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# On the energy of complement of regular line graphs

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## ABSTRACT

Let $G$ be a simple graph with $n$ vertices and let $\lambda_1, \lambda_2, \ldots, \lambda_n$ be its eigenvalues. The energy of $G$ is defined to be

$$E(G) = \sum_{i=1}^{n} |\lambda_i| \cdot$$

In this note, for a given $k$-regular graph we find explicit formulas for the energy of $\overline{L(G)}$, the complement of line graph of $G$. This provides us with some practical ways to compute the energy of a large family of regular graphs.

## 1. Introduction

Let $G$ be a simple graph with $n$ vertices and let $A$ be its adjacency matrix (a $n \times n$ (0,1)-matrix whose $(i,j)$ entry is 1 if and only if two vertices $i$ and $j$ are adjacent). The characteristic polynomial of $G$ is defined as $\varphi(G,x) = \det(xI - A)$, where $I$ is the identity matrix of size $n$. The eigenvalues of $G$ (i.e. the eigenvalues of $A$) are the roots $\lambda_1, \lambda_2, \ldots, \lambda_n$ of $\varphi(G,x)$, all of which are real. The energy of $G$ is defined to be

$$E(G)=\sum_{i=1}^{n}|\lambda_i|\cdot$$

The multiplicity of an eigenvalue $\lambda$ of $G$ is the order of $\lambda$ as a root of $\varphi(G,x)$. The spectrum of $G$ is the array

$$\begin{pmatrix}\lambda_1 & \lambda_2 & \cdots & \lambda_s\\ m_1 & m_2 & \cdots & m_s\end{pmatrix},$$

where $\lambda_1,\lambda_2,\ldots,\lambda_s$ are the distinct eigenvalues of $G$ and $m_i$ is the multiplicity of $\lambda_i$ ($i=1,2,\ldots,s$). It is easy to see that $m_1+\cdots+m_s=n$. On the other hand, because tr($A$) =0, we have $m_1\lambda_1+\cdots+m_s\lambda_s=0$.

It is well-known that if $G$ is a $k$-regular graph, then $k$ is an eigenvalue of $G$ with multiplicity $c$, where $c$ is the number of connected components of $G$, and that the absolute value of each eigenvalue of $G$ is less than or equal to $k$.

In [1], it is shown that the spectrum of a bipartite graph is symmetrical with respect to 0. Further, we have the following theorem. (See for example [11], p.444).

**Theorem 1.1.** *Let G be a connected graph, and let* $\lambda_1$ *be its largest eigenvalue. Then* $-\lambda_1$ *is an eigenvalue of G if and only if G is bipartite.*

The following theorems are used to prove the main results.

**Theorem 1.2.** [1] *Let G be a connected k-regular graph with spectrum*

$$\begin{pmatrix}k & \lambda_2 & \cdots & \lambda_s\\ 1 & m_2 & \cdots & m_s\end{pmatrix}.$$

*Then L(G), the line graph of G, is a* $(2k-2)$*-regular graph with spectrum*

$$\begin{pmatrix}2k-2 & \lambda_2+k-2 & \cdots & \lambda_s+k-2 & -2\\ 1 & m_2 & \cdots & m_s & \dfrac{n(k-2)}{2}\end{pmatrix},$$

*where n is the number of vertices of G.*

**Theorem 1.3.** [1] *Let G be a k-regular graph with spectrum*

$$\begin{pmatrix} k & \lambda_2 & \cdots & \lambda_s \\ 1 & m_2 & \cdots & m_s \end{pmatrix}.$$

*Then* $\overline{G}$*, the complement of G, is a* $(n-1-k)$*-regular graph with spectrum*

$$\begin{pmatrix} n-1-k & -\lambda_2-1 & \cdots & -\lambda_s-1 & 1 \\ 1 & m_2 & \cdots & m_s & \frac{n(k-2)}{2} \end{pmatrix},$$

*where n is the number of vertices of G.*

**Theorem 1.4.** [4] *Let G be a k-regular graph with n vertices (0<k<n), and let* $\tilde{\lambda}$ *be its smallest eigenvalue. Then*

$$\max\{k-n, -k\} \le \tilde{\lambda} \le -1.$$

Using these facts, in the next section we prove formulas for evaluating the energy of complement of line graph of some regular graphs.

