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Bounds on the General Randić Index of Trees with a Given Maximum Degree

Huiqing LIU[a*] Xun YAN[b†] Zheng YAN[a‡]

[a]School of Mathematics and Computer Science, Hubei University, Wuhan 430062, China

[b]Downhole Technical Operation Corporation,

Changqing Petroleum Exploration Bureau, Xi'an 710021, China

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Abstract

The general Randić index of an organic molecule whose molecular graph is G is defined as the sum of $(d(u)d(v))^\alpha$ over all pairs of adjacent vertices of G, where $d(u)$ is the degree of the vertex u in G and α is a real number with $\alpha \neq 0$. In this paper, we characterize the trees with minimal and maximal general Randić indices, respectively, among all trees with a given maximum degree.

1. Introduction

Given a molecular graph G, *the general Randić index*, denoted by $w_\alpha(G)$, is defined as the sum of $(d(u)d(v))^\alpha$ over all pairs of adjacent vertices of G, where $d(u)$

*email: liuhuiqing@eyou.com; Partially supported by NSFC (No. 10571105).

†email: yanxun1@163.com

‡email: yanzhenghubei@yahoo.com.cn

is the degree of the vertex u in G and α is a real number with $\alpha \neq 0$. Recently, the problem concerning graphs with maximal or minimal general Randić indices of a given class of graphs has been studied extensively by many researches, and many results have been achieved (see[3]-[7], [10]-[21],[23]). It is well known that the Randić index $w_{-\frac{1}{2}}(G)$ was proposed by Randić [22] in 1975 and Bollobás and Erdős [3] generalized the index by replacing $-\frac{1}{2}$ with any real number α in 1998. The research background of Randić index together with its generalization appears in chemical field and can be found in the literature (see [8, 9, 22]).

Here, we characterize the trees with minimal and maximal general Randić indices, respectively, among all trees with a given maximum degree.

In order to discuss our results, we first introduced some terminologies and notations of graphs. Other undefined notations may refer to [1, 2]. Let $G = (V, E)$ be a graph. For a vertex u of G, we denote the neighborhood and the degree of u by $N_G(u)$ and $d_G(u)$, respectively. A *pendant vertex* is a vertex of degree 1. A vertex v called *a claw* if all but one of neighbors of v are pendant vertices. Denote $V_0(G) = \{v \in V(G) \ : \ d_G(v) = 1\}$ and $V_1(G) = \{v \in V(G) \ : \ N_G(v) \cap V_0(G) \neq \emptyset\}$. The *maximum degree* of G is denoted by $\Delta = \Delta(G)$. We use $G-u$ or $G-uv$ to denote the graph that arises from G by deleting the vertex $u \in V(G)$ or the edge $uv \in E(G)$. Similarly, $G+uv$ is a graph that arises from G by adding an edge $uv \notin E(G)$, where $u,\ v \in V(G)$. A *pendant chain* $P_s^0 = v_0v_1 \cdots v_s$ of a graph G is a sequence of vertices $v_0, v_1, \ldots, v_s$ such that v_0 is a pendant vertex of G, $d_G(v_1) = \cdots = d_G(v_{s-1}) = 2$ (unless $s = 1$) and $d_G(v_s) \geq 3$. We also call that v_s and s the end-vertex and the length of the pendant chain P_s^0, respectively. If $s = 1$, then the pendant chain P_s^0 is a pendant edge. Let $\mathcal{P}(T)$=$\{P_i^0 \ : \ i \geq 1\}$.

A tree is a connected acyclic graph. Let T be a tree with n vertices and maximum degree Δ. If $\Delta = 2$, then $T \cong P_n$, a path of order n; and if $\Delta = n-1$, then $T \cong K_{1,n-1}$. Therefore, in the following, we assume that $3 \leq \Delta \leq n-2$. Let $\mathscr{T}_{n,\Delta} = \{T \ : \ T$ is a tree with n vertices and maximum degree $\Delta,\ 3 \leq \Delta \leq n-2\}$.

