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UPPER BOUNDS FOR LAPLACIAN ENERGY OF GRAPHS

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Abstract

Let G be a graph on n vertices and m edges. Let $\mu_1, \mu_2, \ldots, \mu_n$ be the eigenvalues of the Laplacian matrix of G. The Laplacian energy of G is defined as $LE(G) = \sum_{i=1}^{n} \left|\mu_i - \frac{2m}{n}\right|$ and some of its properties have recently been established. In this paper we determine a few new upper bounds for $LE(G)$, thus correcting an error in the paper [4].

INTRODUCTION

In this work we consider only simple graphs , i. e., undirected graphs without loops or multiple edges. Let G be such a graph with n vertices. Denote the eigenvalues of the Laplacian matrix of G by $\mu_1, \mu_2, \ldots, \mu_n$. The Laplacian spectrum of G is defined as the set of all Laplacian eigenvalues $\mu_1, \mu_2, \ldots, \mu_n$ [1,6–9].

The concept of Laplacian energy of the graph G has been defined in [3] as:

$$LE(G) = \sum_{i=1}^{n} |\gamma_i|$$

where

$$\gamma_i = \mu_i - \frac{2m}{n} \ .$$

By [3,4], the Laplacian eigenvalues of the graph G satisfy the following relations :

$$\sum_{i=1}^{n} \mu_i = 2m \quad ; \quad \sum_{i=1}^{n} \mu_i^2 = 2m + \sum_{i=1}^{n} d_i^2 \ .$$

Hence, the auxiliary "eigenvalues" γ_i , $i = 1, 2, \ldots, n$, obey the conditions

$$\sum_{i=1}^{n} \gamma_i = 0 \quad ; \quad \sum_{i=1}^{n} \gamma_i^2 = 2M$$

where

$$M = m + \frac{1}{2} \sum_{i=1}^{n} \left(d_i - \frac{2m}{n} \right)^2 \ .$$

The number d_i denotes the degree of the i-th vertex of G and $2m/n$ is the average vertex degree. It is easy to see that $M \geq m$ for all graphs G and $M = m$ for regular graphs. Obviously $LE(G) \geq 0$ and $LE(G) = 0$ if $m = 0$ [3].

The Laplacian energy is a relatively new concept, so the study of its mathematical properties started recently, and the first results were reported by Zhou and Gutman [3,4].

In this paper we observe some new upper bounds for the Laplacian energy of a graph.

THE MAIN RESULTS

1

Let G be an (n, m)-graph possessing p components $(p \geq 1)$. In [3] (Theorem 3) is proven that:

$$LE(G) \leq \frac{2m}{n} p + \sqrt{(n-p) \left[2M - p \left(\frac{2m}{n} \right)^2 \right]} \ . \tag{1}$$

We consider the right-hand side expression in (1) as a function of the parameter p :

$$f(x) = \frac{2m}{n} x + \sqrt{(n-x) \left[2M - x \left(\frac{2m}{n} \right)^2 \right]}, \qquad 0 \leq x \leq n \ .$$

Now,

$$f'(x) = \frac{2m}{n} - \frac{2M - 2x(\frac{2m}{n})^2 + n(\frac{2m}{n})^2}{2\sqrt{(n-x)[2M - x(\frac{2m}{n})^2]}} \ .$$

In the paper [4] it was claimed that for $a = 2m/n$,

$$2M + a^2\, n - 2\, a^2\, x \geq 0$$

holds for $x \leq n$. This, however, is not generally true. In reality, the above inequality is valid only if

$$x \leq M \left(\frac{2m}{n}\right)^{-2} + \frac{n}{2} \ . \tag{2}$$

Consequently, the function $f(x)$ decreases if and only if and condition (2) is obeyed, and

$$4\left(\frac{2m}{n}\right)^2 \left[(n-x)\left(2M - x\left(\frac{2m}{n}\right)^2\right)\right] \leq \left[2M - 2x\left(\frac{2m}{n}\right)^2 + n\left(\frac{2m}{n}\right)^2\right]^2$$

which can further be written as

$$4Mn\left(\frac{2m}{n}\right)^2 \leq 4M^2 + n^2\left(\frac{2m}{n}\right)^4$$

i. e.,

$$\left[2M + n\left(\frac{2m}{n}\right)^2\right]^2 \geq 0 \ .$$

Due to the definition of the function $f(x)$, the following condition is also necessary:

$$x \leq 2M\left(\frac{2m}{n}\right)^{-2} \ .$$

Conclusion 1. For the graphs with the number of components

$$p \leq 2M\left(\frac{2m}{n}\right)^{-2}$$

the upper bound increases with decreasing p . Hence, for such graphs

$$LE(G) \leq \frac{2m}{n} + \sqrt{(n-1)\left[2M - \left(\frac{2m}{n}\right)^2\right]} \ . \tag{3}$$

