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On tricyclic graphs with minimal energy*

Shuchao Li[a], Xuechao Li[b], Zhongxun, Zhu[c,†]

[a]Faculty of Mathematics and Statistics, Central China Normal University, Wuhan 430079, P.R. China

[b]Division of Academic Enhancement, The University of Georgia, GA, USA 30602

[c]Department of Computer Science, South Central University for Nationalities, Wuhan 430074, P.R. China

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Abstract. The energy of a graph is defined as the sum of the absolute values of all the eigenvalues of the graph. Let $\mathscr{G}_n$ be the class of tricyclic graphs G on n vertices and containing no disjoint odd cycles C_p, C_q of lengths p and q with $p+q \equiv 2 \pmod 4$. In this paper, we obtain the minimal and second-minimal values on the energies of the graphs in $\mathscr{G}_n$ and determine the corresponding graphs.

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†Corresponding author. lscmath@mail.ccnu.edu.cn (S. Li), xcli@uga.edu (X. Li), zzxun73@163.com (Z. Zhu)

1. Introduction

Let G be a simple graph with n vertices. Let $A(G)$ be the adjacency matrix of G. The *characteristic polynomial* of G is

$$\phi(G,\lambda) = \det(\lambda I - A) = \sum_{i=0}^{n} a_i \lambda^{n-i},$$

Sachs theorem states that [12] for $i \geqslant 1$,

$$a_i = \sum_{S \in L_i} (-1)^{p(S)} 2^{c(S)},$$

where L_i denotes the set of Sachs graphs of G with i vertices, that is, the graphs in which every component is either a K_2 or a cycle, $p(S)$ is the number of components of S and $c(S)$ is the number of cycles contained in S. In addition $a_0 = 1$. The roots $\lambda_1, \ldots, \lambda_n$ of $\phi(G,\lambda)$ are called the eigenvalues of G. Since $A(G)$ is symmetric, all *eigenvalues* of G are real. Let C_n denote a cycle of length n. Other undefined notation may refer to [2, 12].

The *energy* of G, denoted by $E(G)$, is then defined as $E(G) = \sum_{i=1}^{n} |\lambda_i|$. Since the energy of a graph can be used to approximate the total π-electron energy of the molecule (e.g., see [11, 12]), there are numerous results on $E(G)$ (e.g., see [1,3,4,5-11,13-27,29-33,35-42]), including graphs with extremal energies [3,7,17,18,20,21,23-26,30,31,33,35-40,43-47].

It is known that [12] $E(G)$ can be expressed as the Coulson integral formula

$$E(G) = \frac{1}{\pi}\int_0^{+\infty} \frac{dx}{x^2} \ln\left[\left(\sum_{i=0}^{\lfloor \frac{n}{2} \rfloor} (-1)^i a_{2i} x^{2i}\right)^2 + \left(\sum_{i=0}^{\lfloor \frac{n}{2} \rfloor} (-1)^i a_{2i+1} x^{2i+1}\right)^2\right]. \tag{1.1}$$

Let $b_{2i}(G) = (-1)^i a_{2i}$ and $b_{2i+1}(G) = (-1)^i a_{2i+1}$ for $0 \leqslant i \leqslant \lfloor \frac{n}{2} \rfloor$. Clearly, $b_0(G) = 1$ and $b_2(G)$ equals the number of edges of G. Thus, by (1.1), $E(G)$ is a strictly monotonically increasing function of $b_i(G), i = 1, \ldots, \lfloor n/2 \rfloor$. A *quasi-order* is introduced (see [12]): if G_1 and G_2 are two graphs, then

$$G_1 \succeq G_2 \Leftrightarrow b_i(G_1) \geqslant b_i(G_2) \text{ for all } i \geqslant 0.$$

If $G_1 \succeq G_2$, and there exists one j such that $b_j(G_1) > b_j(G_2)$, then we write $G_1 \succ G_2$. Therefore,

$$G_1 \succ G_2 \Rightarrow E(G_1) > E(G_2).$$

Many results on the minimal energy have been obtained for various classes of graphs. In [3], Caporossi et al. gave the following conjecture.

Conjecture 1.1. *Connected graphs G with $n \geqslant 6$ vertices, $n-1 \leqslant e \leqslant 2(n-2)$ edges and minimum energy are star with $e-n+1$ additional edges all connected to the same vertex for $e \leqslant n + \lfloor \frac{n-7}{2} \rfloor$, and bipartite graphs with two vertices on one side, one of which is connected to all vertices on the other side otherwise.*

This conjecture is true when $e = n-1, 2(n-2)$ [3, Theorem 1], and when $e = n$ for $n \geqslant 6$ [17], $e = n+1$ for $n \geqslant 9$ [39]. In this paper, we consider the above conjecture for the case $e = n+2$ for $n \geqslant 7$.

Figure 1: Graphs G_n^0 and G_n^1

A connected simple graph with n vertices and $e = n+2$ edges is called a *tricyclic graph.* Let $\mathscr{G}_n$ be the class of tricyclic graph G with n vertices and containing no disjoint two odd cycles C_p, C_q with $p+q \equiv 2 \pmod 4$. Let G_n^0 be the graph formed by joining 3 pendent vertices to a vertex of degree one of the $K_{1,n-1}$ (e.g., see Figure 1), and G_n^1 be the graph formed by joining $n-6$ pendent vertices to a vertex of degree 4 of the complete bipartite graph $K_{2,4}$ (e.g., see Figure 1). In this paper, we show that G_n^0, G_n^1 have, respectively, minimal and the second-minimal energies in $\mathscr{G}_n$ for $n \geqslant 11$ and G_n^1 has the minimal energy in $\mathscr{G}_n$ for $7 \leqslant n \leqslant 10$.

The following two lemmas are needed in our paper.

Lemma 1.2 ([39])**.** *Let G be a graph with n vertices and let uv be a pendent edge of G with pendent vertex v. Then for $2 \leqslant i \leqslant n$, $b_i(G) = b_i(G-v) + b_{i-2}(G-u-v)$.*

Lemma 1.3 ([39])**.** *Let G be any graph. Then $b_4(G) = m(G,2) - 2s$, where $m(G,2)$ is the number of 2-matchings of G and s is the number of quadrangles in G.*

2. Lemmas and main results

In this section, we shall determine the tricyclic graphs in $\mathscr{G}_n$ ($n \geqslant 11$) having the minimal and the second-minimal energies. Our idea is, at first, to show $E(G) > E(G_n^1)$ for any $G \in \mathscr{G}_n$ with $G \ncong G_n^0, G_n^1, R_n, W_n, S_n, Q_n$; and proceed to show that $E(G_n^0) < E(G_n^1)$, where R_n, S_n, Q_n are as shown in Figure 2. For $\mathscr{G}_n$ with $7 \leqslant n \leqslant 10$, we characterize the graph with minimal energy. The following fact is immediate.