In order to formulate our results, we need to define three trees $S_{n,\Delta}$ $(n \leq 2\Delta)$, $W_{n,\Delta}$ and $Y_{n,\Delta}$ (shown in Figure 1) as follows:

$S_{n,\Delta}$ $(n \leq 2\Delta)$ is a graph obtained from the star $K_{1,\Delta}$ by attaching one pendant vertex to each of $n-\Delta-1$ pendant vertices of $K_{1,\Delta}$.

$W_{n,\Delta}$ is a graph obtained from the star $K_{1,\Delta}$ by attaching $n-\Delta-1$ pendant vertices to one pendant vertex of $K_{1,\Delta}$.

$Y_{n,\Delta}$ is a graph obtained from the path $P_{n-\Delta+1}$ of order $n-\Delta+1$ by attaching $\Delta-1$ pendant vertices to one end-vertex of $P_{n-\Delta}$.

Note that $S_{n,\Delta}, Y_{n,\Delta} \in \mathscr{T}_{n,\Delta}$, and if $n \leq 2\Delta$, then $W_{n,\Delta} \in \mathscr{T}_{n,\Delta}$.

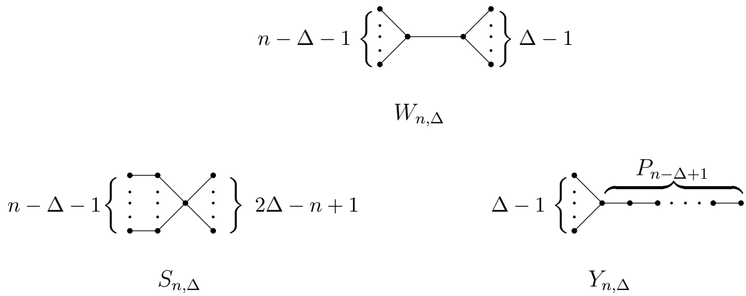


Figure 1

2. Upper Bound

In this section, we first give some lemmas that used in the proof of our results.

Lemma 2.1. *For $\alpha < 0$ (or $\alpha > 1$) and $l > 0$, the function $f(x) = (x+l)^\alpha - x^\alpha$ is monotonously increasing in $x \geq 1$.*

Proof. Note that $\frac{df(x)}{dx} = \alpha[(x+l)^{\alpha-1} - x^{\alpha-1}] > 0$ for $\alpha < 0$ (or $\alpha > 1$), and hence the lemma holds. ■

Lemma 2.2. *Let G be a graph, and let $u, v \in V(G)$ with $d_G(u), d_G(v) \geq 3$. Suppose that u_0u and $v_0v_1 \cdots v_l$ ($v_l = v$) are the pendant chains of G with end vertices u, v, respectively (see Figure 2). Set $G^* = G - v_0v_1 + u_0v_0$. If $l \geq 3$, then, for $\alpha \neq 0$,*

$$w_\alpha(G^*) > w_\alpha(G).$$

Proof. Let $d_G(u) = t$. Then $t \geq 3$. Note that

$$w_\alpha(G^*) - w_\alpha(G) = (2t)^\alpha + 2^\alpha - t^\alpha - 4^\alpha = (t^\alpha - 2^\alpha)(2^\alpha - 1) > 0,$$

and hence the lemma holds. ■

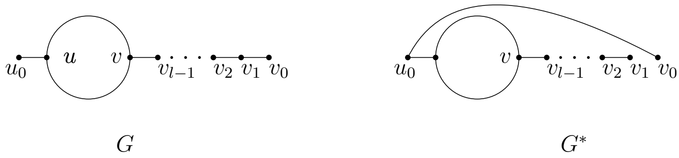


Figure 2

Lemma 2.3. *Suppose that G is a graph and $u, v \in V(G)$ with $d_G(u) > d_G(v) \geq 2$. Let $uu_0, vv_0 \in E(G)$ with $u_0 \in V_0(G)$, $N_G(v_0) \setminus \{v\} = \{v_1, v_2, \ldots, v_s\}$ ($s \geq 1$) and v_0 being not on the path connecting u to v (see Figure 3). Set $G' = G - v_0v_1 - \cdots - v_0v_s + u_0v_1 + \cdots + u_0v_s$. Then, for $\alpha \neq 0$,*

