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Recent results in constrained packing of equal circles on a sphere

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This paper reviews results on how kN non-overlapping equal circles forming N twins (k = 2), triplets (k = 3), or quartets (k = 4) are to be packed on a sphere so that the angular diameter of the circles will be as large as possible, under constraints: for k = 2, within each twinned pair, the circles are in contact; for k = 3, within each triplet, the circle centres lie at the vertices of an equilateral triangle inscribed in a great circle of the sphere; for k = 4, within each quartet, the circle centres lie at the vertices of a regular tetrahedron. Computer solutions to this optimization problem are presented for small numbers of twins, triplets and quartets. The configurations obtained can be characterized as perfect matchings in the contact graph of the unconstrained problem (k = 2), compounds of equilateral triangles (k = 3), or compounds of regular tetrahedra (k = 4).

1. Introduction

Many situations in the physical and life sciences and in technological applications can be modelled as variants of the packing problem – determining the most efficient packing of objects in an appropriate space. In the *unconstrained* version of the problem, the task is to arrange n equal circles (spherical caps) without overlap on a sphere so that their angular radius $r(n)$ is a maximum. This is the Tammes (1930) problem, to which proven solutions are available for n = 1 to 12 and n = 24 (Fejes Tóth 1964), and conjectural solutions for other values of n up to 130 (Sloane *et al.* 2000). Configurations of n packed circles model, for example, distribution of pores on spherical pollen grains (Tammes 1930), certain polyhedral borane frameworks (Greenwood and Earnshaw 1986), Valence-Shell Electron-Pair Repulsion models of molecules (Gillespie & Nyholm 1957), and spherical codes in information theory (Conway & Sloane 1999).

In applications it can be more useful to deal with a *constrained* version of the spherical circle packing problem. For example, the arrangement of packed circles may be required to exhibit a particular overall point-group symmetry. Axially symmetric packing has been investigated by Goldberg (1967). Families of multisymmetric packings, in tetrahedral, octahedral and icosahedral groups, have been studied (Robinson 1969; Tarnai & Gáspár 1987; Tarnai 2002); the icosahedral solutions are tabulated to large numbers of circles (Sloane *et al.* 2000). Restriction to icosahedral symmetry results in the '*fly's eye*' domes of Buckminster Fuller (Baldwin 1997). Centrosymmetric packings, in which every circle has an antipodal partner, have been studied (Fejes Tóth 1965; Conway *et al.* 1996; Fowler *et al*. 2002), and solutions are relevant to the construction of the Gamma Knife® in brain surgery (Leksell 1983). Single- and multi-stranded spiral packings have also been analysed (Gáspár 1990; Székely 1974).

The present note is concerned with three variants of the constrained packing problem.

(a) The first is where circles are locally paired. In the twinned-circle problem, the objects to be packed are N rigidly connected contacting pairs of circles, pairs of 'twins'. Packing of twinned circles can be used as a geometrical model of packing of dimers of protein molecules in core shells of spherical viruses such as hepatitis B (Wynne *et al.* 1999), and can also be seen as packing of 'diatomic molecules' rather than 'atoms' on the spherical shell. Thus, the problem to be solved is: How must $2N$ non-overlapping equal circles forming N twinned pairs be packed on a sphere so that the angular radius $r^*(2N)$ of the circles will be as large as possible under the constraint that contact is maintained within each twinned pair? If the solution is not unique, how many (non-isomorphic) solutions with the same radius $r^*(2N)$ exist?

(b) The second is where circle centres are grouped as rigid equilateral triangles. This triplet-packing problem produces new zero-volume compounds (or *nolids*; Holden 1991), some of which are unexpectedly non-rigid. Here the problem is: How must $3N$ non-overlapping equal circles forming N triplets be packed on a sphere so that the angular radius $r^*(3N)$ of the circles will be as large as possible under the constraint that, within each triplet, the circle centres lie at the vertices of an equilateral triangle inscribed in a great circle of the sphere?

(c) The third is where circles are grouped in regular tetrahedral quartets. The motivation of the quartet-packing problem comes from the Linnett quartet model of valency. Linnett (1964) proposed a departure from the usual thinking in valence theory. Instead of considering electron pairs as the basis of the octet rule, he constructed atomic and molecular valence configurations from quartets of spin-up and spin-down electrons. Spin correlation leads to the idea of rigid tetrahedral quartets of electrons of like spin, and a disposition of opposite quartets that is governed by the balance of charge- and spin-correlation. Predictions of geometric arrangement of bonds, in a VSEPR-like model, can then be made. The mathematical problem is: How must $4N$ non-overlapping equal circles forming N quartets be packed on a sphere so that the angular radius $r^*(4N)$ of the circles will be as large as possible under the constraint that, within each quartet, the circle centres lie at the vertices of a regular tetrahedron?