Fact 1. *For any $G \in \mathscr{G}_n$, there are at most three edge-disjoint cycles contained in G.*

Lemma 2.1. *If $G \in \mathscr{G}_n$, then $b_{2i} \geqslant 0$ for $0 \leqslant i \leqslant \lfloor \frac{n}{2} \rfloor$.*

Proof. Let L_i be the set of Sachs graphs of G with i vertices. By Sachs theorem,

$$b_{2i} = \sum_{S \in L_{2i}} (-1)^{p(S)+i} 2^{c(S)} = \sum_{S \in L_{2i}^1} (-1)^{p(S)+i} + \sum_{S \in L_{2i}^2} (-1)^{p(S)+i} 2^{c(S)},$$

where L_{2i}^1 is the set of graphs with no cycles in L_{2i}, and $L_{2i}^2 = L_{2i} \setminus L_{2i}^1$.

If every S in L_{2i} has no cycle, then $p(S) = i$, and so

$$b_{2i}(G) = \sum_{S \in L_{2i}} 1 \geqslant 0.$$

Otherwise, there exists S' in L_{2i} such that S' contains cycles. If S' has no odd cycles, then $b_{2i}(G) \geqslant 0$ [14]; otherwise, together with Fact 1, S' must contain two edge-disjoint odd cycles, say C_k, C_l. Since $G \in \mathscr{G}_n$, we have $k + l \equiv 0 \pmod 4$. If S' has no cycle except C_k and C_l, then

$$p(S') + i = 2 + \frac{2i - (k+l)}{2} + i \equiv 0 \pmod 2.$$

If S' has another cycle C_m, then C_m must be even. Thus its corresponding term in b_{2i} is the following

$$(-1)^{p(S')+i} 2^3.$$

On the other hand, since C_m is an even cycle, it has exactly two perfect matching, say M_1, M_2, therefore there exist Sachs graphs S_1'', S_2'' in L_{2i} such that $S_1'' := (S' \setminus C_m) \cup C_k \cup C_l \cup M_1$ and $S_2'' := (S' \setminus C_m) \cup C_k \cup C_l \cup M_2$, respectively. Its corresponding term in b_{2i} is the following

$$(-1)^{p(S_1'')+i} \cdot 2^2 + (-1)^{p(S_2'')+i} \cdot 2^2,$$

where $p(S_1'')+i = p(S_2'')+i = 2+\frac{2i-(k+l)}{2}+i \equiv 0 \pmod 2$. It is easy to see $|L_{2i}^1| \geqslant 2|L_{2i}^2|$, and so

$$b_{2i} \geqslant \sum_{M_1,M_2 \subseteq C_m}^{C_m \subseteq S' \in L_{2i}^2} \left[(-1)^{p(S_1'')+i} \cdot 2^2 + (-1)^{p(S_2'')+i} \cdot 2^2 + (-1)^{p(S')+i} \cdot 2^3\right] \geq 0,$$

where $S_1'' = (S' \setminus C_m) \cup C_k \cup C_l \cup M_1$ and $S_2'' = (S' \setminus C_m) \cup C_k \cup C_l \cup M_2$. $\square$

In $\mathscr{G}_n$, there exist four special graphs, namely that R_n, W_n, S_n, Q_n; see Figure 2, where R_n has $n-7$ pendent vertices, W_n has $n-6$ pendent vertices, S_n has $n-5$ pendent vertices and Q_n has $n-4$ pendent vertices. In the following lemmas, we shall repeatedly use these graphs.

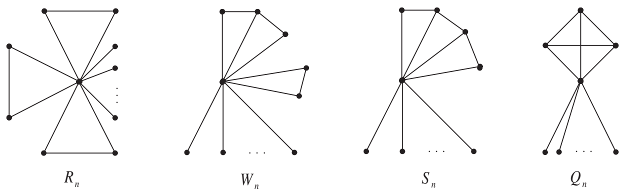


Figure 2: Graphs R_n, W_n, S_n and Q_n.

It is straightforward to check that graph $G \in \mathscr{G}_n$ has at least 3 cycles and at most 7 cycles. Furthermore, there do not exist five cycles in G.

Let $m(G,2)$ denote the number of 2-matchings of a graph G. Obviously, $m(P_n,2) = (n-2)(n-3)/2$ and $m(C_n,2) = n(n-3)/2$.

Lemma 2.2. *If $G \in \mathscr{G}_n$ has exactly three cycles with $G \not\cong R_n$, then $b_4(G) > b_4(G_n^1)$ for $n \geqslant 7$.*

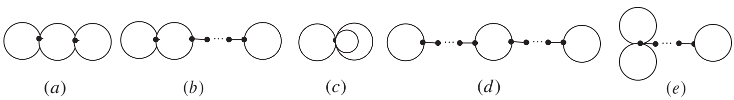


Figure 3: Five possible cases for the arrangement of three cycles in G.

Proof. Since $G \in \mathscr{G}_n$ contains exactly three cycles, say C_a, C_b, C_c, then these must be edge-disjoint; see Figure 3. It is immediate that $n+2-a-b-c \geqslant 0$. By induction on $n+2-a-b-c$. If $n+2-a-b-c=0$, then either C_a, C_b, C_c has exactly one vertex in common (e.g., see (c) in Figure 3), or there exist two pairs of cycles in $\{C_a, C_b, C_c\}$, such that each pair of cycles, say $\{C_a, C_b\}$ (respectively, $\{C_b, C_c\}$) have a vertex in common, say v_1 (respectively, v_2) satisfying $v_1 \neq v_2$ (e.g., see (c) in Figure 3). For the latter, by Lemma 1.2, we have

$$\begin{aligned} b_4(G) &= m(G,2) - 2s \geqslant m(G,2) - 6 \\ &= \frac{a(a-3)}{2} + \frac{b(b-3)}{2} + \frac{c(c-3)}{2} + (a-2)(n+2-a) \\ &\quad +2(n+2-a-2) + (b-2)c + 2(c-2) - 6 \\ &= \frac{1}{2}(a+b+c)^2 - \frac{3}{2}(a+b+c) - 14 \\ &= \frac{1}{2}n^2 + \frac{1}{2}n - 15. \end{aligned}$$

and so

$$b_4(G) - b_4(G_n^1) \geqslant \frac{1}{2}n^2 + \frac{1}{2}n - 15 - (4n-24) = \frac{1}{2}n^2 - \frac{7}{2}n + 9 > 0,$$

and hence $b_4(G) > b_4(G_n^1)$. Similarly, for the former, namely that for (c) in Figure 3, we can also prove $b_4(G) > b_4(G_n^1)$ when $p = k \geqslant 1$.

