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THE LAPLACIAN ENERGY OF SOME LAPLACIAN INTEGRAL GRAPHS

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Abstract

The *energy* of a simple graph G is equal to the sum of the absolute values of the eigenvalues of its adjacency matrix. The *Laplacian energy* of G, recently introduced in the literature, is an analogous graph invariant defined as a function of the eigenvalues of the Laplacian matrix and the average degree of G. We investigate the Laplacian energy of the graphs whose energy was studied in the paper I. Gutman and L. Pavlović, *The energy of some graphs with large number of edges*, Bull. Acad. Serbe Sci. Arts (Cl. Math. Natur.) **118** (1999) 35–50. We prove that all these graphs are Laplacian integral.

INTRODUCTION

Let $G = G(V,E)$ be a simple graph, V its vertex set with cardinality n, and E its edge set with cardinality m. The *spectrum* of G is the non-increasing sequence $\lambda_1, \lambda_2, \ldots, \lambda_n$ of the distinct eigenvalues of the adjacency matrix $A(G)$ of G.

If $D(G)$ is the diagonal matrix of the vertex degrees of G, *L(G)=D(G)-A(G)* is defined to be the *Laplacian matrix* of G. The *spectrum of L(G)* is the sequence of its eigenvalues displayed in non-increasing order, denoted by $\{\mu_1, \ldots, \mu_{n-1}, \mu_n\}$. It is well known that $L(G)$ is a positive semidefinite and singular matrix. So, for $i = 1, 2, \ldots, n-1,\ \ \mu_i \geq 0$ and $\mu_n = 0$. Besides, when each Laplacian eigenvalue is an integer number, G is said to be a *Laplacian integral graph* [13].

The *energy* of the graph G, $E(G)$, is equal to the sum of the absolute values of the eigenvalues of G. This invariant was introduced by one of the authors in [3] and it has been extensively studied (see the reviews [3, 5] and the references therein).

Recently, Gutman and Zhou [8] introduced the concept of *Laplacian energy* of the graph G as

$$LE(G) = \sum_{i=1}^{n} \left| \mu_i - \frac{2m}{n} \right| . \tag{1}$$

For the few subsequent works on Laplacian energy see [6, 18, 20].

While there are many papers concerned with finding the ordinary energy of particular graphs (see [2, 4, 7, 10, 11, 16, 17, 19]), there are no such studies of Laplacian energy. In this work we investigate the Laplacian energy of the graphs with large number of edges for which Gutman and Pavlović studied the ordinary energy [7]. All of these graphs are obtained from the complete graph K_n by the deletion of some edges, according to distinct rules as follows: the $Ka_n(k)$-*graphs* are obtained from K_n by deleting k edges which have a common vertex; the $Kb_n(k)$-*graphs* are obtained from K_n by deleting k independent edges; the $Kc_n(k)$-*graphs* are obtained from K_n by deleting a k-clique, $k < n$; the $Kd_n(k)$-*graphs*, are obtained from K_n by deleting the edges of a k-membered cycle. Following these constructions, we introduce two further kinds of graphs: the $Ke_n(k)$-*graphs*, that are obtained from K_n by deleting the edges of k independent paths P_3 and the $Kf_n(k)$-*graphs*, that are obtained from K_n by deleting the edges of k independent triangles C_3. We investigate the Laplacian energy of these graphs, for all graph-theoretically relevant values of n and k (see [7]). We analyze the behavior of the Laplacian energy as a function of k, for fixed values of n. We prove that, except in the case of the $Kd_n(k)$-graphs, all of them are Laplacian integral.

Analogously to what happens in the study of the ordinary energy, in this paper, we refer to two simple graphs G_1 and G with same order, as to be *equienergetic*, when $LE(G_1) = LE(G_2)$. And we say that G_1 and G_2 are *cospectral graphs*, when both of them have the same Laplacian spectra.

The following well known fact about the Laplacian spectrum of a graph will be used several times in this work. Its proof can be found in [1].

Fact 1. Let G be a graph with n vertices and $\overline{G}$ be its complement. If the Laplacian spectrum of G is $\{\mu_1, \mu_1, \ldots, \mu_n\}$, then the Laplacian spectrum of $\overline{G}$ is $\{n-\mu_{n-1}, n-\mu_{n-2}, \ldots, n-\mu_1, 0\}$.