In all three variants, it is useful to define, for a packing of n circles, the *contact graph*, drawn by taking each circle centre as a vertex, and joining by an edge those pairs of vertices representing circles in contact. This paper presents computer solutions to the three constrained packing problems for twins ($N = 2$ to 12), triplets ($N = 2$ to 7), and quartets ($N = 2$ to 8).

2. Packing of twinned circles

If the contact graph of the unconstrained packing of n circles supports a perfect matching (a set of disjoint edges that covers each vertex exactly once), then the edges of the matching can be considered as representing $n/2$ pairs of twinned circles in a solution of the $N = n/2$ twinned-circle packing problem. In such cases, the radius of circles in the twinned problem is the same as that in the unconstrained problem, $r^*(2N) = r(n)$. Clearly, such a solution cannot be bettered in radius by removal of a constraint. As a given graph may have many perfect matchings, we may find multiple non-isomorphic solutions to the twins problem even when the solution to the unconstrained problem is unique.

Can we expect *all* unconstrained contact graphs for even n to support at least one perfect matching? Figure 1 shows the contact graphs for $n = 4, 6, 8, \ldots, 24$. Table 1 summarises some of their properties, which illustrate some general characteristics of contact graphs. For example, all contact graphs are planar (*i.e.*, they can be embedded on the sphere or plane without edge crossings) and have maximum vertex degree at most 5 (Fejes Tóth 1964). Vertex degrees of 3, 4 and 5 are observed in the proven and conjectured solutions for these

Table 1. *Properties of solutions of the unconstrained problem*

Solutions are characterized by number n and radius $r(n)$ of the spherical circles, edge count e, minimum degree δ, maximum degree Δ, connectivity k of the contact graph, and point-group symmetry G of the solution. For $n = 20$, the solution has two rattling circles, and e and G are specified for the most symmetrical arrangement, when both rattlers are isolated. In considering rigidity of such a graph, we would attach each isolated point to the main body of the graph by two new edges and the effective edge count then rises to 43, as listed in parentheses.

n	$r(n)$ (deg)	E	δ	Δ	k	G	Source
4	109.4712206	6	3	3	3	T_d	Fejes Tóth 1943
6	90	12	4	4	4	O_h	Fejes Tóth 1943
8	74.8584922	16	4	4	4	D_{4d}	Schütte & van der Waerden 1951
10	66.1468220	19	3	4	3	C_{2v}	Danzer 1963
12	63.4349488	30	5	5	5	I_h	Fejes Tóth 1943
14	55.6705700	28	4	4	4	D_{2d}	Schütte & van der Waerden 1951
16	52.2443957	32	4	4	4	D_{4d}	Schütte & van der Waerden 1951
18	49.5566548	34	3	5	3	C_2	Tarnai & Gáspár 1983
20	47.4310362	39 (43)	(2)	5	2	(D_{3h})	van der Waerden 1952
22	44.7401612	42	3	5	3	C_1	Sloane *et al.* 2000
24	43.6907671	60	5	5	5	O	Robinson 1961

and larger values of n. A vertex degree of 2 occurs when the solution includes a rattling circle (as for $n = 20$): the corresponding vertex can be considered to have degree 2, as it is always possible to let the circle move into contact with at least two neighbours within the range of its two-dimensional rattling motion. Exceptionally, a solution may also have one-dimensional freedom, as in the contact graph for $n = 5$, where two circle centres are fixed at the poles and the other three lie on the equator, along which they are able to move, in contact with the polar circles but not necessarily with each other (Tarnai & Gáspár 1983). Allowing for this process of addition of supplementary edges, all contact graphs are connected.

Other empirical observations about contact graphs for the unconstrained problem are that all known examples are bridgeless (there is no edge whose removal disconnects the graph), and (for $n > 3$, $n \neq 5$) all have at least one triangle, but no face of more than six sides.

