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Zagreb M_1 Index, Independence Number and Connectivity in Graphs

Hongbo Hua

Department of Computing Science, Huaiyin Institute of Technology, Huaian, Jiangsu 223000, People's Republic of China
E-mail: hongbo.hua@gmail.com

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Abstract

Let G be a simple, undirected and connected graph. By $M_1(G)$ and $RMTI(G)$ we mean the first Zagreb index and the reciprocal Schultz molecular topological index of G, respectively. In this paper, we determined, respectively, the maximal elements with respect to $M_1(G)$ among all graphs having prescribed vertex-connectivity, edge-connectivity, vertex-independence number, and edge-independence number. As applications, all maximal elements with respect to $RMTI(G)$ are also determined among the above mentioned graph families, respectively.

1 Introduction

Let G be a simple connected graph with vertex set $V(G)$ and edge set $E(G)$. For a vertex $i \in V(G) = \{1, 2, \cdots, n\}$, let $N_G(i)$ denote the set of its (first) neighbors in G and the degree $d_G(i)$ of i is $|N_G(i)|$, or simply denoted as $d(i) = |N(i)|$ if there is no confusion arising. The degree sequence of G is $d = (d_1, \cdots, d_n)$, where $d_i = d(i)$.

The first Zagreb index M_1 and the second Zagreb index M_2 of G are defined as follows:

$$M_1 = M_1(G) = \sum_{i \in V(G)} d_i^2 \text{ and } M_2 = M_2(G) = \sum_{ij \in E(G)} d_i d_j$$

The reader is referred to [2–5] for the main properties of M_1 and M_2. In particular, for some recent results on $M_1(G)$, see for example [6–19].

Let G be a connected graph with n vertices. The distance matrix $\mathbf{D}$ of G is an $n \times n$ matrix ($\mathbf{D_{ij}}$) such that $\mathbf{D_{ij}}$ is the distance between the vertices i and j in G [21].

For a connected graph G with n vertices, the reciprocal distance matrix $\mathbf{R}$, also called the Harary matrix (see for instance [20,22]), is defined as an $n \times n$ matrix ($\mathbf{R_{ij}}$) such that $\mathbf{R_{ij}} = \frac{\mathbf{1}}{\mathbf{D_{ij}}}$ if $i \neq j$ and 0 otherwise. The reciprocal molecular topological index RMTI [23] of G is defined as

$$RMTI = RMTI(G) = d(\mathbf{A} + \mathbf{R})\mathbf{1},$$

where $\mathbf{1}$ is a $n \times 1$ vector with all its entries equal to 1. If we set $R_i = \sum\limits_{j=1}^{n} \mathbf{R_{ij}}$ in the above equation, then we obtain

$$RMTI = RMTI(G) = \sum_{i=1}^{n} R_i^2 + \sum_{i=1}^{n} d_i R_i.$$

In [23], RMTI and other formulations of reciprocal and constant-interval reciprocal Schultz-type topological indices have been discussed and their use illustrated by the QSPR studies on physical constants of alkanes and cycloalkanes.

More recently, Zhou and Trinajstić [24] reported some properties of the reciprocal molecular topological index RMTI. They also derived the upper bounds for RMTI in terms of the number of vertices and the number of edges for various classes of graphs under some restricted conditions.

In this paper, we determined, respectively, the maximal elements with respect to $M_1(G)$ among all graphs having prescribed graph invariants, such as, vertex–connectivity, edge–connectivity, vertex–independence number, and edge–independence number. As applications, all maximal elements with respect to $RMTI(G)$ are also determined among the above mentioned graph families, respectively.

