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# A Note on Graphs with Zero Nullity

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**Abstract**

By nullity of a graph we mean that of its adjacency matrix. We determine the possible size of a graph with zero nullity. For any possible size we also construct a graph with zero nullity.

## Introduction and Preliminaries

Throughout this paper, we consider simple graphs with no loops or multiple edges, and we will use standard terminology and ideas from graph theory. For convenience, we recall a few key notions that will be used in the paper. A vertex in a graph is a *pendant vertex* if it has degree 1. Let $G$ be a graph with vertex set $\{v_1, \ldots, v_n\}$. The *adjacency matrix* of $G$ is the $n \times n$ matrix $A$ such that $A_{ij}$ is 1 if $v_i$ and $v_j$ are adjacent and is 0 otherwise. We will use $\eta(G)$ to denote the algebraic multiplicity 0 as an eigenvalue of adjacency matrix of the graph $G$ and we call it *nullity of* $G$. We denote a path on $n$ vertices by $P_n$. Also we denote the complete graph of order $n$ and the complete bipartite graph with parts of sizes $m$ and $n$, by $K_n$ and $K_{m,n}$, respectively. If $G$ and $H$ are two arbitrary graphs, $u$ a

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vertex of $G$, and $v$ a vertex of $H$, the *coalescence* of $G$ and $H$ on $u$ (or on $v$) is the graph obtained by identifying the vertices $u$ and $v$.

In [2] L. Collatz and U. Sinogowitz posed the problem of characterizing all graphs which have the number zero in their spectrum. This question is a very interesting one in chemistry, because, as has been shown in [9], the occurrence of a zero eigenvalue in the spectrum of a bipartite graph (corresponding to an alternant hydrocarbon) indicates chemical instability of the molecule which such a graph represents. The question is of interest also for non-alternant hydrocarbons (non-bipartite graphs), but a direct connection with chemical stability is in these cases not so straightforward. The problem has not yet been solved completely, but some particular results are known [4–8].

The problem of finding a connection between the graph structure and its nullity is of interest. For instance, this connection has been studied by expressing a set of rules by which we can, after a finite number of steps, determine $\eta(G)$ (see, e.g., [5, 7]).

In this note we deal with a connection between the size and the nullity of a graph. Namely, we determine the possible size of a graph with zero nullity. For any possible size we also construct a graph with zero nullity. Specifically, we determine those graphs that their sizes force them to have a non-zero nullity.

## Results

The following lemma is well-known (see, e.g. [3]).

**Lemma 1.** *Let $G$ be a graph. If $v$ is a pendent vertex of $G$ with unique neighbor $u$, then* $\eta(G) = \eta(G \setminus \{u, v\})$.

Let $H$ be a subgraph of $G$. We denote by $G \setminus E(H)$ the graph obtained by removing the edges of $H$ from $G$. The following lemma is an immediate consequence of Proposition 7 of [1].

**Lemma 2.** *For $n \geq 4$ and $2 \leq t \leq n-1$, the nullity of $K_n \setminus E(K_{1,t})$ is zero.*

In the following we determine the graphs that their sizes force them to have a non-zero nullity.

**Theorem 1.** *Let $G$ be a graph of order $n$ and size $m$. If $m < \frac{n}{2}$, $m = \lceil \frac{n}{2} \rceil$ (for odd $n$), or $m = \binom{n}{2} - 1$, then $G$ has a non-zero nullity.*

**Proof.** We distinguish the following cases:

- If $m < \frac{n}{2}$, then the graph has an isolated vertex and so its nullity is not zero.
- If $n$ is odd and $m = \lceil \frac{n}{2} \rceil$, then the graph has at least $\lfloor \frac{n}{2} \rfloor$ components. So if it has no isolated vertices, it has a $P_3$ as a component. Thus it has a non-zero nullity.
- If $m = \binom{n}{2} - 1$, then the graph has two vertices with the same neighbors, i.e., there exist two equal rows in the adjacency matrix of $G$ so its nullity is not zero.

□

Now we would like to prove that beside the above cases, there exists a graph of arbitrary size and zero nullity.

**Theorem 2.** *Let $n \geq 5$, $\lceil \frac{n}{2} \rceil \leq m \leq \binom{n}{2}$, $m \neq \binom{n}{2} - 1$, and $m \neq \lceil \frac{n}{2} \rceil$ for odd $n$. Then there is a graph of order $n$ and size $m$ with zero nullity.*

**Proof.** If $n$ is even and $\lceil \frac{n}{2} \rceil \leq m \leq n-1$, the graph $P_{2m-n+2} \cup (n-m-1)P_2$ is a graph of order $n$ and size $m$ of nullity zero. If $n$ is odd and $\lceil \frac{n}{2} \rceil + 1 \leq m \leq n-1$, then the graph $P_{2m-n-1} \cup (n-m-1)P_2 \cup K_3$ is the desired one.

Now, by induction on $n$, we prove that for every $n$ and $m \geq n$ there exists a graph $G$ such that for a vertex $v_G$ of $G$ both the graphs $G$ and $G \setminus \{v_G\}$ have zero nullity. We call the graphs with these properties *good*. For $n = 5$, consider the graphs of Figure 1 as well as the graph $K_5$ which are of size $m = 5, 6, 7, 8, 10$. By Lemmas 1 and 2, it is easy to see that these graphs are good. Let $n \geq 6$. By Lemma 2, the graph $G = K_n \setminus E(K_{1,t})$,

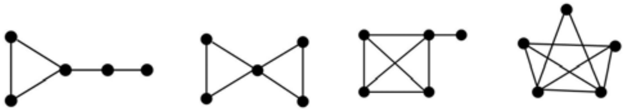

Figure 1: good graphs of order $n = 5$

for $2 \leq t \leq n-1$, in which $v_G$ is one the vertex with maximum degree is good. Thus for any $m$, $\binom{n-1}{2} + 1 \leq m \leq \binom{n}{2}$, and $m \neq \binom{n}{2} - 1$ there exist a good graph. For $m = \binom{n-1}{2}$, by Lemmas 1 and 2 the graph $G$ which is the coalescence of $K_n \setminus E(K_2)$ and $K_2$ on a vertex of $K_n \setminus E(K_2)$ with the least degree is good (take $v_G$ to be one of the vertex with maximum degree). If $n \leq m \leq \binom{n-1}{2} - 1$, then by induction hypothesis (or by the first part of the proof), there is a graph $H$ of order $n-1$ and size $m-1$ which has a vertex $v_H$ such that both $H$ and $H \setminus \{v_H\}$ has zero nullity. Now, by Lemma 1 the graph $G$ which is the coalescence of $H$ and $K_2$ on $v_H$ is a good graph of order $n$ and size $m$ (take $v_G$ to be the pedant vertex of $G$). $\square$

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