THE $Ka_n(k)$-GRAPHS

For fixed integers n and k, $n \geq 3$ and $0 \leq k \leq n-1$, the graph $Ka_n(k)$ is obtained from K_n by the deletion of k edges with a common endpoint. If G is such a graph then $\overline{G}$ is the union of the star S_{k+1} and $n-k-1$ isolated vertices. From Fact 1, since the Laplacian spectrum of $\overline{G}$ is $\{k+1, 1\ (k-1 \text{ times}), 0\ (n-k \text{ times})\}$, the Laplacian spectrum of G is $\{n\ (n-k-1 \text{ times}), n-1\ (k-1 \text{ times}), n-k-1, 0\}$. Therefore, we arrive at our first result:

Theorem 1. For $n \geq 3$ and $0 \leq k \leq n-1$,

$$LE(Ka_n(k)) = 2n - 2 + \left(2 - \frac{8}{n}\right) k \,. \tag{2}$$

Proof. For given integers n and k, $n \geq 3$ and $0 \leq k \leq n-1$, let $G = Ka_n(k)$. This graph has $m = n(n-1)/2 - k$ edges. So, the average degree of G is $2m/n = n - 1 - 2k/n$. From (1) and after some algebraic manipulations we have

$$LE(G) = (n-k-1)\left|1 + \frac{2k}{n}\right| + (k-1)\left|\frac{2k}{n}\right| + \left|-k + \frac{2k}{n}\right| + n - 1 - \frac{2k}{n} \,. \tag{3}$$

Since $n \geq 3$, it is $-k + 2k/n < 0$. Consequently, $|-k + 2k/n| = k - 2k/n$. Applying this inequality to (3), we straightforwardly obtain (2). $\Box$

Remark 1. From Theorem 1 it follows that $LE(Ka_n(k))$ is independent of k, if $n = 4$, and that $LE(Ka_n(k))$ monotonically and linearly increases with k, if $n > 4$, see Fig. 1. From Eq. (2) it is evident that the maximum value of $LE(Ka_n(k))$ is attained for $k = n - 1$, if $n > 3$, whereas for $n = 3$, $LE(K_{an}(k))$ is a decreasing function on k, so that its maximum value is only attained when $k = 0$. In Fig. 2 are displayed the four Laplacian equienergetic and mutually non-cospectral graphs with $n = 4$ vertices.

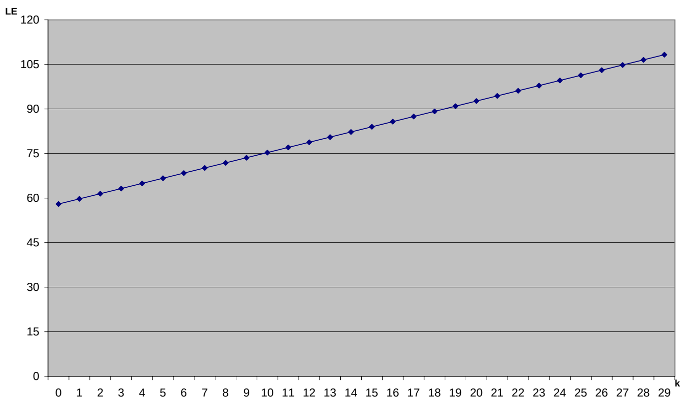

Fig. 1. The dependence of $LE(Ka_n(k))$ on k for $n = 30$ and $0 \leq k \leq n - 1$.

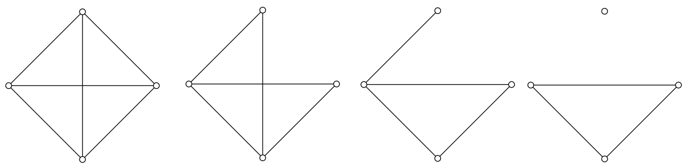

Fig. 2. $LE(Ka_4(0)) = LE(Ka_4(1)) = LE(Ka_4(2)) = LE(Ka_4(3)) = 6$.