2 Lemmas and results

As usual, we begin with some notations and terminology. The diameter of a graph G, denoted by $Diam(G)$, is the maximum cardinality among all distances of any one pair of vertices in G. Suppose that G_1 and G_2 are graphs with $V(G_1) \bigcap V(G_2) = \emptyset$. By $G_1 + G_2$ we denote the sum of G_1 and G_2 with $V(G_1 + G_2) = V(G_1) \cup V(G_2)$ and $E(G_1 + G_2) = E(G_1) \cup E(G_2)$. The join of G_1 and G_2, denoted by $G_1 \vee G_2$, is the graph with vertex set $V(G_1 \vee G_2) = V(G_1) \cup V(G_2)$ and edge set $E(G_1 \vee G_2) = E(G_1) \cup E(G_1) \cup \{xy | x \in V(G_1) \ and \ y \in V(G_2)\}$.

For $S \subseteq V(G)$, the induced subgraph of S, denoted by $G[S]$, is the graph whose vertex set is S and edge set is composed of those edges with both ends in S. For $E' \subseteq E(G)$, the induced subgraph of E', denoted by $G[E']$, is the graph whose edge set is E' and vertex set is composed of those vertices which is an end of edge in E'. For other notations and terminology not defined here, see [1].

We start with a lemma which will be frequently used in the subsequent part of this paper.

Lemma 1.(Zhou and Trinajstić [24]) *Let G be a connected simple graph with n vertices and m edges. Then*

$$RMTI(G) \leq \frac{3}{2}M_1(G) + (n-1)m$$

with equality if and only if $Diam(G) \leq 2$.

A subset S of the vertex set $V(G)$ is said to be a *vertex-independent set* of G if $G[S]$ is an empty graph, namely, a graph having no edges. Let S be a vertex-independent set of G. When G is connected, we have $V(G) - S \neq \emptyset$. If for any vertex x in $V(G) - S$, $N(x) \bigcap S \neq \emptyset$, then S is called a maximal vertex-independent set of G. Let $i(G) = min\{|S| :$ S is a maximal vertex-independent set of $G\}$, $i(G)$ is then said to be the *vertex-independence number* of G. If X is a maximal vertex-independent set of G with $|X| = i(G)$, then X is also said to be an $i(G)-$*set* of G.

A subset T of the edge set $E(G)$ is said to be an *edge-independent set* of G if T contains exactly one edge or any two edges in T (if do exist!) share no common vertices. Let T be an edge-independent set of G. When G is connected, we have $E(G)-T \neq \emptyset$. For any e in $E(G)-T$, if $\{e\} \cup T$ is no longer an edge-independent set of G, then T is called a maximal edge-independent set of G. Let $m(G) = min\{|T| : T$ is a maximal edge-independent set of $G\}$, $m(G)$ is then said to be the *edge-independence number* of G.

For a connected graph G, we evidently have $1 \leq m(G) \leq \lfloor \frac{n}{2} \rfloor$. As for the vertex-independence number $i(G)$, we also have the same property.

Proposition 2. *Let G be a connected simple graph with n vertices and $i(G) = k$. Then $1 \leq k \leq \lfloor \frac{n}{2} \rfloor$.*

Proof. The left-hand side of the above inequality is obvious. Now, we prove the right-hand one. If $G \cong S_n$, the result is evident. Suppose that $G \ncong S_n$ and we proceed by induction on

n. When $2 \leq n \leq 4$, it's easy to check that the result holds. Now let $n \geq 5$. Since $G \ncong S_n$, there must exist an edge, say e, in G such that $G - \{e\} = G_1 \cup G_2$ with $2 \leq n_1 \leq n-2$ and $2 \leq n_2 \leq n-2$, where n_1 and n_2 are orders of G_1 and G_2, respectively. Then by induction hypothesis, we have $i(G) \leq i(G_1) + i(G_2) \leq \lfloor \frac{n_1}{2} \rfloor + \lfloor \frac{n_2}{2} \rfloor = \lfloor \frac{n}{2} \rfloor$. This concludes the proof. ■

For $2 \leq k \leq \frac{n-1}{2}$, we define a graph $G_{n_1, \cdots, n_k}$ as follows:

