MATCH
Communications in Mathematical and in Computer Chemistry

MATCH Commun. Math. Comput. Chem. **60** (2008) 179-182

ISSN 0340 - 6253

An Induced Subgraph of the Dualist of a Hexagonal System

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(Received July 20, 2007)

Abstract

The structure of cycles in an induced subgraph of the dualist of a hexagonal system is explored.

1 Introduction

The material on lattices can be found in [1]. A *lattice* is an array of points in the plane with position vectors $r_\nu = \nu_1 a_1 + \nu_2 a_2$, where a_1 and a_2 are two linearly independent *primitive vectors* and where ν_1 and ν_2 are integers. The *triangular lattice* of lattice spacing a is a lattice with primitive vectors $a_1 = (a, 0)$ and $a_2 = (\frac{1}{2}a, \frac{\sqrt{3}}{2}a)$.

A *generalized lattice* is derived from a lattice of points with position vectors r_ν by replacing the point at r_ν by m points with position vectors $r_\nu^{(l)} = r_\nu + b_l$ $(l = 1, 2, \ldots, m)$, where the b_l's are the *basis vectors*. The *hexagonal lattice* of lattice spacing a is a generalized lattice with primitive vectors $a_1 = (\sqrt{3}a, 0), a_2 = (\frac{\sqrt{3}}{2}a, \frac{3}{2}a)$ and basis vectors $b_1 = (0, 0), b_2 = (0, a)$.

The *triangular lattice graph* of lattice spacing a is the graph whose vertices are the points of the triangular lattice of lattice spacing a, and where two vertices are adjacent if the distance between them is a. Informally, the triangular lattice graph of lattice spacing a is obtained by tiling the plane with regular triangles of side length a.

The *hexagonal lattice graph* of lattice spacing a is the graph whose vertices are the points of the hexagonal lattice of lattice spacing a, and where two vertices are adjacent if the distance between them is a. Informally, the hexagonal lattice graph of lattice spacing a is obtained by tiling the plane with regular hexagons of side length a.

For any cycle C on the hexagonal lattice graph of lattice spacing 1, the vertices and the edges lying on C and in the interior of C form a *hexagonal system* [2]. The *dualist* [3], also called the *skeleton*, of a hexagonal system is obtained by replacing the centers of the hexagons with vertices and two vertices are joined by a line if the corresponding hexagons are adjacent. The dualist of a hexagonal system is a subgraph of the triangular lattice graph of lattice spacing $\sqrt{3}$.

2 The result

Remark 1 ([4])**.** *For each perfect matching M of a hexagonal system H, there exists an M-alternating hexagon.*

Theorem 2. *Let H be a hexagonal system with perfect matchings. Let M be a perfect matching of H. Consider the subgraph of the dualist of H induced by the vertices corresponding to the M-alternating hexagons. For each cycle C of this subgraph, if C is turned into a directed cycle, then the number of right arcs of C equals the number of left arcs of C, the number of up-right arcs of C equals the number of down-left arcs of C and the number of up-left arcs of C equals the number of down-right arcs of C.*

Proof. It suffices to show that the number of right arcs of C equals the number of left arcs of C.

We consider the non-trivial case where there exists a horizontal arc, i.e. a right arc or a left arc, in C. (In fact, it can be shown that there exists a horizontal arc in C, but this is unnecessary.) The cycle C can be directed in two ways, however, it suffices to show the result for exactly one of these. Let the cycle be directed so that it has a right arc.

Let us gain further insight into the structure of C that will prove useful later. In C, note that a right arc is followed by either an up-right arc or a down-right arc, a left arc is followed by either an up-left arc or a down-left arc, an up-right arc is followed by either a right arc or an up-left arc, an up-left arc is followed by either a left arc or an up-right arc, a down-right arc is followed by either a right arc or a down-left arc, and a down-left arc is followed by either a left arc or a down-right arc.

The remainder of the proof is structured as follows. The right arcs of C are classified into two types, type I and type II, and the left arcs of C are also similarly classified into two types, type I and type II. Then it is shown that the number of type II-right arcs of C equals the number of type II-left arcs of C. Finally, it is shown that the number of type I-right arcs of C equals the number of type I-left arcs of C, hence the result.

A right arc of C is of *type I* (*type II*) if the next horizontal arc, as we move along C, is a right (left) arc. Similarly, a left arc of C is of *type I* (*type II*) if the next horizontal arc, as we move along C, is a left (right) arc.

Cut C into directed paths such that each directed path starts with a right arc and ends with the arc preceding the next right arc. A directed path starting with a type I-right arc has exactly one horizontal arc, in particular, it has neither a type II-right arc nor a type II-left arc. A directed path starting with a type II-right arc has the arc sequence right, left, left, …, left, discarding the non-horizontal arcs. Clearly, this directed path has exactly one type II-right arc, the first in the sequence, and exactly one type II-left arc, the last in the sequence. Hence, the number of type II-right arcs of C equals the number of type II-left arcs of C.

Cut C into directed paths such that each directed path starts with a horizontal arc and ends with the arc preceding the next horizontal arc. Consider one such directed path. The first arc of it is either a type I-right arc, a type II-right arc, a type I-left arc, or a type II-left arc.

Case Type I-right arc: The directed path has either the arc sequence right, up-right, up-left, up-right, …, up-left, up-right or the arc sequence right, down-right, down-left, down-right, …, down-left, down-right. As we move along such a directed path, the x-coordinate increases by $1.5d$, where $d = \sqrt{3}$ (the distance between the centers of two adjacent hexagons).

Case Type II-right arc: The directed path has either the arc sequence right, up-right, up-left. …, up-right, up-left or the arc sequence right, down-right, down-left, …, down-right, down-left. A we move along such a directed path, the x-coordinate increases by d, where $d = \sqrt{3}$.

Case Type I-left arc: The directed path has either the arc sequence left, up-left, up-right, up-left, ..., up-right, up-left or the arc sequence left, down-left, down-right, down-left, ..., down-right, down-left. As we move along such a directed path, the x-coordinate decreases by $1.5d$, where $d = \sqrt{3}$.

Case Type II-left arc: The directed path has either the arc sequence left, up-left, up-right, ..., up-left, up-right or the arc sequence left, down-left, down-right, ..., down-left, down-right. As we move along such a directed path, the x-coordinate decreases by d, where $d = \sqrt{3}$.

As we move along C, the x-coordinate does not change. Hence, $1.5d$ times the number of type I-right arcs of C plus d times the number of type II-right arcs of C minus $1.5d$ times the number of type I-left arcs of C minus d times the number of type II-left arcs of C equals zero. Therefore, the number of type I-right arcs of C equals the number of type I-left arcs of C. This completes the proof.

$\sqcup$

Corollary 3 ([5]). *Let H be a hexagonal system with perfect matchings. Let M be a perfect matching of H. The subgraph of the inner dual of H induced by the vertices corresponding to the M-alternating hexagons is bipartite.*

In fact, this corollary can be generalized to 2-connected plane bipartite graphs with perfect matchings [5].

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