THE $Kb_n(k)$-GRAPHS

Denote by G the graph $Kb_n(k)$ for fixed integers $n \geq 3$ and $0 \leq k \leq \lfloor n/2 \rfloor$. Since G is obtained from K_n by deleting k independent edges, $\overline{G}$ is the union of k copies of K_2 and $(n-2k)$ isolated vertices. Then, the Laplacian spectrum of $\overline{G}$ is $\{\ 2 \quad (k \text{ times}),\ 0 \quad (n-k \text{ times})\}$. From Fact 1, the Laplacian spectrum of G is $\{\ n \quad (n-k-1 \text{ times}),\ n-2 \quad (k \text{ times}),\ 0\}$. Therefore, we have the following result:

Theorem 2. For $n \geq 3$ and $0 \leq k \leq \lfloor n/2 \rfloor$,

$$LE(Kb_n(k)) = (2n-2) + \left(2 - \frac{4}{n}\right) k - \frac{4k^2}{n} \ . \tag{4}$$

Proof. Let G be a $Kb_n(k)$-graph, where $n \geq 3$ and $0 \leq k \leq \lfloor n/2 \rfloor$. Then, its average degree is $2m/n = n - 1 - 2k/n$. From (1) and after some simple calculations, we get

$$LE(G) = (n-k-1)\left|1 + \frac{2k}{n}\right| + k\left| -1 + \frac{2k}{n}\right| + n - 1 - \frac{2k}{n} \ . \tag{5}$$

Since $0 \leq k \leq \lfloor n/2 \rfloor$, $-1 + 2k/n < 0$, and then $|-1 + 2k/n| = 1 - 2k/n$. Applying this to (5) we directly obtain (4). $\square$

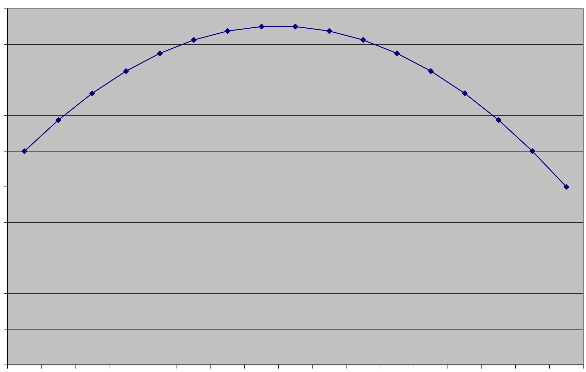

Fig. 3. The dependence of $LE(Kb_n(k))$ on k for $n = 30$ and $0 \leq k \leq 15$.

Remark 2. It is easy to see that the function $f(x) = (2n-2) + (2-4/n)x - 4x^2/n$, $x \in \mathbb{R}$, reaches its maximum value when $x = (n-2)/4$.

Corollary 1. For a fixed integer $n \geq 3$, let $k_* = \lfloor (n-2)/4 \rfloor$ and $k^* = k_* + 1$. Then, for each k, $0 \leq k \leq \lfloor n/2 \rfloor$, the Laplacian energy of $Kb_n(k)$, viewed as a function of k, reaches its maximum value at

- $k = k_*$ and $k = k^*$, when $n \equiv 0 \pmod 4$
- $k = k^* = (n-1)/4$, when $n \equiv 1 \pmod 4$
- $k = k_* = (n-2)/4$, when $n \equiv 2 \pmod 4$
- $k = k_* = (n-3)/4$, when $n \equiv 3 \pmod 4$.

Proof. For a fixed $n \geq 3$ and $0 \leq k \leq \lfloor n/2 \rfloor$, let $f(k) = LE(Kb_n(k)) = (2n-2)+(2-4/n)k - 4k^2/n$. In the first case, when $n \equiv 0 \pmod 4$, then $\lfloor (n-2)/4 \rfloor = n/4 - 1$ and $f(n/4-1) = f(n/4)$. So, $f(k_*) = f(k^*)$. In the second case, when $n \equiv 1 \pmod 4$, then $\lfloor (n-2)/4 \rfloor = (n-5)/4$. Since $f((n-5)/4) < f((n-1)/4)$, then $k^* = (n-1)/4$. When $n \equiv 2 \pmod 4$, then $\lfloor (n-2)/4 \rfloor = (n-2)/4$ and f attains its maximum value at $k = k_*$. Finally, when $n \equiv 3 \pmod 4$, then we have $\lfloor (n-2)/4 \rfloor = (n-3)/4$ and $f((n-3)/4) > f((n+1)/4)$. Consequently, the maximum value of f is reached when $k_* = (n-3)/4$. $\square$