Suppose it is true for all graphs $G \in \mathscr{G}_n$ having exactly three cycles and $G \ncong R_n$ with $n+2-a-b-c < p$ ($p \geqslant 1$), and suppose $n+2-a-b-c = p$.

Case 1. There are no pendent edges in G. Then there are at most two cycles having a vertex in common.

Subcase 1.1. If there are exactly two cycles, say C_b and C_c, having exactly one vertex in common; see (b), (e) in Figure 3. For (b) in Figure 3, if $p = k \geqslant 1$, then C_b (or, C_c) connects C_a by a path of length k. Without loss of generality, let C_b connect C_a by a path of length k. By Lemma 1.2,

$$\begin{aligned} b_4(G) &= m(G,2) - 2s \geqslant m(G,2) - 6 \\ &= \frac{a(a-3)}{2} + \frac{b(b-3)}{2} + \frac{c(c-3)}{2} + \frac{(k-1)(k-2)}{2} + (a-2)(n+2-a) \\ &\quad + 2(n+2-a-1) + (k-1)(b+c) + (b-2+c) \end{aligned}$$

$$
\begin{aligned}
& +(b-2)c+2(c-2)-6 \\
= \;& \frac{1}{2}(a+b+c)^2-\frac{3}{2}(a+b+c)+k(a+b+c)+\frac{(k-1)(k-2)}{2}-14 \\
= \;& \frac{1}{2}n^2+\frac{1}{2}n-14,
\end{aligned}
$$

and thus

$$
b_4(G)-b_4(G_n^1) \;\geqslant\; \frac{1}{2}n^2+\frac{1}{2}n-14-(4n-24)=\frac{1}{2}n^2+\frac{7}{2}n+10>0,
$$

and therefore $b_4(G)>b_4(G_n^1)$. Similarly, for the former, namely that for (e) in Figure 1, we can also prove $b_4(G)>b_4(G_n^1)$ when $p=k\geqslant 1$.

Subcase 1.2 If there does not exist two cycles in $\{C_a,C_b,C_c\}$ having exactly one vertex in common, then set $n+2-a-b-c=k\geqslant 2$. Hence there exist two pairs of cycles in $\{C_a,C_b,C_c\}$, say $\{C_a,C_b\}$ and $\{C_b,C_c\}$, such that each pair of such two cycles are connected by a path. Without loss of generality, assume that C_a (respectively, C_c) connects C_b by P_{r+1} (respectively, P_{l+1}), where $r\geqslant 1$ (respectively, $l\geqslant 1$). Then by Lemma 1.2,

$$
\begin{aligned}
b_4(G) \;&=\; m(G,2)-2s\geqslant m(G,2)-6 \\
&=\; \frac{a(a-3)}{2}+\frac{b(b-3)}{2}+\frac{c(c-3)}{2}+\frac{(r-1)(r-2)}{2}+\frac{(l-1)(l-2)}{2} \\
&+\; (a-2)(n+2-a)+2(n+2-a-1)+(r-1)(b+c+l)-6 \\
&+\; (b+c+l-2)+(b-2)(c+l)+2(c+l-1)+(l-1)c+(c-2) \\
&=\; \frac{1}{2}(a+b+c)^2-\frac{3}{2}(a+b+c)+k(a+b+c)+\frac{1}{2}(k-2)(k-3)-15 \\
&=\; \frac{1}{2}n^2+\frac{1}{2}n-13,
\end{aligned}
$$

and so

$$
b_4(G)-b_4(G_n^1) \;\geqslant\; \frac{1}{2}n^2+\frac{1}{2}n-13-(4n-24)=\frac{1}{2}n^2-\frac{7}{2}n+11>0.
$$

It is immediate that $b_4(G)>b_4(G_n^1)$.

Case 2. uv is a pendent edge of G with pendent vertex v. By Lemma 1.1,

$$
b_4(G)=b_4(G-v)+b_2(G-u-v), \quad b_4(G_n^1)=b_4(G_{n-1}^1)+b_2(K_{1,4}).
$$

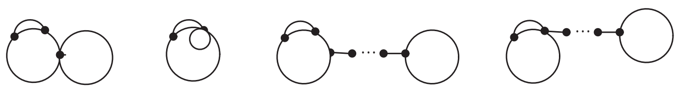
Note that $G-v$ has exactly three cycles on $n-1$ vertices and $G \ncong R_n$, therefore $G-v \ncong G^0_{n-1}, G^1_{n-1}, W_{n-1}, S_{n-1}, Q_{n-1}, R_{n-1}$, by induction hypothesis, $b_4(G-v) > b_4(G^1_{n-1})$. It is easy to see $G \ncong G^0_n, G^1_n, W_n, S_n, Q_n, R_n$, we have $b_2(G-u-v) \geqslant b_2(K_{1,4}) = 4$. It is immediate that $b_4(G) > b_4(G^1_n)$ with $G \ncong R_n$.

By combing Cases 1 and 2, this lemma follows. □

Remark. In Case 2 above, if $G \cong R_n$, then $b_2(G-u-v) = 3 < b_2(K_{1,4}) = 4$. This is just the reason why we consider $G \in \mathscr{G}_n$ with $G \ncong R_n$.

Lemma 2.3. *If $G \in \mathscr{G}_n$ has exactly four cycles with $G \ncong W_n$, then $b_4(G) > b_4(G^1_n)$ for $n \geqslant 7$.*

Proof. Since G has exactly four cycles, by Fact 1, there are two cycles, say C_a and C_b, having t ($t \geqslant 1$) common edges and there exists exactly one cycle, say C_c, which is edge-disjoint with the rest cycles contained in G (e.g., see Figure 4.). So $n+2-a-b+t-c \geqslant 0$. By induction on $n+2-a-b+t-c$. If $n+2-a-b+t-c=0$, then either all of the four cycles have a vertex in common, or there exists exactly one cycle, say C_b, connecting C_c by a vertex, that is, only C_b and C_c have exactly one vertex in common.