$$w_\alpha(G') > w_\alpha(G).$$

Proof. Note that

$$\begin{aligned} w_\alpha(G') - w_\alpha(G) &= (s+1)^\alpha d_G^\alpha(v) + d_G^\alpha(u) - (s+1)^\alpha d_G^\alpha(u) - d_G^\alpha(v) \\ &= (d_G^\alpha(u) - d_G^\alpha(v))((s+1)^\alpha - 1) > 0, \end{aligned}$$

and hence the lemma holds. ∎

Figure 3

Lemma 2.4. *Let G be a connected graph of order $n \geq 4$, and let $v \in V(G)$. Suppose that $u_0, v_0 \in N_G(v) \cap V_0(G)$. Set $G^* = G - vu_0 + u_0v_0$ (see Figure 4). Then, for $\alpha < 0$,*

$$w_\alpha(G^*) > w_\alpha(G).$$

Proof. Let $d_G(v) = t$. Since G is connected and $n \geq 4$, $t \geq 3$. Thus

$$\begin{aligned} w_\alpha(G^*) - w_\alpha(G) &= \sum_{u \in N_G(v) \setminus \{v_0, u_0\}} d_G^\alpha(u) \cdot [(t-1)^\alpha - t^\alpha] + (2t-2)^\alpha + 2^\alpha - 2 \cdot t^\alpha \\ &> 2^\alpha (t-1)^\alpha + 2^\alpha - 2 \cdot t^\alpha = [(2t-2)^\alpha - t^\alpha] - (t^\alpha - 2^\alpha) \\ &> 0. \end{aligned}$$

The last inequality follows by Lemma 2.1 as $2t-2>t$. ■

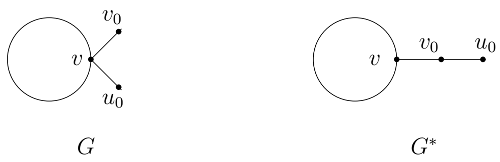


Figure 4

Theorem 2.5. *Let $T \in \mathscr{T}_{n,\Delta}$ and $3 \le \lceil \frac{n}{2} \rceil \le \Delta \le n-2$. Then*

$$w_\alpha(T) \le (2\Delta - n + 1)\Delta^\alpha + 2^\alpha(n-\Delta-1)(1+\Delta^\alpha) \tag{1}$$

and equality holds if and only if $T \cong S_{n,\Delta}$ for $\alpha < 0$.

Proof. First we note that if $T \cong S_{n,\Delta}$, then the equality in (1) holds.

Now we prove that if $T \in \mathscr{T}_{n,\Delta}$ and $3 \le \lceil \frac{n}{2} \rceil \le \Delta \le n-2$, then (1) holds and the equality in (1) holds only if $T \cong S_{n,\Delta}$.

Let $T \in \mathscr{T}_{n,\Delta}$. Let $w \in V(T)$ with $d_T(w) = \Delta \ge 3$. Since $\Delta \ge \lceil \frac{n}{2} \rceil$, we have $N_T(w) \cap V_0(T) \ne \emptyset$. Let $u_0 \in V_0(T)$ with $wu_0 \in E(T)$.

We choose T such that $w^\alpha(T)$ is as large as possible. We will show three facts.

Fact 1. *For any $P_l^0 \in \mathcal{P}(\mathcal{T})$, we have $l \le 2$.*

Proof of Fact 1. Assume $P_l^0 = v_0v_1\cdots v_l \in \mathcal{P}(\mathcal{T})$ with end vertex v_l, where $v_0 \in V_0(T)$ and $l \ge 3$. Let $T' = T - v_0v_1 + u_0v_0$. Then $T' \in \mathscr{T}_{n,\Delta}$. By Lemma 2.2, we have $w_\alpha(T') \ge w_\alpha(T)$, a contradiction with our choice.