Proposition 1. For every even integer n and for all i, $0 \leq i \leq \lfloor n/4 \rfloor - 1$, $Kb_n(i)$ and $Kb_n((n-2-2i)/2))$ are Laplacian equienergetic and non-cospectral graphs.

Proof. Let n be even and $0 \leq i \leq \lfloor n/4 \rfloor - 1$. Consider the graphs $G = Kb_n(i)$ and $H = Kb_n((n-2-2i)/2))$. From (4), it is easy to prove that $EL(G) = EL(H)$. The spectra of G and H are, respectively, equal to $\{\, n \ \ (n-i-1 \text{ times}),\ n-2 \ \ (i \text{ times}), 0\}$ and $\{\, n \ \ (n/2+i \text{ times}),\ n-2 \ \ (n/2-1-i \text{ times}), 0\}$. In order to prove that G and H are Laplacian non-cospectral graphs, it is enough to show that n or k have distinct multiplicities in, at least, one of the spectra. But, if that does not hold, then $n-1-i = n/2+i$ and $i = n/2-1-i$; consequently, $i = (n-2)/4$, which is impossible by the conditions on i. So, G and H are non-cospectral graphs. $\square$

THE $Ke_n(k)$-GRAPHS

For given integers $n \geq 3$ and $0 \leq k \leq \lfloor n/3 \rfloor$, let G be a $Ke_n(k)$-graph, that is, a graph obtained from K_n by the deletion of k independent paths P_3. Then, its complement $\overline{G}$ is the union of k copies of P_3 and $(n-3k)$ isolated vertices. So, the Laplacian spectrum of $\overline{G}$ is $\{\ 3\ \ (k \text{ times}),\ 1\ \ (k \text{ times}),\ 0\ \ (n-2k \text{ times})\}$ and, from Fact 1, the Laplacian spectrum of G is $\{\ n\ \ (n-2k-1 \text{ times}),\ n-1\ \ (k \text{ times}),\ n-3\ (k \text{ times}), 0\}$.

Theorem 3. For $n > 3$ and $0 \leq k \leq \lfloor n/3 \rfloor$,

$$LE(Ke_n(k)) = (2n-2) + \left(4 - \frac{8}{n}\right) k - \frac{8k^2}{n}\ . \tag{6}$$

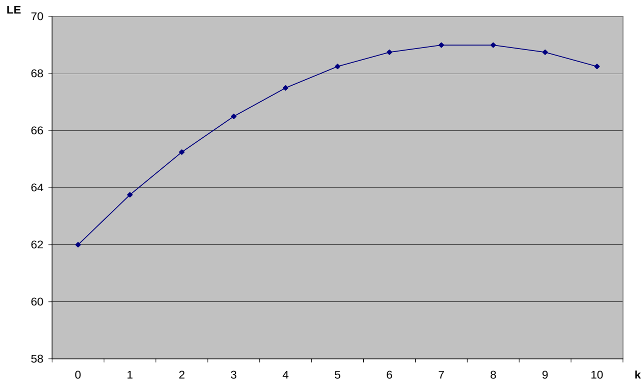

Fig. 4. The dependence of $LE(Ke_n(k))$ on k for $n = 30$ and $1 \leq k \leq 10$.