(a) (b) (c) (d)

Figure 4: Four possible cases for the arrangement of four cycles in G

When $t=1$, by Lemma 1.2,

$$\begin{aligned} b_4(G) &= m(G,2) - 2s \geqslant m(G,2) - 8 \\ &= \frac{(a+b-2)(a+b-5)}{2} + \frac{c(c-3)}{2} + (a+b-2-4+c) \\ &\quad +(a+b-2-2)c + 2(c-2) - 8 \\ &= \frac{(a+b)^2}{2} - \frac{5}{2}(a+b) + (a+b)c + \frac{c^2-5c}{2} - 13. \end{aligned}$$

Since $n+2-a-b+1-c=0$, thus $a+b=n+3-c$. And so

$$\begin{aligned} b_4(G)-b_4(G_n^1) &\geqslant \frac{(n+3-c)^2}{2}-\frac{5}{2}(n+3-c)+(n+3-c)c+\frac{c^2-5c}{2} \\ &\quad -13-(4n-24) \\ &= \frac{1}{2}n^2-\frac{7}{2}n+8>0, \end{aligned}$$

and hence $b_4(G)>b_4(G_n^1)$.

When $t\geqslant 2$, by Lemma 1.2,

$$\begin{aligned} b_4(G) &= m(G,2)-2s\geqslant m(G,2)-8 \\ &= \frac{(a+b-2t)(a+b-2t-3)}{2}+\frac{c(c-3)}{2}+\frac{(t-1)(t-2)}{2}+2(c-2)-8 \\ &\quad +2(a+b-2t-2+c)+(t-2)(a+b-2t+c)+(a+b-2t-2)c \\ &= \frac{(a+b)^2}{2}-(t+\frac{3}{2})(a+b)+(a+b)c+\frac{c^2-3c}{2}+\frac{(t-1)(t-2)}{2} \\ &\quad +3t-tc-16. \end{aligned}$$

Since $n+2-a-b+t-c=0$, thus $a+b=n+2+t-c$. And so

$$b_4(G)-b_4(G_n^1) \geqslant \begin{cases} \frac{1}{2}n^2-\frac{7}{2}n+14, & \text{if } t=2; \\ \frac{1}{2}n^2-\frac{7}{2}n+8, & \text{if } t\geqslant 3. \end{cases}$$

and therefore $b_4(G)>b_4(G_n^1)$ for $n\geqslant 7$. Similarly, in the case that all of the four cycles have exactly one vertex in common when $n\geqslant 7$, the result $b_4(G)>b_4(G_n^1)$ also holds for $n+2-a-b+t-c=0$.

Suppose it is true for all graphs $G\in\mathscr{G}_n$ having exactly four cycles and $G\ncong W_n$ with $n+2-a-b+t-c<p$ ($p\geqslant 1$), and suppose $n+2-a-b+t-c=p$.

Case 1. There is no pendent edges in G. Then either there exactly one cycle, say C_b, connecting the separated cycle, say C_c, by a path of length p, or the separated cycle connects the rest three cycles by a path of length p, that is, one end vertex of this path is in the separated cycle, the other end vertex is a common vertex of the rest cycles. Here we show our result only for the former, similarly we can prove our result for the latter case.

Subcase 1.1. $t = 1$. Then by Lemma 1.2,

$$\begin{aligned}
b_4(G) &= m(G,2) - 2s \geqslant m(G,2) - 8 \\
&= \frac{(a+b-2)(a+b-5)}{2} + \frac{c(c-3)}{2} + \frac{(p-1)(p-2)}{2} + (a+b-2-2) - 8 \\
&\quad +(a+b-2)(c+p-1) + (a+b-2-4+c+p) + (p-1)c + (c-2) \\
&= \frac{(a+b)^2}{2} - \frac{5}{2}(a+b) + (a+b)(c+p) + \frac{c^2-5c}{2} + \frac{p^2-5p}{2} + pc - 12.
\end{aligned}$$

Since $n+2-a-b+1-c = p$, we $a+b = n+3-p-c$. And so

$$\begin{aligned}
b_4(G) - b_4(G_n^1) &\geqslant \frac{(n+3-p-c)^2}{2} - \frac{5}{2}(n+3-p-c) + (n+3-p-c)(c+p) \\
&\quad + \frac{c^2-5c}{2} + \frac{p^2-5p}{2} + pc - 12 - (4n-24) \\
&= \frac{1}{2}n^2 - \frac{7}{2}n + 9 > 0,
\end{aligned}$$

and thereby $b_4(G) > b_4(G_n^1)$.

Subcase 1.2. $t = 2$. Then by Lemma 1.2,

$$\begin{aligned}
b_4(G) &= m(G,2) - 2s \geqslant m(G,2) - 8 \\
&= \frac{(a+b-4)(a+b-7)}{2} + \frac{c(c-3)}{2} + \frac{(p-1)(p-2)}{2} + 2(a+b-4-2+c+p) \\
&\quad +(a+b-4-2)(c+p) + 2(c+p-1) + (p-1)c + (c-2) - 8 \\
&= \frac{(a+b)^2}{2} - \frac{7}{2}(a+b) + (a+b)(c+p) + \frac{c^2-7c}{2} + \frac{p^2-7p}{2} + pc - 9.
\end{aligned}$$

Since $n+2-a-b+2-c = p$, we have $a+b = n+4-c-p$. And so

$$\begin{aligned}
b_4(G) - b_4(G_n^1) &\geqslant \frac{(n+4-c-p)^2}{2} - \frac{7}{2}(n+4-c-p) + (n+4-c-p)(c+p) \\
&\quad + \frac{c^2-7c}{2} + \frac{p^2-7p}{2} + pc - 9 - (4n-24) \\
&= \frac{1}{2}n^2 - \frac{7}{2}n + 9 > 0,
\end{aligned}$$

and hence $b_4(G) > b_4(G_n^1)$.

Subcase 1.3. $t \geqslant 3$. Then by Lemma 1.2,

$$\begin{aligned}
b_4(G) &= m(G,2) - 2s \geqslant m(G,2) - 8 \\
&= \frac{(a+b-2t)(a+b-2t-3)}{2} + \frac{c(c-3)}{2} + \frac{(t-1)(t-2)}{2} + \frac{(p-1)(p-2)}{2} \\
&\quad +2(a+b-2t-2+c+p) + (t-2)(a+b-2t+c+p) \\
&\quad +(a+b-2t-2)(c+p) + 2(c+p-1) + (p-1)c + (c-2) - 8
\end{aligned}$$

$$= \frac{(a+b)^2}{2} - (t+\frac{3}{2})(a+b) + (a+b)(c+p) + \frac{c^2-3c}{2} + \frac{t^2+3t}{2} + \frac{p^2-3p}{2} + pc - tc - pt - 14.$$

Since $n+2-a-b+t-c=p$, we have $a+b=n+2+t-c-p$. And so

$$\begin{aligned} b_4(G) - b_4(G_n^1) &\geqslant \frac{(n+2+t-c-p)^2}{2} - (t+\frac{3}{2})(n+2+t-c-p) \\ &\quad +(n+2+t-c-p)(c+p) + \frac{c^2-3c}{2} + \frac{t^2+3t}{2} + \frac{p^2-3p}{2} \\ &\quad +pc - tc - pt - 14 - (4n-24). \\ &= \frac{1}{2}n^2 - \frac{7}{2}n + 9 > 0, \end{aligned}$$

and therefore $b_4(G) > b_4(G_n^1)$.

