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# Note on Laplacian Energy of Graphs

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**Abstract**

Let $G$ be a graph with $n$ vertices and $m$ edges. Denote by $\mu_1, \mu_2, \cdots, \mu_n$ the Laplacian eigenvalues of $G$. Recently, I. Gutman and B. Zhou [17] proposed an energy-like quantity, *Laplacian energy*, defined to be $LE(G) = \sum_{i=1}^{n} |\mu_i - \frac{2m}{n}|$, and some bounds concerning this quantity have been obtained there. In this note, a few more novel bounds for $LE(G)$ are presented.

## INTRODUCTION

The concept of graph energy was introduced by Gutman long time ago. Recently this concept started to attract considerable attention of mathematicians and

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mathematical chemists; for recent mathematical works on the energy of graphs see [11, 12, 13, 14, 15, 16, 19, 20, 21, 22] and the references cited therein.

Let $\mu_1, \mu_2, \cdots, \mu_n$ be the Laplacian eigenvalues of $G$. Recently, Gutman and Zhou [17] proposed an energy-like quantity $LE(G)$, called *Laplacian energy*, which is based on the eigenvalues of the Laplacian matrix of $G$ and defined to be $LE(G) = \sum_{i=1}^{n} |\mu_i - \frac{2m}{n}|$.

In [17], numerous similarities and dissimilarities between the ordinary energy and the Laplacian energy are analyzed. Also, various bounds for $LE(G)$ have been obtained in [17, 18, 23]. In this note, some new bounds for $LE(G)$ have been obtained.

Let $G$ be a simple graph with $n$ vertices and $m$ edges. The ordinary spectrum of $G$, consisting of the numbers $\lambda_1, \lambda_2, ..., \lambda_n$, is the spectrum of the adjacent matrix $A(G)$ of $G$. The energy of the graph G is then defined as

$$E(G) = \sum_{i=1}^{n} |\lambda_i|. \tag{1}$$

The Laplacian spectrum of $G$, consisting of the numbers $\mu_1, \mu_2, \cdots, \mu_n$, is the spectrum of the Laplacian matrix $L(G)$ of $G$. Then

$$LE(G) = \sum_{i=1}^{n} |\gamma_i|, \tag{2}$$

where

$$\gamma_i = \mu_i - \frac{2m}{n}. \tag{3}$$

The ordinary graph eigenvalues obey the following relations:

$$\sum_{i=1}^{n} \lambda_i = 0 \qquad \text{and} \qquad \sum_{i=1}^{n} (\lambda_i)^2 = 2m. \tag{4}$$

Analogous relations hold for the Laplacian energy as given below:

$$\sum_{i=1}^{n} \gamma_i = 0 \qquad \text{and} \qquad \sum_{i=1}^{n} (\gamma_i)^2 = 2M. \tag{5}$$

Here

$$M = m + \frac{1}{2} \sum_{i=1}^{n} (\delta_i - \frac{2m}{n})^2 \tag{6}$$

and $\delta_i$ denotes the degree of the $i-th$ vertex of $G$.

## UPPER AND LOWER BOUNDS ON $LE(G)$

D. de Caen reported the following result in [25].

**Lemma 1** *Let $G$ be a simple graph with $n$ vertices and $m$ edges. Then*

$$\sum_{i=1}^{n}(\delta_i)^2 \leq m\left[\frac{2m}{n-1}+n-2\right].$$

The following three Lemmas are due to K. C. Das [24].

**Lemma 2** *Let $G$ be a simple graph with $n$ vertices and $m$ edges. Then*

$$\sum_{i=1}^{n}(\delta_i)^2 \leq m\left(\frac{2m}{n-1}+n-2\right)-\delta_1\left[\frac{4m}{n-1}-2m_1-\frac{n+1}{n-1}\delta_1+(n-1)\right],$$

where $\delta_1$ is the highest degree and $m_1$ is the average degree of the vertices adjacent to one of the vertices with degree $\delta_1$.

**Lemma 3** *Let $G$ be a simple graph with $n$ vertices and $m$ edges. Then*

$$\sum_{i=1}^{n}(\delta_i)^2 \leq 2mn-n(n-1)\delta_n+2m(\delta_n-1),$$

where $\delta_n$ is the lowest degree.

**Lemma 4** *Let $G$ be a simple graph with $n$ vertices and $m$ edges. Then*

$$\sum_{i=1}^{n}(\delta_i)^2 \leq m(m+1).$$

Now we are ready to give an upper bound on $LE(G)$.