Proof. Similarly as in the proof of Theorem 1, we fix integers $n \geq 3$ and $0 \leq k \leq \lfloor n/3 \rfloor$. The graph $G \cong Ke_n(k)$ has $m = n(n-1)/2 - 2k$ edges and average degree $2m/n = n - 1 - 4k/n$. From (1) and after simple calculations, we have

$$LE(G) = (n-2k-1)\left|1 + \frac{4k}{n}\right| + k\left|\frac{4k}{n}\right| + k\left| -2 + \frac{4k}{n}\right| + n - 1 - \frac{4k}{n}\ . \tag{7}$$

Since $0 \leq k \leq \lfloor n/3 \rfloor$, then $-2 + 4k/n < 0$ and therefore $|-2 + 4k/n| = 2 - 4k/n$. Applying this to (7), we get (6). $\square$

Remark 3. From Theorems 1 and 2, $LE(Ke_n(k)) = LE(Kb_n(k)) + (2 - 4/n)k - 4k^2/n$. Thus, Corollary 1 holds for every $Ke_n(k)$-graph, $0 \leq k \leq \lfloor n/3 \rfloor$. Besides, we have the following result, the proof of which is analogous to that of Proposition 1.

Proposition 2. For every even n and $i = 0, 1, \ldots, \lfloor n/12 \rfloor$, $Ke_n(\lfloor n/3 \rfloor) - i)$ and $Ke_n(\lfloor n/6 \rfloor - 1 + i)$ are Laplacian equienergetic and non-cospectral graphs.

THE $Kc_n(k)$-GRAPHS

Let $n \geq 3$ and k, $1 \leq k \leq n-1$, be fixed integers, and let $G \cong Kc_n(k)$, that is, the graph G is obtained from K_n by deleting a k-clique. Its complement $\overline{G}$ is the union of the complete graph K_k and $(n-k)$ isolated vertices. Since the Laplacian spectrum of $\overline{G}$ is $\{\, k \;\; (k-1 \text{ times}),\; 0 \;\; (n-k+1 \text{ times})\}$, it follows from Fact 1 that the Laplacian spectrum of G is $\{\, n \;\; (n-k \text{ times}),\; n-k \;\; (k-1 \text{ times}), 0\}$. Therefore, the Laplacian energy of the $Kc_n(k)$-graph G is given by the following theorem, the proof of which is analogous those of Theorems 1, 2, and 3.

Theorem 4. For $n \geq 3$ and $1 \leq k \leq n-1$,

$$LE(Kc_n(k)) = \frac{2(n-k)(n+k^2-k)}{n} \; .$$

Proof. Similar to the proof of Corollary 1. □

Remark 4. For $n > 3$, it is easy to see that the function $f(x) = 2(n-x)(n+x^2-x)/n$, $x \in \mathbb{R}$, reaches its maximum value when $x = [n+1+\sqrt{(n-1)^2-2n}\,]/3$.

Corollary 2. For a fixed integer $n > 3$, let $k_* = \left\lfloor [n+1+\sqrt{(n-1)^2-2n}\,]/3 \right\rfloor$ and $k^* = k_* + 1$. Then the maximum value of the Laplacian energy of the $Kc_n(k)$-graph, $1 \leq k \leq n-1$, as a function of k, is attained at

- $k = k_* = k^*$, when $n = 6$
- $k = k^* = 2n/3$, when $n > 6$ and $n \equiv 0 \pmod 3$
- $k = k_* = (2n-2)/3$, if $n \equiv 1 \pmod 3$
- $k = k_* = (2n-1)/3$, if $n \equiv 2 \pmod 3$.

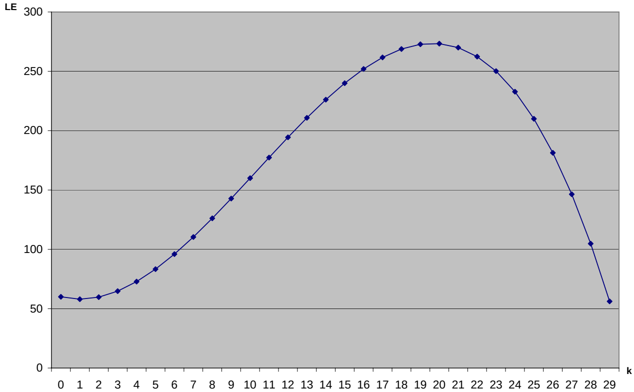

Fig. 5. The dependence of $LE(Kc_n(k))$ on k for $n = 30$ and $1 \leq k \leq 29$.

Remark 5. In the class of $Kc_n(k)$-graphs, there is at least one Laplacian equienergetic non-coespectral pair. Let $n = 6$ and $k_1 = 4$ and $k_2 = 3$. Then the graphs $Kc_6(3)$ and $Kc_6(4)$ have this property.