Subcase 2. uv is a pendent edge of G with pendent vertex v. By Lemma 1.1,

$$b_4(G) = b_4(G-v) + b_2(G-u-v), \quad b_4(G_n^1) = b_4(G_{n-1}^1) + b_2(K_{1,4}).$$

Note that $G-v$ has exactly four cycles on $n-1$ vertices and $G \ncong W_n$, therefore $G-v \ncong G_{n-1}^0, G_{n-1}^1, R_{n-1}, S_{n-1}, Q_{n-1}, W_{n-1}$, by induction hypothesis, $b_4(G-v) > b_4(G_{n-1}^1)$. It is easy to see $G \ncong G_n^0, G_n^1, W_n, S_n, Q_n, R_n$, we have $b_2(G-u-v) \geqslant b_2(K_{1,4}) = 4$. It is immediate that $b_4(G) > b_4(G_n^1)$ with $G \ncong W_n$.

By combing Cases 1 and 2, this lemma follows. $\square$

Lemma 2.4. *If $G \in \mathscr{G}_n$ has exactly six cycles and $G \ncong G_n^0$, G_n^1, S_n, then $b_4(G) > b_4(G_n^1)$ for $n \geqslant 7$.*

Proof. Since G has six cycles, then it is straightforward to check that either any two of the six cycles have exactly two vertices in common, or there are two cycles either having exactly one vertex in common, or having no vertex in common; see Figure 5.

Case 1. Any two of the six cycles have exactly two vertices in common. Thus choose three cycles C_a, C_b and C_c having t edges in common. Then $n+2-a-b-c+2t \geqslant 0$. By induction on $n+2-a-b-c+2t$. Assume $n+2-a-b-c+2t = 0$,

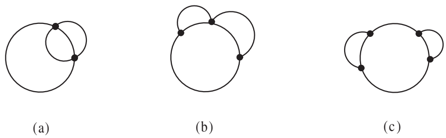


Figure 5: Three possible cases for the arrangement of six cycles in G

When $t=1$, by Lemma 1.2,

$$\begin{aligned} b_4(G) &= m(G,2)-2s \geqslant m(G,2)-12 \\ &= \frac{(a+b-2)(a+b-5)}{2}+\frac{c(c-3)}{2}+(a+b-2-4) \\ &\quad +2(a+b-2-2)+(c-3)(a+b-2)-12 \\ &= \frac{(a+b)^2}{2}-\frac{7}{2}(a+b)+(a+b)c+\frac{c^2-7c}{2}-15. \end{aligned}$$

Since $n+2-a-b+2-c=0$, we have $a+b=n+4-c$. And so

$$\begin{aligned} b_4(G)-b_4(G_n^1) &\geqslant \frac{(n+4-c)^2}{2}-\frac{7}{2}(n+4-c)+(n+4-c)c+\frac{c^2-7c}{2} \\ &\quad -15-(4n-24) \\ &= \frac{1}{2}n^2-\frac{7}{2}n+3, \end{aligned}$$

and hence $b_4(G)>b_4(G_n^1)$ for $n \geqslant 7$.

When $t=2$. Let $c=2+l$, if $l=1$, by Lemma 1.2,

$$\begin{aligned} b_4(G) &= m(G,2)-2s \geqslant m(G,2)-12 \\ &= \frac{(a+b-4)(a+b-7)}{2}+\frac{c(c-3)}{2}+2(a+b-4-2)+(a+b-4-4)-12 \\ &= \frac{(a+b)^2}{2}-\frac{5}{2}(a+b)+\frac{c^2-3c}{2}-14. \end{aligned}$$

Since $n+2-a-b+4-3=0$, then $a+b=n+3$. And so

$$\begin{aligned} b_4(G)-b_4(G_n^1) &\geqslant \frac{(n+3)^2}{2}-\frac{5}{2}(n+3)+\frac{c^2-3c}{2}-14-(4n-24) \\ &= \frac{1}{2}n^2-\frac{7}{2}n+7>0, \end{aligned}$$

and therefore $b_4(G)>b_4(G_n^1)$.

If $l \geqslant 2$, by Lemma 1.2,

$$\begin{aligned} b_4(G) &= m(G,2) - 2s \geqslant m(G,2) - 12 \\ &= \frac{(a+b-4)(a+b-7)}{2} + \frac{c(c-3)}{2} + 2(a+b-4-2) \\ &\quad +2(a+b-4-2) + (c-4)(a+b-4) - 12 \\ &= \frac{(a+b)^2}{2} - \frac{11}{2}(a+b) + (a+b)c + \frac{c^2-11c}{2} - 6. \end{aligned}$$

Since $n+2-a-b+4-c=0$, we have $a+b=n+6-c$. And so

$$\begin{aligned} b_4(G) - b_4(G_n^1) &\geqslant \frac{(n+6-c)^2}{2} + (n+6-c)(c-\frac{11}{2}) + \frac{c^2-11c}{2} - 6 - (4n-24) \\ &= \frac{1}{2}n^2 - \frac{7}{2}n + 3, \end{aligned}$$

and hence $b_4(G) > b_4(G_n^1)$ for $n \geqslant 7$.

When $t \geqslant 3$. Let $c = t+l$, then $c = t+l$. By Lemma 1.2,

$$\begin{aligned} b_4(G) &= m(G,2) - 2s \geqslant m(G,2) - 12 \\ &= \frac{(a+b-2t)(a+b-2t-3)}{2} + \frac{c(c-3)}{2} + 2(a-t+b-t-2) \\ &\quad +(t-2)(a-t+b-t) + 2(a+b-2t-2) + (l-2)(a+b-2t) - 12 \\ &= \frac{(a+b)^2}{2} - (t+\frac{3}{2})(a+b) + (a+b)l + \frac{c^2-3c}{2} + 3t - 2tl - 20. \end{aligned}$$

Since $n+2-a-b+2t-c=0, c=t+l$, thus we have $a+b=n+2+t-l$. And so

$$\begin{aligned} b_4(G) - b_4(G_n^1) &\geqslant \frac{(n+2+t-l)^2}{2} - (t+\frac{3}{2})(n+2+t-l) + (n+2+t-l)l \\ &\quad +\frac{c^2-3c}{2} + 3t - 2tl - 20 - (4n-24) \\ &= \frac{n^2}{2} - \frac{7}{2}n + 3, \end{aligned}$$

and hence $b_4(G) > b_4(G_n^1)$ for $n \geqslant 7$.

