem follows. $\square$

BOUNDS FOR THE SPECTRAL RADIUS AND DISTANCE ENERGY

Theorem 4. Let G be a (p,q)-graph of diameter 2 and μ_1 be its greatest D-eigenvalue. Then $\mu_1 \geq (2\,p^2 - 2q - 2p)/p$. Equality holds if and only if G is a regular graph.

Proof. Let G be a connected graph of diameter 2, and let its vertices be labelled as $v_1, v_2, \ldots, v_p$. Let d_i denote the degree of v_i. Then, as G is of diameter 2, it is easy to observe that the i-th row of D consists of d_i one's and $p - d_i - 1$ two's. Let $x = [1, 1, 1, \ldots, 1]$, the all one vector. Then by the Raleigh Principle

$$\mu_1 \geq \frac{x\,D\,x^T}{x\,x^T} = \frac{1}{p}\sum_{i=1}^{p}(2p - d_i - 2) = \frac{2\,p^2 - 2q - 2p}{p}\ .$$

If G is r-regular, then each row sum of D is equal to $2p - r - 2$ and hence $\mu_1 = 2p - r - 2$ and equality holds. Conversely, if equality holds then x is the eigenvector corresponding to μ_1 and this happens when all row sums of D are equal. Since the i-th row sum is equal to $2p - d_i - 2$, this occurs only when d_i has the same value for all i, i. e., only when G is regular. $\square$

The following theorem gives upper and lower bounds for the energy of graphs of diameter 2.

Theorem 5. Let G be a (p,q)-graph of diameter 2 and let Δ be the absolute value of the determinant of its distance matrix. Then

$$\sqrt{4p\,(p-1) - 6q + p\,(p-1)\,\Delta^{2/p}} \leq E_D(G) \leq \sqrt{2p\,(2\,p^2 - 3q - 2p)}\ .$$

Proof. This proof is fully analogous to what McClelland [24] has done in the case of the ordinary graph energy (see pp. 147-148 in the book [11]). In view of the definition (1) of D-energy and bearing in mind Lemma 7,

$$\begin{aligned} E_D^2 &= \left(\sum_{i=1}^{p}|\mu_i|\right)^2 = \sum_{i=1}^{p}\mu_i^2 + \sum_{i\neq j}|\mu_i|\ |\mu_j| \\ &= 4p\,(p-1) - 6q + \sum_{i\neq j}|\mu_i|\ |\mu_j|\ . \end{aligned} \tag{3}$$

By using the the inequality between the arithmetic and geometric means we have

$$\frac{1}{p(p-1)} \sum_{i \neq j} |\mu_i| \, |\mu_j| \;\geq\; \left(\prod_{i \neq j} |\mu_i| \, |\mu_j| \right)^{1/[p(p-1)]} = \left(\prod_{i \neq j} |\mu_i|^{2(p-1)} \right)^{1/[p(p-1)]}$$

$$= \prod_{i \neq j} |\mu_i|^{2/p} = \Delta^{2/p} \, . \qquad \Box \qquad (4)$$

Combining Equations (3) and (4) we arrive at the lower bound of Theorem 5.

By expanding $\sum_{i=1}^{p} \sum_{j=1}^{p} [|\mu_i| - |\mu_j|]^2$ and by taking into account (1), we obtain

$$p \sum_{i=1}^{p} \mu_i^2 - 2\, E_D(G)^2 + p \sum_{j=1}^{p} \mu_j^2$$

This expression is necessarily non-negative. The upper bound for E_D follows now from Lemma 7. $\Box$

Theorem 6. Let G be an r-regular graph of diameter 2. Then

$$E_D \leq 2p - r - 2 + \sqrt{(p-1)\,[p\,(r+4) - (r+2)^2]} \; .$$

Proof. Let G be an r-regular graph with p vertices and q edges. Then by Theorem 4, the greatest D-eigenvalue is $\mu_1 = 2p - r - 2$. By applying the Cauchy–Schwarz inequality to the two $p-1$ vectors $(1, 1, \dots, 1)$ and $(\mu_2, \mu_3, \dots, \mu_p)$ we get

$$\left(\sum_{i=2}^{p} |\mu_i| \right)^2 \leq (p-1) \sum_{i=2}^{p} \mu_i^2$$

i. e.,

$$\left(E_D - \mu_1\right)^2 \leq (p-1)\left(4\,p^2 - 6q - 4p - \mu_1^2\right)$$

i. e.,

$$E_D \leq \mu_1 + \sqrt{(p-1)\;(4\,p^2 - 6q - 4p - \mu_1^2)} \; .$$

Since $\mu_1 = 2p - r - 2$ and $2q = pr$, we have

$$E_D \leq 2p - r - 2 + \sqrt{(p-1)\;[p(r+4) - (r+2)^2]} \; . \qquad \Box$$

Theorem 7. For any graph G of diameter 2,

$$E_D \leq \frac{1}{p}\left[2\,p^2 - 2q - 2p + \sqrt{(p-1)\,\left[(2p+q)\,(2\,p^2-4q) - 4\,p^2\right]}\right] .$$

Proof. This proof follows the ideas of Koolen and Moulton [22,23], used for obtaining an analogous upper bound for the ordinary graph Energy E. By the Cauchy–Schwarz inequality we have

$$E_D \leq \mu_1 + \sqrt{(p-1)\,[4\,p^2 - 6q - 4p - \mu_1^2]} .$$

Define a function

$$f(x) := x + \sqrt{(p-1)(4\,p^2 - 6q - 4p - x^2)}$$

for

$$\frac{2\,p^2 - 2q - 2p}{p} \leq x \leq \sqrt{4\,p^2 - 6q - 4p}$$

Then $(2\,p^2 - 2q - 2p)/p \geq 1$ and hence $f(x)$ is a decreasing function for $2\,p^2 - 2q - 2p/p \leq x^2$. But $(2\,p^2 - 2q - 2p)/p \leq x \leq x^2$ as $x \geq 1$. Hence $f(x) \leq f((2\,p^2 - 2q - 2p)/p)$, proving the theorem. $\square$