Corollary 1. Let G be a connected (n,m)-graph. Then

$$LE(G) \leq \frac{2m}{n} + \sqrt{(n-1)\left[2M - \left(\frac{2m}{n}\right)^2\right]} \ .$$

Proof. It follows directly from the Conclusion 1. ■

2

Let G be an (n,m)-graph with $n \geq 3$. In [4] (Proposition 1) has been proven that

$$LE(G) \leq \sqrt{\frac{2M - \left(\frac{2m}{n}\right)^2}{n-1}} + \frac{2m}{n} + \sqrt{(n-2)\left[2M - \frac{2M - \left(\frac{2m}{n}\right)^2}{n-1} - \left(\frac{2m}{n}\right)^2\right]} \,. \tag{4}$$

Now, we show that the bound (3) and the bound (2) are equal, i. e.,

$$\sqrt{\frac{2M - \left(\frac{2m}{n}\right)^2}{n-1}} + \frac{2m}{n} + \sqrt{(n-2)\left[2M - \frac{2M - \left(\frac{2m}{n}\right)^2}{n-1} - \left(\frac{2m}{n}\right)^2\right]}$$

$$= \frac{2m}{n} + \sqrt{(n-1)\left[2M - \left(\frac{2m}{n}\right)^2\right]} \,. \tag{5}$$

Let $a = 2M - (2m/n)^2$. Then

$$\sqrt{\frac{a}{n-1}} + \frac{2m}{n} + \sqrt{(n-2)\left[a - \frac{a}{n-1}\right]} = \frac{2m}{n} + \sqrt{(n-1)a}$$

holds if and only if

$$\frac{a}{n-1} + 2\frac{(n-2)}{n-1}\sqrt{a^2} + a\frac{(n-2)^2}{n-1} = (n-1)a \,.$$

It is elementary to verify that the above identity is satisfied for all values of n.

Corollary 2. For every (n,m)-graph G,

$$LE(G) \leq \frac{2m}{n} + \sqrt{(n-1)\left[2M - \left(\frac{2m}{n}\right)^2\right]} \,. \tag{6}$$

Proof. For $n \geq 3$, inequality (3) and equality (4) directly lead to inequality (5). For connected graphs with $n = 2$ inequality (5) holds by Corollary 1. For the graph with $n = 1$ and the disconnected graph with $n = 2$ equality in (5) is satisfied in a trivial manner. ■

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References

[1] D. Cvetković, M. Doob, H. Sachs, *Spectra of Graphs – Theory and Application*, Academic Press, New York, 1980.

[2] I. Gutman, Some properties of Laplacian eigenvectors, *Bull. Acad. Serbe Sci. Arts (Cl. Math. Natur.)* **28** (2003) 1–7.

[3] I. Gutman, B. Zhou, Laplacian energy of a graph, *Lin. Algebra Appl.* **414** (2006) 29–37.

[4] I. Gutman, B. Zhou, On Laplacian energy of graphs, *MATCH Commun. Math. Comput. Chem.* **57** (2007) 211–220.

[5] M. Fiedler, Algebraic connectivity of graphs, *Czehoslov. Math. J.* **23** (1973) 298–305.

[6] R. Grone, R. Merris, V. S. Sunder, The Laplacian spectrum of graph, *SIAM J. Matrix Anal. Appl.* **11** (1990) 218–238.

[7] R. Grone, R. Merris, The Laplacian spectrum of graph II, *SIAM J. Discr. Math.* **7** (1994) 221–229.

[8] R. Merris, Laplacian matrices of graphs. A survey, *Lin. Algebra Appl.* **197-198** (1994) 143–176.

[9] R. Merris, A survey of graph Laplacians, *Lin. Multilin. Algebra* **39** (1995) 19–31.