Conditions necessary and sufficient for a perfect matching are known from Tutte's (1947) 1-factor theorem, but do not seem to offer an immediate connection with the foregoing summary of known properties of contact graphs. Other theorems prove the existence of perfect matchings in some classes of contact graphs: Petersen's theorem (see *e.g.*, Biggs *et al.*

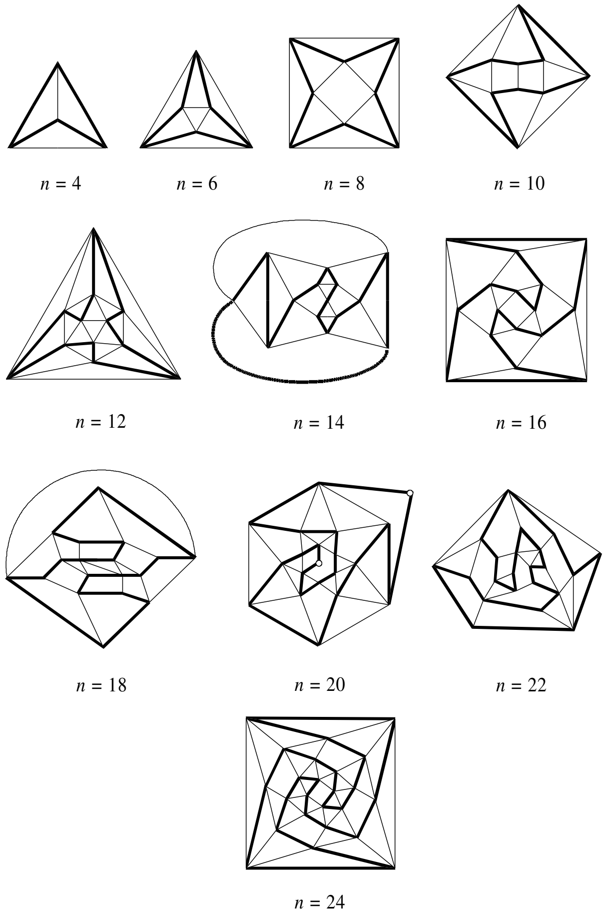

Figure 1. Contact graphs for the best packing of n equal circles on a sphere for even n, $4 \leq n \leq 24$, with examples of Hamiltonian circuits. Vertices correspond to circle centres, and edges to pairs of circles in contact. Symmetries and edge counts are given in Table 1. Open circles for n = 20, representing the centres of rattling circles, are connected to the main graph by two supplementary edges. A perfect matching results from taking every second edge of the Hamiltonian circuit.

1986; Tutte 1998) states that every bridgeless cubic graph has a perfect matching. Amongst our contact graphs, $N = 2$ (the tetrahedron) is covered by this theorem.

Another useful idea comes from consideration of Hamiltonian walks and circuits. A *walk* of length l is an alternating sequence $v_0,e_1,v_1,e_2, \ldots e_l,v_l$ such that v_{j-1} and v_j are the endpoints

Table 2. *Properties of solutions of the twins problem*

Solutions are characterized by the number N of pairs of twinned circles, radius $r^*(2N) = r(n)$ of the spherical circles, point-group symmetry G of the solution of the unconstrained problem, total number of perfect matchings of the contact graph for the unconstrained problem, P_{tot} , and number of non-isomorphic perfect matchings, P_{red} , by symmetry group. The notation $6 = 2{\times}C_2 + 4{\times}C_1$, means that there are six non-isomorphic perfect matchings, of which two have C_2 and four C_1 symmetry.

$N = n/2$	$r^*(2N) = r(n)$ (deg)	G	P_{tot}	P_{red}
2	109.4712206	T_d	3	$1 = D_{2d}$
3	90	O_h	8	$1 = D_3$
4	74.8584922	D_{4d}	14	$3 = D_4 + D_2 + C_2$
5	66.1468220	C_{2v}	20	$6 = 2{\times}C_2 + 4{\times}C_1$
6	63.4349488	I_h	125	$5 = T_h + D_{3d} + D_3 + D_2 + C_2$
7	55.6705700	D_{2d}	64	$8 = 8{\times}C_1$
8	52.2443957	D_{4d}	92	$11 = D_4 + S_8 + 2{\times}D_2 + 4{\times}C_2 + 3{\times}C_1$
9	49.5566548	C_2	142	$76 = 10{\times}C_2 + 66{\times}C_1$
10	47.4310362	(D_{3h})	558	$54 = 15{\times}C_2 + 39{\times}C_1$
11	44.7401612	C_1	120	$120 = 120{\times}C_1$
12	43.6907671	O	7744	$385 = O + T + D_4 + 7{\times}D_3 + 10{\times}D_2$ $+3{\times}C_4 + 6{\times}C_3 + 80{\times}C_2 + 276{\times}C_1$

of edge e_j, and a *Hamiltonian walk* is one that includes every vertex of the graph exactly once and no edge more than once. A *Hamiltonian circuit* includes an additional edge e_{l+1} joining v_l back to v_0. A graph is *Hamiltonian connected* if every pair of vertices are the ends of some Hamiltonian walk, and a graph is *Hamiltonian* if it contains a Hamiltonian circuit. The connection with perfect matchings is that, if a graph with an even number of vertices is Hamiltonian, two perfect matchings are immediately available, as we may take alternate edges of the Hamiltonian circuit as the set of independent edges for a matching. If the even-vertex graph has a Hamiltonian walk, then one perfect matching is available by selecting every second edge of that walk, starting from e_1.