Suppose it is true for all graphs $G \in \mathscr{G}_n$ having exactly six cycles and $G \ncong G_n^0, G_n^1, S_n$ with $n+2-a-b+2t-c < p$ $(p \geqslant 1)$, and suppose $n+2-a-b+2t-c = p$, then G must have pendent edge, say uv, with pendent vertex v. By Lemma 1.1,

$$b_4(G) = b_4(G-v) + b_2(G-u-v), \quad b_4(G_n^1) = b_4(G_{n-1}^1) + b_2(K_{1,4}).$$

Notice that $G-v$ has exactly six cycles on $n-1$ vertices and $G \ncong G_n^0, G_n^1, S_n$, therefore $G-v \ncong G_{n-1}^0, G_{n-1}^1, R_{n-1}, W_{n-1}, Q_{n-1}, S_{n-1}$, by induction hypothesis, $b_4(G-v) > b_4(G_{n-1}^1)$. It is easy to see $G \ncong G_n^0, G_n^1, W_n, S_n, Q_n, R_n$, we have $b_2(G-u-v) \geqslant b_2(K_{1,4}) = 4$. It is immediate that $b_4(G) > b_4(G_n^1)$ with $G \ncong G_n^0, G_n^1, S_n$.

Case 2. There are two cycles contained in G having exactly one vertex in common. Then choose three cycles, say C_a, C_b, C_c, such that C_a, C_b have exactly one vertex in common and the number of edges in $C_c \setminus (C_a \cup C_b)$ is l. Then $n+2-(a+b+l) \geqslant 0$, by induction on $n+2-(a+b+l)$. If $n+2-(a+b+l) = 0$, then by Lemma 1.2,

$$\begin{aligned} b_4(G) &= m(G,2) - 2s \geqslant m(G,2) - 12 \\ &= \frac{a(a-3)}{2} + \frac{b(b-3)}{2} + \frac{(l-1)(l-2)}{2} + (a-2)b + 2(b-2) \\ &\quad +(l-2)(a+b) + 2(a+b-2) - 12 \\ &= \frac{(a+b)^2}{2} - \frac{3}{2}(a+b) + l(a+b) + \frac{l^2-3l}{2} - 19. \end{aligned}$$

Since $n+2-a-b-l = 0$, so $a+b = n+2-l$. And so

$$\begin{aligned} b_4(G) - b_4(G_n^1) &\geqslant \frac{(n+2-l)^2}{2} - \frac{3}{2}(n+2-l) + l(n+2-l) + \frac{l^2-3l}{2} - 9 - (4n-24) \\ &= \frac{n^2}{2} - \frac{7}{2}n + 4, \end{aligned}$$

thus $b_4(G) > b_4(G_n^1)$ is immediate for $n \geqslant 7$..

Suppose it is true for all graphs $G \in \mathscr{G}_n$ having exactly six cycles and $G \ncong G_n^0, G_n^1, S_n$ with $n+2-a-b-l < p$ $(p \geqslant 1)$, and suppose $n+2-a-b-l = p$, then G must have a pendent edge, say uv, with pendent vertex v. By Lemma 1.1, we have

$$b_4(G) = b_4(G-v) + b_2(G-u-v), \quad b_4(G_n^1) = b_4(G_{n-1}^1) + b_2(K_{1,4}).$$

Note that $G-v$ has exactly six cycles on $n-1$ vertices and $G \ncong G_n^0, G_n^1, S_n$, therefore $G-v \ncong G_{n-1}^0, G_{n-1}^1, R_{n-1}, W_{n-1}, Q_{n-1}, S_{n-1}$, by induction hypothesis, $b_4(G-v) > b_4(G_{n-1}^1)$. It is easy to see $G \ncong G_n^0, G_n^1, W_n, S_n, Q_n, R_n$, we have $b_2(G-u-v) \geqslant b_2(K_{1,4}) = 4$. It is immediate that $b_4(G) > b_4(G_n^1)$ with $G \ncong G_n^0, G_n^1, S_n$.

Case 3. There are two cycles contained in G having no vertex in common. Then choose three cycles, say C_a, C_b, C_c, such that C_a, C_b have t edges in common and C_c, C_b have l

edges in common. Then $n+2-(a+b+c)+t+l \geqslant 0$, by induction on $n+2-(a+b+c)+t+l$. If $n+2-(a+b+c)+t+l=0$, i.e., $a+b+c=n+2+t+1$, then by Lemma 1.2,

$$\begin{aligned} b_4(G) &= m(G,2)-2s \geqslant m(G,2)-12 \\ &= \frac{(a+b+c-2t-2l)(a+b+c-2t-2l-3)}{2}+\frac{(t-1)(t-2)}{2} \\ &\quad +(t-2)(n+2-t)+2(n+2-t-2)+\frac{(l-1)(l-2)}{2} \\ &\quad +(l-2)(n+2-t-l)+2(n+2-t-l-2). \end{aligned}$$

Since $n+2-(a+b+c)+t+l=0$, so $a+b+c=n+2+t+l$. And so

$$\begin{aligned} b_4(G)-b_4(G_n^1) &\geqslant \frac{(n+2+t+l-2t-2l)(n+2+t+l-2t-2l-3)}{2} \\ &\quad +\frac{(t-1)(t-2)}{2}+(t-2)(n+2-t)+2(n+2-t-2)-(4n-24) \\ &\quad +(l-2)(n+2-t-l)+2(n+2-t-l-2+\frac{(l-1)(l-2)}{2}) \\ &= \frac{n^2}{2}-\frac{7}{2}n+17>0, \end{aligned}$$

thus $b_4(G)>b_4(G_n^1)$ is immediate.