For $2 \leq n_i \leq n - 2k + 2$, $i = 1, \cdots, k$, let $K_{n_1}, \cdots, K_{n_k}$ be k complete graphs of orders $n_1, \cdots, n_k$, respectively. Also, we let $V(K_{n_i}) = \{v_{i1}, \cdots, v_{in_i}\}$ for $i = 1, \cdots, k$. Then we write $G_{n_1, \cdots, n_k} = (K_{n_1} - \{v_{11}\}) \vee (K_{n_2} - \{v_{21}\}) \vee \cdots \vee (K_{n_k} - \{v_{k1}\})$. For $k = 2$, let $\bar{G}_{n_1, n_2}$ be the graph obtained from G_{n_1, n_2} by adding the edge $v_{11}v_{21}$. Now, we have the following:

Theorem 3. *Let G be a connected simple graph with n vertices and $i(G) = k (1 \leq k \leq \lfloor \frac{n}{2} \rfloor)$. Then we have*

(i). If $k = 1$, then $M_1(G) \leq n(n-1)^2$ with equality if and only if $G \cong K_n$.

(ii). If $k = 2$, then $M_1(G) \leq (n-1)(n-2)^2 + 4$ with equality if and only if $G \cong \bar{G}_{2, n-2}$.

(iii). If $3 \leq k \leq \frac{n-1}{2}$, then $M_1(G) \leq (n-k)^3 + (n-2k+1)^2 + k - 1$ with equality if and only if $G \cong G_{2, \cdots, 2, n-2k+2}$.

(iv). If $k = \frac{n}{2}$, then $M_1(G) \leq \frac{n^3}{4}$ with equality if and only if $G \cong K_{k,k}$.

Proof. We first prove a claim stated as follows.

Claim. *Let G be a connected simple graph with n vertices and $i(G) = k (1 \leq k \leq \lfloor \frac{n}{2} \rfloor)$. Then $d(x) \leq n - k$ for any vertex x in G.*

Proof. Assume to the contrary that there exists a vertex y in G such that $d(y) \geq n - k + 1$. Then we can obtain a maximal vertex-independent set $Y = \{y\} \cup Z(G - N[y])$ of G, where $Z(G - N[y])$ denotes a maximal vertex-independent set of $G - N[y]$. But then

$$i(G) \leq |Y| \leq 1 + (n - d(y) - 1) = n - d(y) \leq n - (n - k + 1) = k - 1,$$

which contradicts the fact that $i(G) = k$. So for each vertex y in G, we have $d(y) \leq n - k$, which completes the proof. ■

The proofs of (iv) **and** (i). When $n = 2k$, we clearly have $d(x) = \frac{n}{2} = n - k$ for each vertex x in $K_{k,k}$, a complete bipartite graph of a bipartition (k, k). By the above claim, we know that $M_1(K_{k,k})$ achieves the maximum cardinality among the set of graphs with $k = \frac{n}{2}$

since each vertex attains its largest degree. Moreover, $i(K_{k,k}) = k$. This proves (iv). Similarly, we can prove (i) holds. ■

The proofs of (ii) **and** (iii). Let G be the graph that among all connected simple graphs with n vertices and $i(G) = k$ has maximum cardinality of $M_1(G)$.

Let $X = \{v_{11}, \cdots, v_{k1}\}$, $2 \leq k \leq \frac{n-1}{2}$, be an $i(G)-$set of G. We clearly have $M_1(G[G-X]) \leq M_1(K_{n-k})$ with equality holding if and only if $G[G-X] \cong K_{n-k}$. Note that every vertex in K_{n-k} is of degree $n-k-1$. Thus, if $G[G-X] \cong K_{n-k}$, then every vertex in $V(G)-V(X)$ is adjacent to at most one vertex in X by the Claim. Furthermore, every vertex in $V(G)-V(X)$ must be adjacent to one vertex X by the choice of G. Now, we must have $G \cong G_{n_1, \cdots, n_k}$, where $n_i = d(v_{i1}) + 1$ for $i = 1, 2, \cdots, k$. Since G is connected, then $n_i = d(v_{i1}) + 1 \geq 2$ for all $i = 1, 2, \cdots, k$.