The energy of regular graph was studied in the papers [2,3,7,10]. Also the energy of the iterated line graphs of regular graphs [5,8,9] as well as of their complements [6] were investigated. However, the results communicated in the papers [5,6,8,9] pertain to second and higher iterated line graphs. It seems that until now no result was obtained for the complement of the (first) line graph of a regular graph.

## 2. Main results

**Theorem 2.1.** *Let G be a connected k-regular graph with n vertices.*

(i) *If G is not bipartite and its smallest eigenvalue is greater than or equal to* $-k+1$*, then*

$$E(\overline{L(G)}) = (2n-4)(k-1)-2.$$

(ii) *If G is bipartite and its second smallest eigenvalue is greater than or equal to* $-k+1$*, then*

$$E(\overline{L(G)}) = (2n-4)(k-1).$$

**Proof.** Let $k > \lambda_2 > \cdots > \lambda_s$ be the distinct eigenvalues of $G$.

(i)The spectrum of $G$ is

$$\begin{pmatrix} k & \lambda_2 & \cdots & \lambda_s \\ 1 & m_2 & \cdots & m_s \end{pmatrix}.$$

By Theorem 1.2, the spectrum of $L(G)$ is

$$\begin{pmatrix} 2k-2 & \lambda_2+k-2 & \cdots & \lambda_s+k-2 & -2 \\ 1 & m_2 & \cdots & m_s & \dfrac{n(k-2)}{2} \end{pmatrix}.$$

By Theorem 2.2, therefore, the spectrum of $\overline{L(G)}$ is

$$\begin{pmatrix} \dfrac{nk}{2}-1-(2k-2) & -\lambda_2-k+1 & \cdots & -\lambda_s-k+1 & 1 \\ 1 & m_2 & \cdots & m_s & \dfrac{n(k-2)}{2} \end{pmatrix}.$$

Note that by the assumption, $-\lambda_i - k + 1 \le 0$; $(i = 2,\ldots,s)$. So we have:

$$E(\overline{L(G)}) = \frac{nk}{2} - 2k + 1 + \sum_{i=2}^{s} m_i(\lambda_i + k - 1) + \frac{n(k-2)}{2}$$
$$= nk - 2k - n + 1 + \sum_{i=2}^{s} m_i\lambda_i + (k-1)\sum_{i=2}^{s} m_i .$$

Using the equations $1\cdot k + \sum_{i=2}^{s} m_i\lambda_i = 0$ and $1 + \sum_{i=2}^{s} m_i = n$, we have

$$E(\overline{L(G)}) = nk - 2k - n + 1 - k + (k-1)(n-1)$$
$$= n(k-1) - 3(k-1) - 2 + (k-1)(n-1)$$
$$= (2n-4)(k-1) - 2.$$

(ii) The spectrum of $G$ is

$$\begin{pmatrix} k & \lambda_2 & \cdots & \lambda_{s-1} & -k \\ 1 & m_2 & \cdots & m_{s-1} & 1 \end{pmatrix}.$$

By Theorem 1.2, the spectrum of $L(G)$ is

$$\begin{pmatrix} 2k-2 & \lambda_2+k-2 & \cdots & \lambda_{s-1}+k-2 & -2 \\ 1 & m_2 & \cdots & m_{s-1} & \dfrac{n(k-2)}{2}+1 \end{pmatrix}.$$

By Theorem 2.2, therefore, the spectrum of $\overline{L(G)}$ is

$$\begin{pmatrix} \dfrac{nk}{2}-1-(2k-2) & -\lambda_2-k+1 & \cdots & -\lambda_{s-1}-k+1 & 1 \\ 1 & m_2 & \cdots & m_{s-1} & \dfrac{n(k-2)}{2}+1 \end{pmatrix}.$$