THE $Kf_n(k)$-GRAPHS

Let $n \geq 3$ and $1 \leq k \leq \lfloor \frac{n}{3} \rfloor$ be fixed integers, and $G \cong Kf_n(k)$, that is, the graph G is obtained from K_n by the deletion of the edges belonging to k disjoint 3-cycles (triangles). So $\overline{G}$ is the union of the k copies of the triangle, $k\,C_3$, and $(n-3k)$ isolated vertices. Since the Laplacian spectrum of $\overline{G}$ is $\{\ 3\ \ (2k \text{ times}),\ 0\ \ (n-2k \text{ times})\}$, the Laplacian spectrum of G is $\{\ n\ \ (n-2k-1 \text{ times}),\ n-3\ \ (2k \text{ times}), 0\}$, from Fact 1. Therefore, the Laplacian energy of G is given by the following theorem, the proof of which is similar to those of Theorems 1–4.

Theorem 5. For $n \geq 3$ and $0 \leq k \leq \lfloor n/3 \rfloor$,

$$LE(Kf_n(k)) = 2n - 2 + \left(8 - \frac{12}{n}\right) k - \frac{24\,k^2}{n}\ .$$

Remark 6. For $n > 3$, the function $f(x) = (2n-2) + (8 - 12/n)x - 24x^2/n$, $x \in \mathbb{R}$, reaches its maximum value at $x = (2n-3)/12$.

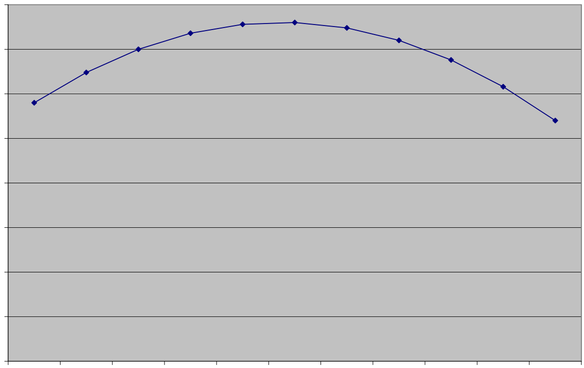

Fig. 6. The dependence of $LE(Kf_n(k))$ on k for $n = 30$ and $0 \le k \le 10$.

Corollary 3. For a fixed integer $n > 3$, let $k_* = \lfloor (2n-3)/12 \rfloor$ and $k^* = k_* + 1$. Then the maximum value of the Laplacian energy of $Kf_n(k)$, viewed as a function of k, $0 \le k \le \lfloor n/3 \rfloor$, is reached at

- $k = k^* = n/6$ when n is even and $n \equiv 0 \pmod 3$
- $k = k_* = (n-3)/6$, when n is odd and $n \equiv 0 \pmod 3$
- $k = k_* = (n-4)/6$, when n is even and $n \equiv 1 \pmod 3$
- $k = k^* = (n-1)/6$, when n is odd and $n \equiv 1 \pmod 3$
- $k = k_* = (n-2)/6$, when n is even and $n \equiv 2 \pmod 3$
- $k = k^* = (n+1)/6$, when n is odd and $n \equiv 2 \pmod 3$.

Proof. Let n be even and $f(k) = LE(Kf_n(k))$, $0 \le k \le \lfloor n/3 \rfloor$. If $n \equiv 0 \pmod 3$, we have $f(n/6 - 1) < f(n/6)$. If $n \equiv 1 \pmod 3$, then $\lfloor (2n-3)/12 \rfloor = (n-4)/6$.

For $n > 3$, it is easy to verify that $f((n-4)/6) > f((n+2)/6)$. If $n \equiv 2 \pmod 3$, then $\lfloor (2n-3)/12 \rfloor = (n-2)/6$ and $f((n-2)/6) > f((n+4)/6)$.