Suppose it is true for all graphs $G \in \mathscr{G}_n$ having exactly six cycles and $G \ncong G_n^0, G_n^1, S_n$ with $n+2-a-b-l<p$ $(p \geqslant 1)$, and suppose $n+2-a-b-l=p$, then G must have a pendent edge, say uv, with pendent vertex v. By Lemma 1.1, we have

$$b_4(G)=b_4(G-v)+b_2(G-u-v), \quad b_4(G_n^1)=b_4(G_{n-1}^1)+b_2(K_{1,4}).$$

Note that $G-v$ has exactly six cycles on $n-1$ vertices and $G \ncong G_n^0, G_n^1, S_n$, therefore $G-v \ncong G_{n-1}^0, G_{n-1}^1, R_{n-1}, W_{n-1}, Q_{n-1}, S_{n-1}$, by induction hypothesis, $b_4(G-v)>b_4(G_{n-1}^1)$. It is easy to see $G \ncong G_n^0, G_n^1, W_n, S_n, Q_n, R_n$, we have $b_2(G-u-v) \geqslant b_2(K_{1,4})=4$. It is immediate that $b_4(G)>b_4(G_n^1)$ with $G \ncong G_n^0, G_n^1, S_n$.

Combining with Cases 1, 2 and 3, this lemma follows. □

Lemma 2.5. *If $G \in \mathscr{G}_n$ has exactly seven cycles and $G \ncong Q_n$, then $b_4(G)>b_4(G_n^1)$ for $n \geqslant 7$.*

Proof. The configuration of the seven cycles contained in G is as in Figure 6. Choose

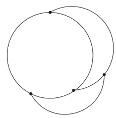


Figure 6: One possible case for the arrangement of seven cycles in G

three cycles, say C_a, C_b and C_c, such that C_a, C_b have exactly t edges in common, but all of t edges are not contained in C_c. Denote the number of edges of C_c not contained in C_a and C_b by l. Then $n+2-(a+b-t+l) \geqslant 0$. By induction on $n+2-(a+b-t+l)$. If $n+2-(a+b-t+l)=0$.

When $t=1$, by Lemma 1.2

$$\begin{aligned}
b_4(G) &= m(G,2)-2s \geqslant m(G,2)-14\\
&= \frac{(a+b-2)(a+b-2-3)}{2}+(a-1+b-1-4+l)+\frac{(l-1)(l-2)}{2}\\
&\quad +(l-2)(a+b-2)+2(a+b-2-2)-14\\
&= \frac{(a+b)^2}{2}-\frac{5}{2}(a+b)+\frac{l^2-5l}{2}+l(a+b)-18.
\end{aligned}$$

Since $n+2-a-b+1-l=0$, we have $a+b=n+3-l$. And so

$$\begin{aligned}
b_4(G)-b_4(G_n^1) &\geqslant \frac{(n+3-l)^2}{2}-\frac{5}{2}(n+3-l)+\frac{l^2-5l}{2}+l(n+3-l)\\
&\quad -18-(4n-24)\\
&= \frac{n^2}{2}-\frac{7}{2}n+3,
\end{aligned}$$

it follows that $b_4(G)>b_4(G_n^1)$ for $n \geqslant 7$.

When $t=2$, by Lemma 1.2

$$\begin{aligned}
b_4(G) &= m(G,2)-2s \geqslant m(G,2)-14\\
&= \frac{(a+b-4)(a+b-4-3)}{2}+\frac{(l-1)(l-2)}{2}+2(a-2+b-2-2+l)\\
&\quad +(l-2)(a-2+b-2)+2(a-2+b-2-2)-14\\
&= \frac{(a+b)^2}{2}-\frac{7}{2}(a+b)+l(a+b)+\frac{l^2-7l}{2}-15.
\end{aligned}$$

Since $n+2-(a+b-2+l)=0$, we have $a+b=n+4-l$. Thus

$$\begin{aligned} b_4(G)-b_4(G_n^1) &\geqslant \frac{(n+4-l)^2}{2}-\frac{7}{2}(n+4-l)+l(n+4-l) \\ &\quad +\frac{l^2-7l}{2}-15-(4n-24) \\ &= \frac{n^2}{2}-\frac{7}{2}n+3, \end{aligned}$$

and hence $b_4(G)>b_4(G_n^1)$ for $n\geqslant 7$.

When $t>2, l=1$, by Lemma 1.2,

$$\begin{aligned} b_4(G) &= m(G,2)-2s\geqslant m(G,2)-14 \\ &= \frac{(a+b-2t)(a+b-2t-3)}{2}+\frac{(t-1)(t-2)}{2}+2(a-t+b-t-2+1) \\ &\quad +(t-2)(a-t+b-t+1)+(a-t+b-t-4)-14 \\ &= \frac{(a+b)^2}{2}-(t+\frac{1}{2})(a+b)+\frac{t^2+t}{2}-21. \end{aligned}$$

Since $n+2-(a-t+b+1)=0$, we have $a+b=n+1+t$. And so

$$\begin{aligned} b_4(G)-b_4(G_n^1) &\geqslant \frac{(n+1+t)^2}{2}-(t+\frac{1}{2})(n+1+t)+\frac{t^2+t}{2}-21-(4n-24) \\ &= \frac{n^2}{2}-\frac{7}{2}n+3, \end{aligned}$$

and hence $b_4(G)>b_4(G_n^1)$ for $n\geqslant 7$.

When $t>2, l\geqslant 2$, by Lemma 1.2,

$$\begin{aligned} b_4(G) &= m(G,2)-2s\geqslant m(G,2)-14 \\ &= \frac{(a+b-2t)(a+b-2t-3)}{2}+\frac{(t-1)(t-2)}{2}+\frac{(l-1)(l-2)}{2} \\ &\quad +2(a-t+b-t-2)+(t-2)(a-t+b-t+l)+(l-2)(a-t+b-t) \\ &\quad +2(a-t+b-t-2+l)-14 \\ &= \frac{(a+b)^2}{2}-(t+\frac{3}{2})(a+b)+l(a+b)+\frac{l^2-3l}{2}+\frac{t^2+3t}{2}-tl-6. \end{aligned}$$

Since $n+2-(a+b-t+l)=0$, we have $a+b=n+t+2-l$. And so

$$\begin{aligned} b_4(G)-b_4(G_n^1) &\geqslant \frac{(n+t+2-l)^2}{2}-(t+\frac{3}{2})(n+t+2-l)+l(n+t+2-l) \\ &\quad +\frac{l^2-3l}{2}+\frac{t^2+3t}{2}-tl-6-(4n-24) \\ &= \frac{n^2}{2}-\frac{7}{2}n+17>0. \end{aligned}$$

Namely that $b_4(G) > b_4(G_n^1)$.