Without loss of generality, we may suppose now that $2 \leq n_1 \leq \cdots \leq n_k$. We claim that for $3 \leq k \leq \frac{n-1}{2}$, $G \cong G_{2, \cdots, 2, n-2k+2}$. If not so, there must exist two integers n_i and n_j such that $n_i \geq n_j \geq 3$, and then we delete the edge between v_{jn_j} and v_{j1}, and add one edge between v_{jn_j} and v_{i1}. Then we obtain the graph $G_{n_1, \cdots, (n_j-1), \cdots (n_i+1), \cdots, n_k}$, and thus

$$\begin{aligned} M_1(G_{n_1, \cdots, (n_j-1), \cdots (n_i+1), \cdots, n_k}) - M_1(G_{n_1, \cdots, n_k}) &= [(n_i-1)+1]^2 + [(n_j-1)-1]^2 \\ &\quad -(n_j-1)^2 - (n_i-1)^2 > 0, \end{aligned}$$

which contradicts the choice of G.

It's easy to check that for $3 \leq k \leq \frac{n-1}{2}$, $i(G_{2, \cdots, 2, n-2k+2}) = k$. So, for $3 \leq k \leq \frac{n-1}{2}$, $M_1(G) \leq M_1(G_{2, \cdots, 2, n-2k+2})$ with equality if and only if $G \cong G_{2, \cdots, 2, n-2k+2}$.

Similarly, when $k = 2$ and $G \ncong \bar{G}_{2, n-2}$, we must have $G \cong G_{2, n-2}$, otherwise, we can prove that $M_1(G) < M_1(G_{2, n-2})$, a contradiction to the choice of G. Obviously, $i(\bar{G}_{2, n-2}) = 2$ and $M_1(G_{2, n-2}) < M_1(\bar{G}_{2, n-2})$, and thus $M_1(G) \leq M_1(\bar{G}_{2, n-2})$ with equality if and only if $G \cong \bar{G}_{2, n-2}$. This concludes the proofs of (ii) and (iii). ■

Therefore, the proof of the present theorem is completed. ■

Corollary 4. *Let G be a connected simple graph with n vertices and $i(G) = k (1 \leq k \leq \lfloor \frac{n}{2} \rfloor)$. Then we have*

(i). If $k = 1$, then $RMTI(G) \leq 2n(n-1)^2$ with equality if and only if $G \cong K_n$.

(ii). If $k = 2$, then $RMTI(G) \leq \frac{1}{2}(4n^3 - 21n^2 + 31n - 2)$ with equality if and only if $G \cong \bar{G}_{2, n-2}$.

(iii). If $3 \leq k \leq \frac{n-1}{2}$, then $RMTI(G) < \frac{1}{2}(4n-3k-1)(n-k)^2 + \frac{1}{2}(n-1)(n-k) + \frac{3}{2}(n-2k+1)^2 + \frac{3}{2}(k-1)$.

(iv). If $k = \frac{n}{2}$, then $RMTI(G) \leq \frac{n^2}{8}(5n-2)$ with equality if and only if $G \cong K_{k,k}$.

Proof. When $k = 1, 2$, or $\frac{n}{2}$, $Diam(K_n) = Diam(K_{k,k}) = Diam(\bar{G}_{2,n-2}) = 2$. Then by Lemma 1 and Theorem 3, the result is immediate. But for $3 \leq k \leq \frac{n-1}{2}$, $Diam(G_{2,\cdots,2,n-2k+2}) = 3$, so the bound in (iii) can not be achieved. ■

Theorem 5. *Let G be a connected simple graph with n vertices and $m(G) = k(1 \leq k \leq \lfloor \frac{n}{2} \rfloor)$. Then*

$$M_1(G) \leq 2k(n-1)^2 + 4k^2(n-2k)$$

with equality if and only if $G \cong K_{2k} \vee (n-2k)K_1$.