A theorem of Tutte (1956) states that every 4-connected planar graph is Hamiltonian connected, and hence Hamiltonian. (A k-connected graph is one where k is the minimum number of vertices whose removal either disconnects the graph or reduces it to the trivial one-vertex graph.) Our contact graphs are all planar and some are 4-connected. The connectivities are listed in Table 1. Whether 4-connected or not, all the contact graphs for the even numbers in the range n = 4 to 24 are Hamiltonian and hence have perfect matchings (see Figure 1).

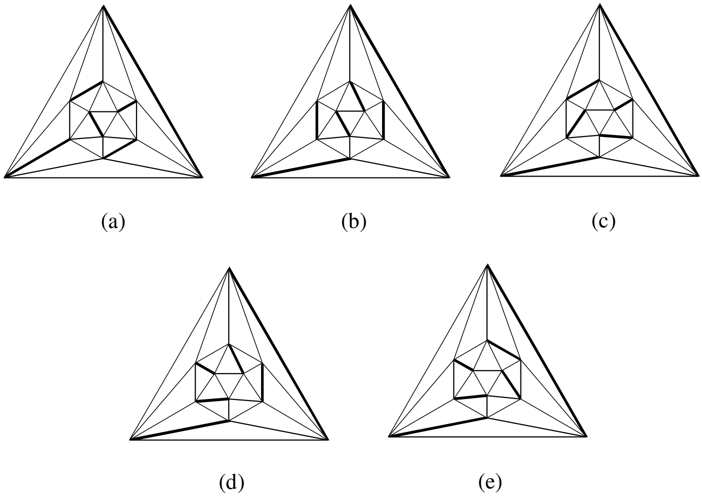

Figure 2. The complete set of non-isomorphic perfect matchings of the contact graph for the best packing of 12 equal circles on a sphere. The symmetries of the graphs decorated by the perfect matchings are (a) T_h, (b) D_{3d}, (c) D_3, (d) D_2, (e) C_2, respectively.

However, Hamiltonicity is not a necessary condition for possession of a perfect matching. The contact graph for n = 32 circles (Danzer 1963), is found on checking to be non-Hamiltonian (and does not even support a Hamiltonian path). Nevertheless, this graph has multiple perfect matchings and hence gives solutions to the twins problem at $r^*(32) = r(32) \approx 37.4752140°$ (see Sloane *et al*. 2000).

Contact graphs for the solutions of the unconstrained problem on even numbers of circles from $n = 4, 6, 8, \ldots, 24$ were constructed from literature data and perfect matchings listed. Although sophisticated techniques are available for counting Kekulé structures in trivalent graphs and perfect matchings in more general graphs, here the graphs are all small, and no special algorithm are needed. All eleven contact graphs were found to have multiple perfect matchings and therefore to give multiple solutions to the twins problem with $r(n) = r^*(2N)$. The elements of the automorphism group of the contact graph (the point group of the contact polyhedron), represented as permutations of the edges, were used to reduce the grand list of perfect matchings to the shortlist of non-isomorphic, symmetry unique, cases and to identify their individual point-group symmetries. The results are shown in Table 2. As an example, for the case $N = 6$ pairs of twins, the complete set of perfect matchings (packings of twins) is presented in Figure 2. Further details, including a discussion of rigidity properties of the packings can be found in Tarnai and Fowler (2006).

3. Packing of regular triplets

In the packing problem for $n = 3N$ equal circles grouped as N equilateral triangles inscribed into great circles of the unit sphere, the positions of the circles can be described by three angular variables per triplet. Each triangle is fixed by the two polar angles of an arbitrarily chosen apical vertex together with the angle of a conical rotation about the radius vector to this apex that fixes the two base vertices. The first triangle can be fixed in a standard position (e.g. apex at the North Pole, one base vertex on the Greenwich meridian), leaving $3(N - 1)$ degrees of freedom to be optimized.