Fact 2. *Let $P_l^0 = v_0v_1\cdots v_l \in \mathcal{P}(\mathcal{T})$ with end vertex v_l and $v_0 \in V_0(T)$. If $v_l \ne w$, then $l = 1$.*

Proof of Fact 2. Assume that $l \ge 2$. Then by Fact 1, $l = 2$. Since $\Delta \ge \lceil \frac{n}{2} \rceil$ and $v_l \ne w$, we have $d_T(v_l) \le n - \Delta - 1 \le \lfloor \frac{n}{2} \rfloor - 1 < \Delta = d_T(w)$. Set

$$T' = T - v_0v_1 + u_0v_0.$$

Then $T' \in \mathscr{T}_{n,\Delta}$. By Lemma 2.3, $w_\alpha(T') \ge w_\alpha(T)$, a contradiction with our choice. ■

Fact 3. *For any vertex $v \in V(T) \setminus \{w\}$, we have $d_T(v) \le 2$.*

Proof of Fact 3. Assume that $d_T(v) \geq 3$ for some $v \in V(T) \setminus \{w\}$. We choose v such that $d_T(w, v)$ is as large as possible. Then $|N_T(v) \cap V_0(T)| \geq 2$ by Fact 2. Let $u', v' \in N_T(v) \cap V_0(T)$. Set

$$T' = T - u'v + u'v'.$$

Then $T' \in \mathscr{T}_{n,\Delta}$. By Lemma 2.4, we have $w_\alpha(T') \geq w_\alpha(T)$, a contradiction with our choice. ■

By Fact 3, the proof of the theorem is complete. ■

By Theorem 2.5, we have $w_\alpha(S_{n,\Delta-1}) \geq w_\alpha(S_{n,\Delta})$ for $\alpha < 0$ and $\lceil \frac{n}{2} \rceil + 1 \leq \Delta \leq n - 1$. Thus we obtain the following result.

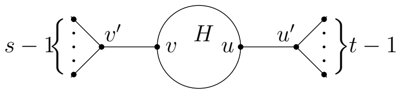
Corollary 2.6. *Let $T \in \mathscr{T}_{n,\Delta}$ and $\Delta \geq l \geq \lceil \frac{n}{2} \rceil$. Then, for $\alpha < 0$, $w_\alpha(T) \leq w_\alpha(S_{n,l})$ with equality if and only if $T \cong S_{n,l}$.*

In [21], Pan, Liu and Xu has shown the following result.

Lemma 2.7 [21]. *Let T be a tree with n vertices and m-matching, where $n \geq 2m$. Then, for $-\frac{1}{2} \leq \alpha < 0$, $w_\alpha(T) \geq w_\alpha(S_{n,n-m})$ with equality if and only if $T \cong S_{n,n-m}$.*

By Lemma 2.7 and Theorem 2.5, we have the following result.

Corollary 2.8. *Let T_1 and T_2 be trees of order n, $n \geq 4$. If T_1 has m-matchings and $\Delta(T_2) = \Delta' \geq n - m$, then $w_\alpha(T_1) \geq w_\alpha(T_2)$ with equality if and only if $T_1 \cong T_2 \cong S_{n,\Delta'}$ for $-\frac{1}{2} \leq \alpha < 0$.*



$Q_{s,t}$

Figure 5

Lemma 2.9. *Let $Q_{s,t}$ be a graph shown in Figure 5, where H is a connected graph. If $s \geq t \geq 2$ and $d_G(v) \geq d_G(u)$, then, for $\alpha \geq 1$,*

$$w_\alpha(Q_{s,t}) < w_\alpha(Q_{s+1,t-1}).$$

Proof. Set $d_G(v) = p$, $d_G(u) = q$. Then $p \geq q$ and

$$w_\alpha(Q_{s+1,t-1}) - w_\alpha(Q_{s,t})$$

$$
\begin{aligned}
&= (s+p^\alpha)(s+1)^\alpha + (t-2+q^\alpha)(t-1)^\alpha - (s-1+p^\alpha)s^\alpha - (t-1+q^\alpha)t^\alpha \\
&= [(s+1)^{\alpha+1} - s^{\alpha+1}] - [t^{\alpha+1} - (t-1)^{\alpha+1}] \\
&\quad +(p^\alpha - 1)[(s+1)^\alpha - s^\alpha] - (q^\alpha - 1)[t^\alpha - (t-1)^\alpha].
\end{aligned}
$$