Proof. If $n = 2k$, the result is evident. Suppose that $n \geq 2k+1$ hereinafter. Let G be the graph having maximum cardinality of $M_1(G)$ among all connected simple graphs with $m(G) = k$. We now take a maximal edge-independent set X from G with $|X| = k$. By the choice of G, we have $G[X] \cong K_{2k}$. Moreover, the graph induced by the vertices in $V(G) - V(G[X])$ must be an empty graph. Otherwise, if $G[V(G) - V(G[X])]$ has an edge e, then $\{e\} \cup X$ is an edge-independent set of G, contradicting that X is a maximal edge-independent set of G. Also, any vertex in $V(G) - V(G[X])$ must be adjacent to every vertex in $V(G[X])$ by the choice of G. Thus $G \cong K_{2k} \vee (n-2k)K_1$. It can be easily seen that $m(K_{2k} \vee (n-2k)K_1) = k$, and $M_1(K_{2k} \vee (n-2k)K_1) = 2k(n-1)^2 + 4k^2(n-2k)$. This proves the theorem. ■

Corollary 6. *Let G be a connected simple graph with n vertices and $m(G) = k$. Then*

$$RMTI(G) \leq (n-1)(5kn-4k-2k^2) + 6k^2(n-2k)$$

with equality if and only if $G \cong K_{2k} \vee (n-2k)K_1$.

Proof. The proof is immediate from the combination of Lemma 1 and Theorem 5 since $Diam(K_{2k} \vee (n-2k)K_1) = 2$. ■

If G is a connected simple graph of order n, which is different from the complete graph K_n, the *vertex-connectivity* of G is then equal to k if all subgraphs of G, obtained from G by deleting fewer than k vertices are connected, and a subgraph obtained from G by deleting exactly k vertices is no longer connected. In this case G is said to be a $k-vertex-$connected graph.

Obviously, the vertex-connectivity concept is not applicable to any complete graph. Thus, if k is the vertex-connectivity of G, then $1 \leq k \leq n-2$. Likewise, if G is a connected graph of order n, the *edge-connectivity* of G is then equal to k if all subgraphs of G, obtained by deleting from G fewer than k edges are connected, and a subgraph obtained from G by deleting exactly k edges is disconnected. In this case G is also said to be a $k-edge-$connected graph. If k is the edge-connectivity of G, then $1 \leq k \leq n-1$. Here we can easily see that $k = n-1$ if and only if $G \cong K_n$.

Theorem 7. *Let G be a simple graph with n vertices and vertex-connectivity $k(1 \leq k \leq n-2)$. Then*

$$M_1(G) \leq k(n-1)^2 + k^2 + (n-k-1)(n-2)^2$$

with equality if and only if $G \cong K_k \vee (K_1 + K_{n-1-k})$.

Proof. Let G be the graph having maximum cardinality of $M_1(G)$ among all k-vertex-connected graphs. Let X be a vertex cut of G with $|X| = k$. Set $G - X = \bigcup_{i=1}^{s} G_i$. Then $s \geq 2$. If $s \geq 3$, we may connect any two components of $G - X$ by a newly added edge, and then we get a graph G' with the same vertex connectivity as that of G, but $M_1(G') > M_1(G)$, a contradiction. Therefore, $s = 2$. By a similar argument, we can prove that any vertex in $G_1 \cup G_2$ is adjacent to every vertex in X. Also, $G[X]$, G_1 and G_2 should be complete graphs by the choice of G. Thus, $G \cong K_k \vee (K_{n_1} + K_{n_2})$, where n_1 and n_2 are respectively the orders of G_1 and G_2. Suppose without loss of generality that $1 \leq n_2 \leq n_1 \leq n-k-1$.