Note that by the assumption, $-\lambda_i-k+1\le 0$; $(i=2,\ldots,s-1)$. So we have:

$$\begin{aligned} E(\overline{L(G)}) &= \frac{nk}{2}-2k+1+\sum_{i=2}^{s-1} m_i(\lambda_i+k-1)+\frac{n(k-2)}{2}+1 \\ &= nk-2k-n+2+\sum_{i=2}^{s-1} m_i\lambda_i+(k-1)\sum_{i=2}^{s-1} m_i. \end{aligned}$$

Using the equations $1\cdot k+\sum_{i=2}^{s-1} m_i\lambda_i+1\cdot(-k)=0$ and $1+\sum_{i=2}^{s-1} m_i+1=n$, we have

$$\begin{aligned} E(\overline{L(G)}) &= nk-2k-n+2+0+(k-1)(n-2) \\ &= n(k-1)-2(k-1)+(k-1)(n-2) \\ &= (2n-4)(k-1). \end{aligned}$$

■

In the case that the eigenvalues of $G$ are integer (for example when $G$ is a complete graph), clearly the smallest eigenvalue (in non-bipartite graphs) and the second smallest eigenvalue (in bipartite graphs) are at least one unite far from $-k$. More generally

**Corollary 2.2.** *Let G be a connected k-regular graph with n vertices.*

(i) *If G is not bipartite and its smallest eigenvalue is integer, then*

$$E(\overline{L(G)})=(2n-4)(k-1)-2.$$

(ii) *If G is bipartite and its second smallest eigenvalue is integer, then*

$$E(\overline{L(G)}) = (2n-4)(k-1).$$

**Lemma 2.3.** *Let G be a connected k-regular graph with n vertices that is not bipartite, and let* $k \geq (n+1)/2$. *Then the smallest eigenvalue of G is greater than or equal to* $-k+1$.

**Proof.** Let $k > \lambda_2 > \cdots > \lambda_s$ be the distinct eigenvalues of *G*. Since $k \geq (n+1)/2$, it follows that $k - n \geq -k+1 > -k$, and that $\max\{k-n, -k\} = k-n$. Therefore, by Theorem 1.4, $\lambda_s \geq k-n \geq -k+1$.

■

**Corollary 2.4.** *Let G be a connected k-regular graph with n vertices.*

(i) *If G is not bipartite and* $k \geq (n+1)/2$, *then*

$$E(\overline{L(G)}) = (2n-4)(k-1)-2.$$

(ii) *If G is bipartite and* $k \geq n/2$, *then*

$$E(\overline{L(G)}) = (2n-4)(k-1).$$

**Proof.** Part (i) follows easily from Theorem 2.1(i) and Lemma 2.3 above.
To prove part (ii), note that in this case *n* is even, that the only possible value for *k* is $n/2$, and that *G* is isomorphic to $K_{k,k}$, which has the spectrum

$$\begin{pmatrix} k & 0 & -k \\ 1 & n-2 & 1 \end{pmatrix}.$$

Now the corollary follows using Theorem 2.1(ii). ■

**Corollary 2.5.** $E(\overline{L(K_n)}) = (n-3)E(K_n),\ n \geq 3.$

**Proof.** Since for $n \geq 3$, $K_n$ is not bipartite, by Corollary 2.4(i), we have

$$E(\overline{L(K_n)}) = (2n-4)(n-2) - 2 = 2(n^2 - 4n + 4) - 2 = (n-3)(2n-2) = (n-3)E(K_n). \qquad \blacksquare$$

Therefore this provides us with a practical way to calculate the energy of some families of graphs. As an interesting example, because $P_{10}$, the Petersen graph, is the complement of line graph of $K_5$, we have $E(P_{10}) = 16$.

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