ON A PAIR OF D-EQUIENERGETIC GRAPHS

The problem of constructing non-cospectral graph having equal energies E, Eq. (2), has been much discussed and numerous examples of this kind were put forward [1,2,17–21,25–28]. Such pairs of graphs are referred to as "equienergetic" (the name first time used in [2]). Motivated by this, in this section we discuss the construction of D-equienergetic graphs. We succeed to do this for every $p \equiv 1\ (mod 3)$ and $p \equiv 0\ (mod\ 6)$.

Evidently, two graphs G_1 and G_2 are said to be D-equienergetic if $E_D(G_1) = E_D(G_2)$.

The graph $G \nabla G$ is obtained by joining every vertex of G to every vertex of another copy of G.

Theorem 8. Let G be a connected r-regular graph on p vertices with $spec(G) = \{r, \lambda_2, \ldots, \lambda_p\}$. Then

$$spec_D(G\nabla G) = \begin{pmatrix} 3p-r-2 & p-r-2 & -2\,(\lambda_i+2) \\ 1 & 1 & 2 \end{pmatrix} \quad , \; i = 2, \ldots, p\,.$$

Proof. The distance matrix of $G\nabla G$ can be written as

$$\begin{bmatrix} A+2\overline{A} & J \\ J & A+2\overline{A} \end{bmatrix}.$$

Then the theorem follows from Lemma 2. $\square$

Theorem 9. For every $p \equiv 0\ (mod\ 6) \geq 18$, there exists a pair of D-equienergetic regular graphs.

Proof. Let $p = 6t$, $t \geq 3$. Let G_1 and G_2 be non-cospectral cubic graphs on $2t$ vertices as specified in Lemma 6. Then their line graphs $L(G_1)$ and $L(G_2)$ are 4-regular on $3t$ vertices. By Lemma 4, the only positive D-eigenvalues of $L(G_1)\nabla L(G_1)$ are $9t-6$ and $3t-6$. The same is true for $L(G_2)\nabla L(G_2)$. Thus $E_D(L(G_1)\nabla L(G_1)) = E_D(L(G_2)\nabla L(G_2)) = 24(t-1)$. The theorem follows now from the fact that both $L(G_1)\nabla L(G_1)$ and $L(G_2)\nabla L(G_2)$ have $6t$ vertices. $\square$

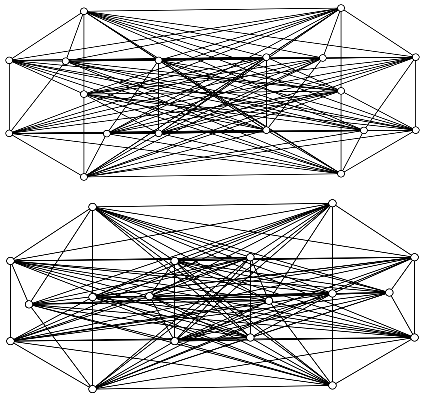

Figure 2: D-equienergetic graphs on 18 vertices with $E_D = 48$.

Theorem 10. For every $p \equiv 1\ (mod\ 3) \geq 10$, there exists a pair of D-equienergetic graphs.

Proof. Let $p = 3t + 1$. Let G_1 and G_2 be non-cospectral cubic graphs on $2t$ vertices as specified by Lemma 6. The line graphs $L(G_1)$ and $L(G_2)$ possess $3t$ vertices and are regular of degree 4. Then by a similar argument as in Theorem 3, we have

$$spec_D(L(G)\nabla K_1) = \begin{pmatrix} 3t - 3 \pm \sqrt{9\,t^2 - 15t + 9} & -(\lambda_i + 2) \\ 1 & 1 \end{pmatrix} \;,\; i = 2, \ldots, 3t$$

where λ_i , $i = 2, 3, \ldots, 3t$, are the (ordinary) eigenvalues of $L(G)$, different from its regularity. Since $\lambda_i + 2 \geq 0$ for $i = 2, \ldots, 3t$, and $3t - 3 \leq \sqrt{9\,t^2 - 15t + 9}$, we have

$$\begin{aligned} E_D(L(G)\nabla K_1) &= 2\sqrt{9\,t^2 - 15\,t + 9} + \sum_{i=2}^{3t}(\lambda_i + 2) \\ &= 2\sqrt{9\,t^2 - 15\,t + 9} - \lambda_1 + 2(3t - 1) \\ &= 2\sqrt{9\,t^2 - 15\,t + 9} - 4 + 2(3t - 1)\ . \end{aligned}$$

Thus

$$E_D(L(G_1)\nabla K_1) = E_D(L(G_2)\nabla K_2) = 2(3t - 3) + 2\sqrt{9\,t^2 - 15\,t + 9}$$

i. e., $L(G_1)\nabla K_1$ and $L(G_2)\nabla K_1$ are D-equienergetic. $\square$

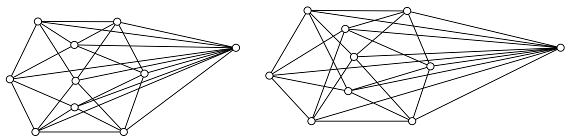

Figure 3: D-equienergetic graphs on 10 vertices with $E_D = 2(6 + 3\sqrt{5})$.

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