Circle packing is one limit of the *intermediate* problem (Fowler & Tarnai 1996) in which n equal circles of radius r are arranged on a sphere, with overlap, so as minimize the proportion of the spherical surface that is left uncovered. Solutions exist for a range $r_p(n) \le r \le r_c(n)$, where $r_c(n)$ represents the *covering limit*, in which the circles have the minimum radius necessary to ensure that every point of the spherical surface is covered by at least one circle, and $r_p(n)$ represents the *packing limit*, in which the circles have the maximum radius compatible with absence of circle overlap. At intermediate r, the optimum solutions can be found by minimizing the penalty function, defined as the uncovered area of the spherical surface.

The strategy adopted here for finding the packing radius for the constrained problem, $r_p^*(n = 3N)$, where $r_p^*(n) \leq r_p(n)$, is to begin by solving (numerically) the constrained intermediate problem in the $3(N - 1)$ angular variables, setting the initial value for r to the known unconstrained packing radius $r_p(n)$. The radius is then decreased, re-optimizing all angles at each step. The penalty increases with falling radius and, at the point where all overlap contributions vanish, the computed penalty intersects the function $4\pi - 6N\pi(1 - \cos r)$ for non-overlapping circles, and the radius is $r_p^*(n)$. The algorithm is a simple downhill simplex search (Press *et al.* 1986), applied with the usual precautions against trapping in local minima.

A computed packing arrangement is summarized as a co*ntact graph*, with *vertices* representing circle centres, and *edges* joining the centres of circles in contact. This graph may

Table 3. *The current best packings of equilateral triangular triplets of circles on a sphere*

Configurations are characterized by the number N of triplets of circles, radius $r_p^*(3N)$ of the spherical circles, ratio $\rho = r_p^*(3N) / r_p(3N)$ of the packing radii for constrained and unconstrained problems, edge count e of the contact graph, and point group symmetries G_p, G_a and G_c of contact graph, abstract packing graph, and packing compound, respectively. In the case $N = 2$, the packing configuration is non-rigid and has alternative solutions, as indicated. In the case $N = 7$, the point-group assignments refer to the most symmetrical placing of the rattling triangle.

N	$r_p^*(3N)$ (deg)	ρ	e	G_p	G_a	G_c	Packing compound
2	30	0.666667	4(6)	D_2, D_{2d}, D_{6h}	D_{2d}, D_{6h}	D_2, D_{2d}, D_{6h}	Two triangles of the four removed from the cuboctahedron
3	30	0.850716	12	D_3	D_{3h}	D_3	One triangle of the four removed from the cuboctahedron
4	30	0.945851	24	O_h	O_h	O	Four triangles in a Cuboctahedron
5	23.662 6	0.881981	20	D_5	D_{5h}	D_5	Five triangles in a pentagonal barrel
6	21.7762	0.878840	24	D_6	D_{6h}	D_6	Six triangles in a hexagonal barrel
7	20.5574	0.901379	27	D_3	D_{3h}	D_3	One rattling triangle

be embedded in three dimensions, or taken as a purely combinatorial description of the packing, the *abstract contact graph*. This second graph may itself be represented on the sphere, with edge lengths idealized for maximal symmetry, or in the plane as a Schlegel or similar diagram.

In the unconstrained problem, the contact graph may be fully connected or may have disconnected components, corresponding to ‘rattling’ circles that have some range of freedom of movement within an area defined by rigidly fixed circles. In the constrained problem, rattling may also occur as a concerted motion of a set of individual vertices, but there is another possibility that can produce disconnected vertices, even when all circles are fixed. A rigid triangular array of circles on a sphere can be fixed by a minimum of four contacts to the other circles, no one circle of the triangular set making more than two of the four necessary contacts. It may happen, for example, that two circles of a particular triplet are fixed by contacts against other rigidly fixed circles, which then fixes the third circle, even if that circle itself is out of contact with all nearest neighbours. For similar reasons, vertices of degree two