For odd n, if $n \equiv 0 \pmod 3$, then $\lfloor (2n-3)/12 \rfloor = (n-3)/6$ and $f((n-3)/6) > f((n+3)/6)$. Consider now the case $n \equiv 1 \pmod 3$. Then $\lfloor (2n-3)/12 \rfloor = (n-7)/6$ and $f((n-7)/6) < f((n-1)/6)$. If $n \equiv 2 \pmod 3$, then $\lfloor (2n-3)/12 \rfloor = (n-5)/6$ and $f((n-5)/6) < f((n+1)/6)$. $\square$

Theorem 6. For fixed integer $n \geq 3$, no pair of non-cospectral $Kf_n(k)$-graphs are Laplacian equienergetic.

Proof. Let $n \geq 3$ and k_1 and k_2 be integers, with $0 < k_1, k_2 \leq \lfloor n/3 \rfloor$. Suppose that contrary to the claim of Theorem 6, the graphs $G_1 \cong Kf_n(k_1)$ and $G_2 \cong Kf_n(k_2)$ are non-cospectral and satisfy $LE(G_1) = LE(G_2)$. Then it must be $k_1 \neq k_2$, and from Theorem 5 we obtain

$$\left(8 - \frac{12}{n}\right) k_1 - \frac{24}{n}\, k_1^2 = \left(8 - \frac{12}{n}\right) k_2 - \frac{24}{n}\, k_2^2 \;. \tag{8}$$

Let $x = 8 - 12/n$ and $y = 24/n$. Then the equation above becomes

$$x k_1 - y k_1^2 = x k_2 - y k_2^2 \;. \tag{9}$$

Using simple calculations, we get

$$k_1 = \frac{2n-3}{6} - k_2 \;. \tag{10}$$

So we have $(2n-3)/6 \in \mathbb{Z}$, that is, there exists $q \in \mathbb{Z}$, such that $2n - 3 = 6q$. Consequently, $n = 3(2q+1)/2$. Since $6q + 3$ is an odd integer, then $n \notin \mathbb{Z}$, which is not possible! $\square$

LAPLACIAN INTEGRAL GRAPHS

A graph G is said to be *Laplacian integral* if its Laplacian spectrum is a subset of the integer number set. The Laplacian integral graphs are extensively studied and there are many available distinct classes of them in the literature, see [13, 14, 15].

The main results in this section are Theorems 7 and 8. In the former we show that every graph considered in this paper is Laplacian integral; the latter asserts that, among all these graphs, only $Ka_n(k)$ is threshold. In order to get there, we repeat some definitions and classical results.

Let be $G = G(V, E)$ and $H = H(V', E')$ be graphs with disjoint vertex sets. The *union* of G and H is the graph $G \oplus H = (V \cup V', E \cup E')$ and the *join* of G and H is the graph $G \vee H$ obtained from $G \oplus H$ by adding edges from each vertex of G to every vertex of H. A graph G is *decomposable* if G can be expressed as joins and unions of isolated vertices, see [14]. A graph G is called *threshold* if and only if G does not have any induced subgraph isomorphic to $2K_2$, P_4 or C_4. If G is a threshold (respectively, a decomposable) graph then $\overline{G}$ is also a threshold (respectively, a decomposable) graph. These graphs have been extensively studied in the literature, see [9, 12, 13, 14, 15].

Fact 2. [14]. A graph G is decomposable if and only if G does not have any induced subgraph isomorphic to P_4.

Fact 3. [14]. If G is a decomposable graph then G is Laplacian integral.

Theorem 7. For graph-theoretically relevant integers n and k, the graphs $Ka_n(k)$, $Kb_n(k)$, $Kc_n(k)$, $Ke_n(k)$ and $Kf_n(k)$ are Laplacian integral.

Proof. First, let G be a $Ka_n(k)$-graph with $n \geq 4$ vertices and $0 \leq k \leq n-1$. Since $\overline{G}$ is the union of the star S_{k+1} and $n-k-1$ isolated vertices, there is no induced subgraph of $\overline{G}$ isomorphic to P_4. From Fact 2, $\overline{G}$ is a decomposable graph and so is G. Then, from Fact 3, G is a Laplacian integral graph. The same arguments are used to prove the assertion in the other cases. □

Theorem 8. For $n \geq 4$ and appropriate integers k, among the graphs $Ka_n(k)$, $Kb_n(k)$, $Kc_n(k)$, $Ke_n(k)$, and $Kf_n(k)$, only the $Ka_n(k)$-graphs are threshold.