**Proposition 5** *Let $G$ be a simple graph with $n$ vertices and $m$ edges. Then*

$$LE(G) \leq \sqrt{mn^2-\frac{4m^2}{n}+\frac{2nm^2}{n-1}}.$$

**Proof.** In view of Eq.(6), we obtain

$$\begin{aligned} M &= m + \frac{1}{2}\left(\sum_{i=1}^{n}\delta_i^2 - \frac{4m}{n}\sum_{i=1}^{n}\delta_i + \frac{4m^2}{n^2}\right) \\ &= \left(m - \frac{2m^2}{n}\right) + \frac{1}{2}\sum_{i=1}^{n}\delta_i^2 . \end{aligned}$$

Using *Cauchy-Schwartz inequality*, and bearing in mind Eqs.(2), (3) and (5), we deduce that

$$LE(G) = \sum_{i=1}^{n}|\gamma_i| \leq \sqrt{n\sum_{i=1}^{n}\gamma_i^2} = \sqrt{2nM}.$$

Now applying Lemma 1 gives directly

$$LE(G) \leq \sqrt{2n\left[\left(m - \frac{2m^2}{n^2}\right) + \frac{1}{2}\sum_{i=1}^{n}\delta_i^2\right]},$$

which implies readily the result of Proposition 5. □

Using the same reasoning as employed in Proposition 5, and by Lemmas 2–4, one can obtain the following

**Proposition 6** *Let $G$ be a simple graph with $n$ vertices and $m$ edges. Then*

$$LE(G) \leq \sqrt{mn\left(\frac{2m}{n-1} + n\right) - \frac{4m^2}{n} + n\delta_1\left[\frac{4m}{n-1} - 2m_1 - \frac{(n+1)}{n-1}\delta_1 + (n-1)\right]}.$$

**Proposition 7** *Let $G$ be a simple graph with $n$ vertices and $m$ edges. Then*

$$LE(G) \leq \sqrt{(-n^3 + n^2 + 2mn)\delta_n + 2mn^2 - \frac{4m^2}{n}}.$$

**Proposition 8** *Let $G$ be a simple graph with $n$ vertices and $m$ edges. Then*

$$LE(G) \leq \sqrt{3mn + nm^2 - \frac{4m^2}{n}}.$$

In the subsequent part of this section, we introduce some lower bounds for $LE(G)$. As usual, we state some lemmas originated from [24].

**Lemma 9** *Let $G$ be a simple graph with $n$ vertices and $m$ edges. Then*

$$\sum_{i=1}^{n}(\delta_i)^2 \geq (\delta_1)^2 + (\delta_n)^2 + \frac{(2m-\delta_1-\delta_n)^2}{n-2},$$

*where $\delta_1$ and $\delta_n$ denote resp. the maximum and minimum degree of vertices in $G$.*

**Lemma 10** *Let $G$ be a simple graph with $n$ vertices and $m$ edges. Then*

$$\sum_{i=1}^{n}(\delta_i)^2 \geq 2m(2p+1) - pn(1+p),$$

where $p = [\frac{2m}{n}]$.

**Lemma 11** *Let $G$ be a simple graph with $n$ vertices and $m$ edges. Then*

$$\sum_{i=1}^{n}(\delta_i)^2 \geq (\delta_1-\delta_n)^2 + \delta_1 + \delta_n(2m-\delta_n).$$

**Proposition 12** *Let $G$ be a simple graph with $n$ vertices and $m$ edges. Then*

$$LE(G) \geq \sqrt{4\left(m - \frac{2m^2}{n^2}\right) + 2\left[\delta_1^2 + \delta_n^2 + \frac{(2m-\delta_1-\delta_n)^2}{n-2}\right]}.$$

**Proof.** Recall that $LE(G) \geq 2\sqrt{M}$ ( see [17] ). By Eq.(6) and Lemma 9, the result follows immediately. $\square$

Similar to Proposition 12, we get

**Proposition 13** *Let $G$ be a simple graph with $n$ vertices and $m$ edges. Then*

$$LE(G) \geq \sqrt{4\left(m - \frac{2m^2}{n^2}\right) + 4m(2p+1) - 2pn(1+p)},$$

where $p = [\frac{2m}{n}]$.

**Proposition 14** *Let $G$ be a simple graph with $n$ vertices and $m$ edges. Then*

$$LE(G) \geq \sqrt{4\left(m - \frac{2m^2}{n^2}\right) + 2(\delta_1 - \delta_n)^2 + 2\delta_1 + 2\delta_n(2m - \delta_n)}.$$

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