Suppose it is true for all graphs $G \in \mathscr{G}_n$ having exactly seven cycles and $G \ncong Q_n$ with $n+2-(a+b-t+l) < p\,(p \geqslant 1)$, and suppose $n+2-(a+b-t+l) = p$, then G must have a pendent edge, say uv, with pendent vertex v. By Lemma 1.1, we have

$$b_4(G) = b_4(G-v) + b_2(G-u-v), \quad b_4(G_n^1) = b_4(G_{n-1}^1) + b_2(K_{1,4}).$$

Notice that $G-v$ has exactly seven cycles on $n-1$ vertices and $G \ncong Q_n$, therefore $G-v \ncong G_{n-1}^0, G_{n-1}^1, W_{n-1}, S_{n-1}, R_{n-1}, Q_{n-1}$, by induction hypothesis, $b_4(G-v) > b_4(G_{n-1}^1)$. It is easy to see $G \ncong G_n^0, G_n^1, W_n, S_n, Q_n, R_n$, we have $b_2(G-u-v) \geqslant b_2(K_{1,4}) = 4$. It is immediate that $b_4(G) > b_4(G_n^1)$ with $G \ncong Q_n$. □

By Lemmas 2.2-2.5, we obtain the following proposition.

Proposition 2.6. *If $G \in \mathscr{G}_n$ and $G \ncong G_n^0, G_n^1, R_n, W_n, S_n, Q_n$, then $E(G) > E(G_n^1)$ for $n \geqslant 7$.*

Proof. By Sachs theorem, $b_0(G) = b_0(G_n^1) = 1$, $b_1(G) = b_1(G_n^1) = 0$, $b_2(G) = b_2(G_n^1) = n+2$, $b_3(G_n^1) = 0, b_i(G_n^1) = 0$ for $i \geqslant 5$. By Lemmas 2.2-2.5, $b_4(G) > b_4(G_n^1)$ for $n \geqslant 7$. By Lemma 2.1, $b_{2i}(G) \geqslant 0$ for $0 \leqslant i \leqslant \lfloor n/2 \rfloor$. Hence by Coulson integral formula (1.1)

$$\begin{aligned} E(G) &= \frac{1}{\pi}\int_0^{+\infty} \frac{dx}{x^2} \ln\left[\left(\sum_{i=0}^{\lfloor\frac{n}{2}\rfloor} b_{2i}(G)x^{2i}\right)^2 + \left(\sum_{i=0}^{\lfloor\frac{n}{2}\rfloor} b_{2i+1}(G)x^{2i+1}\right)^2\right], \\ E(G_n^1) &= \frac{1}{\pi}\int_0^{+\infty} \frac{dx}{x^2} \ln\left[\left(\sum_{i=0}^{\lfloor\frac{n}{2}\rfloor} b_{2i}(G_n^1)x^{2i}\right)^2\right]. \end{aligned}$$

From these formulas it is immediate to show that $E(G) > E(G_n^1)$. □

Proposition 2.7. (i) *$E(G_n^0) < E(G_n^1)$ for $n \geqslant 11$.*

(ii) *$E(G_n^1) < E(G_n^0), E(R_n), E(W_n), E(S_n), E(Q_n)$ for $7 \leqslant n \leqslant 10$.*

Proof. (i) We want to determine the characteristic polynomial of G_n^0 (respectively, G_n^1). In fact, by Sachs theorem we can obtain $a_0 = 1, a_1 = 0, a_2 = -(n+2), a_3 = -6, a_4 = 3n-15$ and $a_i = 0$ for $i \geqslant 5$. It follows that

$$\phi(G_n^0, \lambda) = \lambda^n - (n+2)\lambda^{n-2} - 6\lambda^{n-3} + (3n-15)\lambda^{n-4}.$$

Similarly, for G_n^1, we obtain $a_0' = 1, a_1' = 0, a_2' = -(n+2), a_3' = 0, a_4' = 4n - 24$ and $a_i' = 0$ for $i \geqslant 5$, and so

$$\phi(G_n^1, \lambda) = \lambda^n - (n+2)\lambda^{n-2} + (4n-24)\lambda^{n-4}.$$

By (1.1),

$$E(G_n^1) - E(G_n^0) = \frac{1}{\pi}\int_0^{\infty} \frac{dx}{x^2} \ln \frac{\left[1+(n+2)x^2+(4n-24)x^4\right]^2}{\left[1+(n+2)x^2+(3n-15)x^4\right]^2+36x^6}.$$

Let

$$\begin{aligned} f(x) &= \left[1+(n+2)x^2+(4n-24)x^4\right]^2 - \left[1+(n+2)x^2+(3n-15)x^4\right]^2 - 36x^6 \\ &= 2(n-9)x^4 + (n-9)(7n-39)x^8 + 2\left[\left(n-\frac{7}{2}\right)^2 - \frac{193}{4}\right]x^6. \end{aligned}$$

It follows that $f(x) > 0$ for $n \geqslant 11$. Hence $E(G_n^0) < E(G_n^1)$ for $n \geqslant 11$.

(ii) By direct calculation (rounded to four decimal places), we have

$E(G_7^0) = 7.5238$,	$E(G_8^0) = 8.0455$,	$E(G_9^0) = 8.5019$,	$E(G_{10}^0) = 8.9134$,
$E(G_7^1) = 6.0000$,	$E(G_8^1) = 6.3246$,	$E(G_9^1) = 6.6332$,	$E(G_{10}^1) = 6.9282$,
$E(R_7) = 10.0000$,	$E(R_8) = 10.5461$,	$E(R_9) = 11.0158$,	$E(R_{10}) = 11.4354$,
$E(W_7) = 8.8703$,	$E(W_8) = 9.4027$,	$E(W_9) = 9.8647$,	$E(W_{10}) = 10.2795$,
$E(S_7) = 6.2548$,	$E(S_8) = 6.8284$,	$E(S_9) = 7.2524$,	$E(S_{10}) = 7.6402$,
$E(Q_7) = 8.2653$,	$E(Q_8) = 8.7446$,	$E(Q_9) = 9.1638$,	$E(Q_{10}) = 9.5662$.

It is immediate that the results hold. □

Unfortunately, using above method, we can not compare the values of $E(G)$ with $E(G_n^0)$ for $G \in \mathscr{G}_n$ and $G \ncong R_n, S_n, W_n, Q_n, G_n^0$. Hence, combining Propositions 2.6 and 2.7, we obtain the following main results of this paper.

Theorem 2.8. (i) *G_n^1 has minimal energy in $\mathscr{G}_n$ for $7 \leqslant n \leqslant 10$.*

(ii) *If $G \in \mathscr{G}_n$ and $G \ncong R_n, W_n, S_n, Q_n, G_n^0, G_n^1$, then $E(G_n^0) < E(G_n^1) < E(G)$ for $n \geqslant 11$.*

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