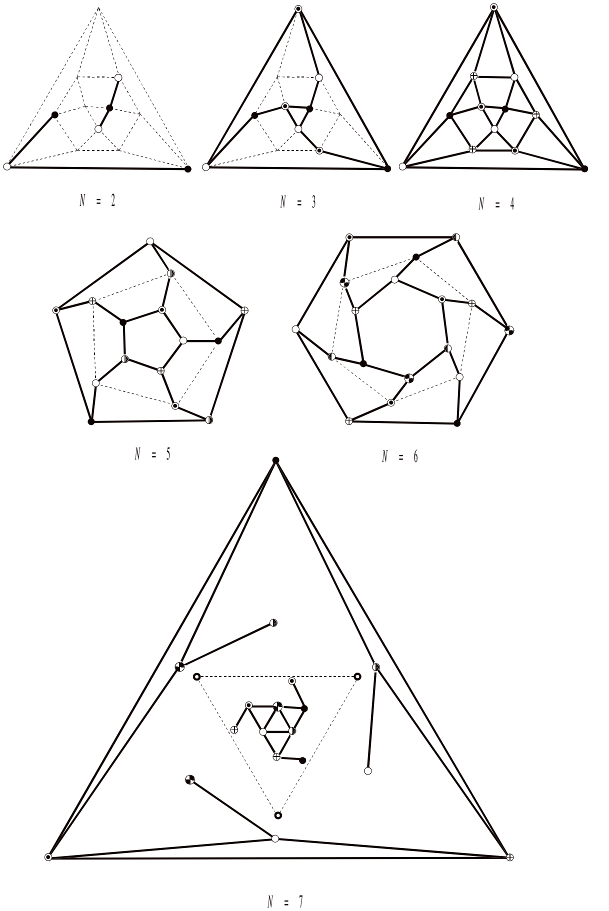

Figure 3. Schlegel-like diagrams for the contact graphs of *N* equilateral triangular triplets of equal circles on a sphere. Within each diagram, different circular symbols are used to show the individual triangular sets. Full lines indicate edges of the contact graph. For *N* = 2 and *N* = 3, the packing graph can be constructed by deletion of one triplet from the graphs for *N* = 3 and *N* = 4, respectively; here, dashed lines indicate the position of the deleted triplets. For *N* = 5 and *N* = 6, the dashed lines show near contacts along the equator of the barrel-shaped contact graphs. For *N* = 7, dashed lines represent a rattling triangle.

or one may occur in the graph, which is therefore not always the skeleton of a polyhedron. It is useful to label the vertices triangle by triangle to help to identify such cases. Though not all packing graphs are polyhedral, a polyhedron-like object can always be produced by taking the union of all the triangles; the resulting object is a *packing compound.* An assembly of faces that encloses no volume is termed a *nolid* by Holden (1991). All our packing compounds are nolids.

The packing *compound* has the symmetry G_c of the union of the triangles, with all edges, vertices and interior points, that are defined by the circle centres in their optimized positions. The contact *graph* embedded on the spherical surface and defined by the same optimized circle positions, but without taking account of the interior of the sphere, has point group symmetry G_p. The *abstract* graph, when embedded in the same spherical surface, and formed by retaining the connectivity of the packing graph but disregarding its specific geometrical properties, has a maximal symmetry G_a.

The results for the packing of $N = 2$ to 7 triplets are summarized in Table 3. The contact graphs embedded on the sphere are represented as Schlegel-like diagrams in Figure 3. Further details not discussed here (e.g. rigidity of packings, 3D shapes of the nolids) can be found in Fowler *et al.* (2005).

4. Packing of regular quartets

In the quartet problem, where $n = 4N$ equal circles are grouped as N rigid regular tetrahedra with vertices on the unit sphere, an exactly similar strategy for finding the packing radius $r_p{}^*(n = 4N)$ was adopted. The results for the packing of $N = 2$ to 8 quartets are summarized in Table 4. The contact graphs embedded on the sphere are represented as Schlegel-like diagrams in Figure 4. Further details (e.g. rigidity of packings, shapes of the compounds of regular tetrahedra) can be found in Tarnai *et al.* (2003).

5. Discussion

(*i*) *The twins problem*

By examining contact graphs for proven and conjectural literature solutions (Table 1) for the unconstrained packing problem for $n = 2N$ circles, we have found solutions for the twinned-circle packing problem for $N = 2$ to 12 pairs. As the solutions for the unconstrained problem for $n = 2N = 4$, 6, 8, 10, 12, 24 are mathematically proven, the twins solutions are exact in these cases. For $N = 7$, 8, 9, 10, 11 pairs, the solutions have the same conjectural status as the underlying solution of the unconstrained problem. From the discussion of perfect matchings given above, if we have found one solution we have found all solutions at the given radius. The set of non-isomorphic solutions has been classified by point-group symmetry in all cases.

Table 4. *The best known packings of regular tetrahedral quartets of equal circles on a sphere*

Configurations are characterized by the number N of the quartets, radius $r_p^*(4N)$ of the spherical circles, ratio $\rho = r_p^*(4N)/r_p(4N)$ of the packing radii for the constrained and unconstrained problems, edge count e of the contact graph, and by point group symmetries G_p, G_a, and G_c of the packing graph, the abstract graph, and the packing compound, respectively. Where the assignment of G_a is ambiguous, the most natural group is listed in brackets.