We claim that $n_2 = 1$. If $n_2 \geq 2$, we have

$M_1(K_k \vee (K_{(n_1+1)} + K_{(n_2-1)})) - M_1(K_k \vee (K_{n_1} + K_{n_2}))$

$$\begin{aligned} &= (n_1+1)[(n_1-1+k)+1]^2 + (n_2-1)[(n_2-2)+k]^2 - n_1(n_1-1+k)^2 \\ &\quad -n_2(n_2-1+k)^2 \\ &= \cdots \\ &= (3n+k-2)(n_1-n_2) - 4 + n + 3k + 4n_2 > 0, \end{aligned}$$

a contradiction to the choice of G. Hence, $G \cong K_k \vee (K_1 + K_{n-1-k})$. An easy computation gives $M_1(K_k \vee (K_1 + K_{n-1-k})) = k(n-1)^2 + k^2 + (n-k-1)(n-2)^2$. The proof is thus completed. ■

Corollary 8. *Let G be a simple graph with n vertices and vertex-connectivity $k(1 \leq k \leq$*

$n-2$). Then

$$RMTI(G) \leq \frac{3}{2}[k(n-1)^2 + k^2 + (n-k-1)(n-2)^2] + (n-1)[k + \frac{(n-1)(n-2)}{2}]$$

with equality if and only if $G \cong K_k \vee (K_1 + K_{n-1-k})$.

Proof. The proof is immediate from the combination of Lemma 1 and Theorem 7 since $Diam(K_k \vee (K_1 + K_{n-1-k})) = 2$. ■

Theorem 9. *Let G be a simple graph with $n \geq 4$ vertices and edge-connectivity $k(1 \leq k \leq n-1)$. Then*

$$M_1(G) \leq k(n-1)^2 + k^2 + (n-k-1)(n-2)^2$$

with equality if and only if $G \cong K_k \vee (K_1 + K_{n-1-k})$.

Proof. If $k = n-1$, the result is evident. So we may assume that $1 \leq k \leq n-2$ in what follows. Let G be the $k-$edge-connected graph such that $M_1(G)$ is maximum. We shall prove that $G \cong K_k \vee (K_1 + K_{n-1-k})$.

Let Y be an edge cut of G with $|Y| = k$. Then $G-Y$ has exactly two components, say G_1 and G_2. Let the orders of G_1 and G_2 be n_1 and n_2, respectively. By the maximality of $M_1(G)$, we must have $G_1 \cong K_{n_1}$ and $G_2 \cong K_{n_2}$. Set $|V(G[Y]) \bigcap V(G_1)| = b_1$ and $|V(G[Y]) \bigcap V(G_2)| = b_2$, where $G[Y]$ is the subgraph of G induced by Y. Obviously, $n_1 \geq b_1 \geq 1$ and $n_2 \geq b_2 \geq 1$. If $n_1 = 1$ or $n_2 = 1$, the result is obvious. Now, suppose that $n_1 \geq 2$ and $n_2 \geq 2$. If $b_1 = 1$ (or $b_2 = 1$), then one vertex, say v_1 in G_1, has exactly k neighbors in G_2 and all other vertices in G_1 have no neighbors in G_2, and thus we can obtain a new graph G' by adding edges between vertices in $V(G_1) - \{v_1\}$ and vertices in $V(G_2)$ and thus $M_1(G') > M_1(G)$, a contradiction to the choice of G.

So it will be assumed that $b_1 \geq 2$ and $b_2 \geq 2$ below. We distinguish between two cases.

Case 1. $\{b_1, b_2\} \neq \{n_1, n_2\}$.

We may suppose that $b_2 \neq n_2$.