Proof. Let G be a $Ka_n(k)$-graph with $n \geq 4$ vertices. Since $\overline{G}$ is the union of the star S_{k+1} and $n-k-1$ isolated vertices, it does not have any induced subgraph isomorphic to $2K_2, P_4$ or C_4. Then, $\overline{G}$ is a threshold graph and so is G. Now, let H be a $Kb_n(k)$-graph with $n \geq 4$ vertices. Its complement $\overline{H}$ is the union of k copies of K_2 and $n-2k$ isolated vertices, and then neither $\overline{H}$ nor H are threshold graphs. The proofs for the $Kc_n(k)$, $Ke_n(k)$, and $Kf_n(k)$-graphs are analogous. □

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References

[1] D. Cvetković, D. Doob, H. Sachs, Spectra of Graphs – Theory and Application, Academic Press, New York, (1980).

[2] J. A. de la Peña, L. Mendoza, Moments and π-electron energy of hexagonal systems in 3-space, *MATCH Commun. Math. Comput. Chem.* **56** (2006) 113–129.

[3] I. Gutman, The energy of a graph, *Ber. Math. Statist. Sekt. Forschungsz. Graz*, **103** (1978) 1–22.

[4] I. Gutman, Hyperenergetic molecular graphs, *J. Serb. Chem. Soc.* **64** (1999) 199–205.

[5] I. Gutman, Topology and stability of conjugated hydrocarbons. The dependence of total π-electron energy on molecular topology, *J. Serb. Chem. Soc.* **70** (2005) 441–456.

[6] I. Gutman, N. M. M. de Abreu, C. T. M. Vinagre, A. S. Bonifácio, S. Radenković, Relations between energy and Laplacian energy, *MATCH Commun. Math. Comput. Chem.* **59** (2008) 343–354.

[7] I. Gutman, L. Pavlović, The energy of some graphs with large number of edges, *Bull. Acad. Serbe Sci. Arts (Cl. Math. Natur.)* **118** (1999) 35–50.

[8] I. Gutman, B. Zhou, Laplacian energy of a graph, *Lin. Algebra Appl.* **414** (2006) 29–37.

[9] P. L. Hammer, A. K. Kelmans, Laplacian spectra and spanning trees of threshold graphs, *Discrete Appl. Math.* **65** (1996) 255–273.

[10] Y. Hou, I. Gutman, Hyperenergetic line graphs, *MATCH Commun. Math. Comput. Chem.* **43** (2001) 29–39.

[11] G. Indulal, A Vijayakumar, On a pair of equienergetic graphs, *MATCH Commun. Math. Comput. Chem.* **55** (2006) 83–90.

[12] N. V. R. Mahadev, U. N. Peled, *Threshold Graphs and Related Topics*, Elsevier, Amsterdam, 1995.

[13] R. Merris, Degree maximal graphs are Laplacian integral, *Lin. Algebra Appl.* **199** (1994) 381–389.

[14] R. Merris, *Graph Theory*, Wiley, New York. 2001.

[15] R. Merris, T. Roby, The lattice of threshold graphs, *J. Inequal. Pure Appl. Math.* **6** (2005) article 2, 1–38.

[16] H. S. Ramane, H. B. Walikar, Construction of equienergetic graphs, *MATCH Commun. Math. Comput. Chem.* **57** (2007) 203–210.

[17] H. S. Ramane, H. B. Walikar, S. B. Rao, B. D. Acharya, P. R. Hampiholi, S. R. Jog, I. Gutman, Spectra and energies of iterated line graphs of regular graphs, *Appl. Math. Lett.* **18** (2005) 679–682.

[18] H. Wang, H. Hua, Note on Laplacian energy of graphs, *MATCH Commun. Math. Comput. Chem.* **59** (2008) 373–380.

[19] L. Xu, Y. Hou, Equienergetic bipartite graphs, *MATCH Commun. Math. Comput. Chem.* **57** (2007) 363–370.

[20] B. Zhou, I. Gutman, On Laplacian energy of graphs, *MATCH Commun. Math. Comput. Chem.* **57** (2007) 211–220.