N	$r_p^*(4N)$ (deg)	ρ	e	G_p	G_a	G_c	Packing compound
1	54.73561	1	6	T_d	T_d	T_d	Single tetrahedron
2	35.26439	0.9421613	12	O_h	O_h	O_h	Stella octangula
3	24.09485	0.7596711	12	D_{6d}	(D_{6d})	D_{6d}	3 tetrahedra in an elongated hexagonal antiprism
4	20.90516	0.8002833	18	T	T_d	T	1 removed from 5 tetrahedra in a dodecahedron
5	20.90516	0.8814970	30	I_h	I_h	I	5 tetrahedra in a dodecahedron
6	18.90715	0.8654986	18	D_6	(D_{6h})	D_6	6 tetrahedra in a hexagonal barrel
7	17.63439	0.8961669	36	C_3	C_3	C_3	1 removed from 8 tetrahedra in an octahedral compound
8	17.63439	0.9411228	48	O	O_h	O	8 tetrahedra in an octahedral compound

In the case $N = 10$, the solution to the corresponding unconstrained problem has rattling circles. A single rattling circle equates to a wagging pair in the twins problem, comprising the rattler and any one of its contactable neighbours A strategy for dealing with such cases is that an augmented contact graph is constructed by joining each rattler simultaneously to all its individually contactable neighbours (feasibility of the contact requiring a geometric calculation) and the set of solutions of all the possible twins problems is given by the perfect matchings of the augmented contact graph.

In all the cases examined, we have been able to find a solution of the twins problem with the same circle diameter as that in the solution of the corresponding unconstrained problem. If all contact graphs for the unconstrained problem have perfect matchings, then it will always be possible to find such a solution. However, in the absence of a proof that *all* unconstrained contact graphs are of this type, there is an open question: What is the smallest number of twins of circles (if any) for which the solution of the twins problem is *not* obtained from a perfect matching in the contact graph of the unconstrained problem?

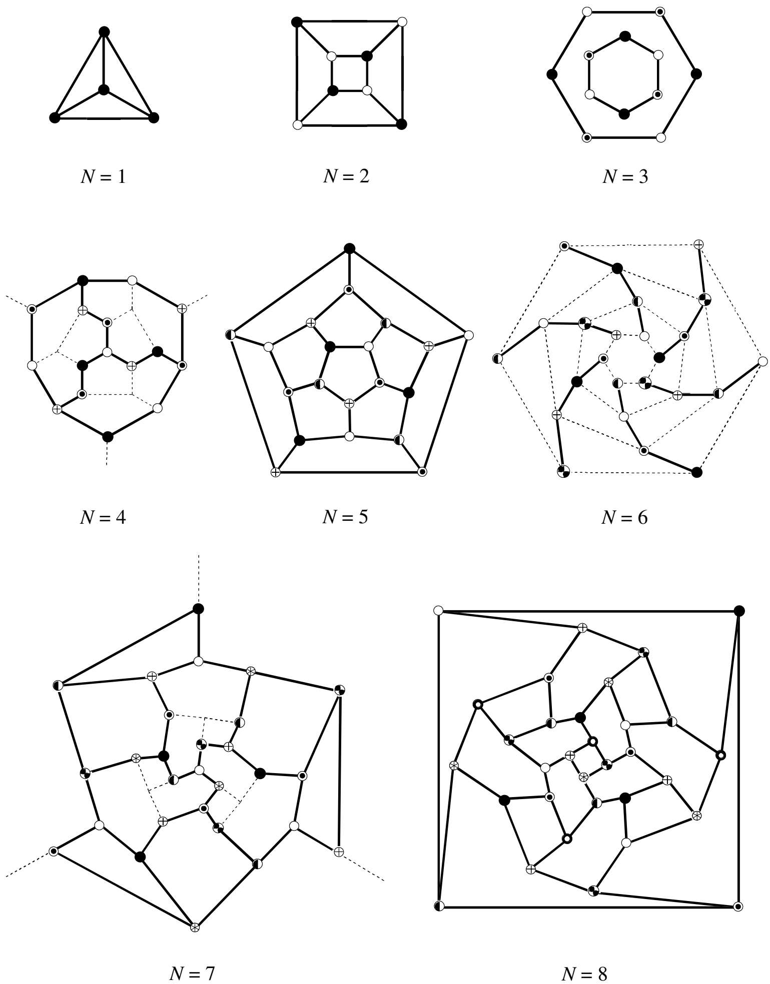

Figure 4. Schlegel-like diagrams for the contact graphs of N regular tetrahedral quartets of equal circles on a sphere. Within each diagram, different circular symbols are used to show the individual tetrahedral sets. Full lines indicate edges of the graph. For $N = 6$, dashed lines are added to show the near contacts along latitudes of the barrel-shaped graph. For $N = 4$ and $N = 7$ the graph can be constructed by deletion of one quartet from the graphs for $N = 5$ and $N = 8$, respectively; here dashed lines indicate the position of the deleted quartet in each case.