Set $V(G[Y]) \bigcap V(G_1) = \{v_1, \cdots, v_{b_1}\}$ and $V(G[Y]) \bigcap V(G_2) = \{u_1, \cdots, u_{b_2}\}$. Let X denote the union of the set of neighbors of all vertices in $\{v_2, \cdots, v_{b_1}\}$ in G_2. Let m be the number of neighbors of v_1 in G_2. Then $1 \leq m \leq k-1$ since $b_1 \geq 2$. Moreover, we have $m \leq n_2 - 1$ since v_1 is not adjacent to at least one vertex in $V(G_2)$ (by the fact that $b_2 \neq n_2$). Now, we delete all edges between v_1 and all other vertices in $V(G_1) - \{v_1\}$, and we add all possible edges between vertices in $V(G_1)-\{v_1\}$ and those in G_2. Also, we can always find any $k-m$ vertices from $(V(G_2)-Z)\bigcup(V(G_1)-\{v_1\})$ and add edges between these vertices and v_1, where Z is the set of neighbors of v_1 in G_2. In fact, we have now "turned" the graph G into $K_k \vee (K_1+K_{n-1-k})$. From the above process, we can find that each vertex in $V(G)-\{v_1\}$ is now of degree $n-2$ or $n-1$ in $K_k \vee (K_1+K_{n-1-k})$ which is not less than that in G. Let $w \in V(G_2)-\{u_1,\cdots,u_{b_2}\}$, then w is adjacent to no vertex in G_1, that is, $d(w)=n_2-1$. since $k \leq n-2$, we may think that w is one of the vertices not adjacent to v_1 in $K_k \vee (K_1+K_{n-1-k})$.

So $M_1(K_k \vee (K_1+K_{n-1-k})) - M_1(G)$

$$\begin{aligned}
&\geq [k^2-(n_1-1+m)^2]+[(n-2)^2-(n_2-1)^2] \\
&= (k^2-m^2)-(n_1-1)^2-2(n_1-1)m+(n_1-1)^2+2(n_2-1)(n_1-1) \\
&= (k^2-m^2)+2(n_1-1)(n_2-m-1)>0,
\end{aligned}$$

a contradiction to the choice of G.

Case 2. $\{b_1, b_2\}=\{n_1, n_2\}$.

In this case, all vertices in both G_1 and G_2 are incident with at least one cut edge. It can be seen that for any $2 \leq n_2 \leq n-2$, there exists $k \leq n-2 \leq (n-n_2-1)n_2=(n_1-1)n_2$. Without loss of generality, we may suppose that $d(v_1)=min\{d(x): x \in V(G_1)\}$. Now, we delete all cut edges between v_1 and its neighbors in $V(G_2)$ and add the equal number of cut edges between the neighbors of v_1 in $V(G_2)$ and some vertices, say $v_{j1},\cdots,v_{js}$, in $V(G_1)-\{v_1\}$. Since $k \leq (n_1-1)n_2$, we can guarantee that the above procedure is effective. Now, we have obtained a new graph G^* which is still $k-edge-$connected. Let the number of neighbors of v_1 in $V(G_2)$ be m. Also, we let $d_{G^*}(v_{ji})=d(v_{ji})+t_i$, $i=1,\cdots,s$. Note that $\sum_{i=1}^{s} t_i=m$. Thus $M_1(G^*)-M_1(G)=[(d(v_1)-m)^2-d^2(v_1)]+\sum_{i=1}^{s}[(d(v_{ji})+t_i)^2-d^2(v_{ji})]>-2md(v_1)+2\sum_{i=1}^{s} d(v_{ji})t_i+m^2 \geq 2\sum_{i=1}^{s} d(v_1)t_i-2md(v_1)+m^2=m^2>0$. Thus, $M_1(G)<M_1(G^*)$, a contradiction once again.

From the combination of Cases 1 and 2 it follows the theorem as desired. ■

Corollary 10. *Let G be a simple graph with n vertices and edge-connectivity $k(1 \leq k \leq n-2)$. Then*

$$RMTI(G) \leq \frac{3}{2}[k(n-1)^2+k^2+(n-k-1)(n-2)^2]+(n-1)[k+\frac{(n-1)(n-2)}{2}]$$

with equality if and only if $G \cong K_k \vee (K_1 + K_{n-1-k})$.

Proof. Since $Diam(K_k \vee (K_1 + K_{n-1-k})) = 2$, the proof is immediate from the combination of Lemma 1 and Theorem 9. ■

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