(*ii*) *The triplet problem*

It is often the case that a solution to the unconstrained problem has rattling circles, able to move on the spherical surface with two degrees of freedom. It is also known that for $n = 5$ the circles in the packing can move with one degree of freedom. Similar properties have been observed for the solutions to the triplet problem. In the best packing of $N = 7$ triplets there is a rattling triangle having a two-degree-of-freedom motion, while in that of $N = 2$ triplets, one triangle is able to move relative to the other with one degree of freedom.

For the unconstrained packing problem, the packing radius typically decreases with increase in the circle number: $r_p(n-1) > r_p(n)$. Proven exceptions are for $n = 6$ and 12, where equality holds, that is, the best arrangement of $n-1$ circles is obtained by removal of one circle from the best packing of n circles. Some years ago, when fewer numerical results were available, Robinson (1969) thought that perhaps there were additional exceptions for $n = 24$, 48, 60, 120. Székely (1974) referred to comments by Molnár who supposed many other exceptions to exist (not only those mentioned by Robinson), and who even did not exclude cases where $r_p(n-i) = r_p(n)$ for $i > 1$ and thought that perhaps 48 was the first value of n for which $i = 2$. Later numerical results refuted all these suggestions, and it is conjectured that the property $r_p(n-1) = r_p(n)$ holds only for $n = 6$ and 12 (Tarnai & Gáspár 1991). If this conjecture is correct, no packings exist for which $r_p(n-2) = r_p(n-1) = r_p(n)$.

However, for constrained packing problems, where congruent sets of k circles are packed on the sphere, it also sometimes occurs that the best packing of $N-1$ sets of circles is obtained by removal of one set from the best packing of N sets of circles, i.e. $r_p^*(k(N-1)) = r_p^*(kN)$. This happens for antipodal packing ($k = 2$) in the cases of $N = 3$ and 6 (Fejes Tóth 1965). In the present case, such cases were found also with triangular triplets ($k = 3$), for $N = 3$ and 4. Surprisingly, however, the removal property is valid for two consecutive values of the number of triplets, so $r_p^*(k(N-2)) = r_p^*(k(N-1)) = r_p^*(kN)$, $N = 4$. This is therefore the first example where the radius in the best packing of $N-2$, $N-1$ and N sets of equal circles is equal, that is, the removal operation can be applied twice in succession. According to the conjecture, no analogous result can exist for unconstrained packing. It is an interesting question whether there exists a constrained problem with a longer removal sequence.

Packings of triangular triplets produce zero-volume compounds called nolids. Interestingly, the packing of N = 4 equilateral triangles gives as packing compound the assembly of four-triangles-in-a-cuboctahedron, noted by Holden (1991) as a chiral 'curious nolid'.

(*iii*) *The quartet problem*

The packing graphs of tetrahedral quartets in the investigated cases (N = 2 to 8) are connected. Beyond N = 8, however, the graphs tend to be disconnected and apart from sporadic exceptions are of low symmetry. Some of them such as the cases N = 9 and N = 10, for instance, have rattling tetrahedra.

Cases with the removal property $r_p*(k(N-1)) = r_p*(kN)$ occur for regular tetrahedral quartets (k = 4), for N = 5 and 8.

Considering the tetrahedra as solid bodies defined by the centres of their respective member circles, the procedure of optimising the packing lead us to a generalised family of compound polyhedra. Some members of this family are well known. For instance, for N = 2, Kepler's stella octangula is obtained, and for N = 5 the regular compound of five tetrahedra is yielded. The packing of N = 8 regular tetrahedra has lead us to an apparently new, octahedrally symmetric compound.

Packing problems are of interest because they come from natural and easily stated questions with wide applicacations. Of the three constrained problems reviewed here, in some ways, the twins problem is the most appealing, partly because abstract graph theoretic properties (perfect matching, Hamiltonicity) of known packing graphs were used to gain the new results. This problem also has a generalisation to that of the densest packing of equal rings of circles, where within each ring the circle centres lie at the vertices of a regular spherical k-gon, and the neighbouring circles are in contact. For k = 5, the solution could provide a good geometrical model for the structure of all-pentamer viruses, for example.

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