If $\alpha = 1$, then

$$
\begin{aligned}
w_\alpha(Q_{s+1,t-1}) - w_\alpha(Q_{s,t}) &= [(s+1)^2 - s^2] - [t^2 - (t-1)^2] + p - q \\
&\geq [(s+1)^2 - s^2] - [t^2 - (t-1)^2] > 0,
\end{aligned}
$$

if $\alpha > 1$, then

$$
\begin{aligned}
w_\alpha(Q_{s+1,t-1}) - w_\alpha(Q_{s,t}) &= [(s+1)^{\alpha+1} - s^{\alpha+1}] - [t^{\alpha+1} - (t-1)^{\alpha+1}] \\
&\quad +(p^\alpha - 1)[(s+1)^\alpha - s^\alpha] - (q^\alpha - 1)[t^\alpha - (t-1)^\alpha] \\
&> (p^\alpha - 1)[((s+1)^\alpha - s^\alpha) - (t^\alpha - (t-1)^\alpha)] > 0,
\end{aligned}
$$

the last inequality follows by Lemma 2.1 as $s > t-1,\ p \geq q \geq 2$. Hence the lemma holds. ■

Theorem 2.10. *Let $T \in \mathscr{T}_{n,\Delta}$ and $3 \leq \lceil \frac{n}{2} \rceil \leq \Delta \leq n-2$. Then*

$$
w_\alpha(T) \leq (\Delta - 1)\Delta^\alpha + (n - \Delta - 1)(n-\Delta)^\alpha + \Delta^\alpha(n-\Delta)^\alpha \tag{2}
$$

and equality holds if and only if $T \cong W_{n,\Delta}$ for $\alpha \geq 1$.

Proof. First we note that if $T \cong W_{n,\Delta}$, then the equality in (2) holds.

Now we prove that if $T \in \mathscr{T}_{n,\Delta}$, then (2) holds and the equality in (2) holds only if $T \cong W_{n,\Delta}$ for $3 \leq \lceil \frac{n}{2} \rceil \leq \Delta \leq n-2$.

Let $T \in \mathscr{T}_{n,\Delta}$. Let $w \in V(T)$ with $d_T(w) = \Delta \geq 3$. Since $\Delta \geq \lceil \frac{n}{2} \rceil$, we have $N_T(w) \cap V_0(T) \neq \emptyset$. We choose T such that $w_\alpha(T)$ is as large as possible. Let $u_0 \in V_0(T)$ with $wu_0 \in E(T)$. We first show two facts.

Fact 1. *For any vertex $u \in N_T(w) \setminus V_0(T)$, u is a claw.*

Proof of Fact 1. Assume that $u \in N_T(w) \setminus V_0(T)$ is not a claw. Then there is a vertex $u' \in N_T(u) \setminus \{w\}$ such that $u' \notin V_0(T)$. Denote $N_T(u') \setminus \{u\} = \{u_1, \ldots, u_s\} (s \geq 1)$. Since $\Delta \geq \lceil \frac{n}{2} \rceil$ and $u \neq w$, $d_T(u) \leq n - \Delta - 1 \leq \lfloor \frac{n}{2} \rfloor - 1 < \Delta = d_T(w)$. Set

$$
T' = T - u'u_1 - \cdots - u'u_s + u_0u_1 + \cdots + u_0u_s.
$$

Then $T' \in \mathscr{T}_{n,\Delta}$. By Lemma 2.3, $w_\alpha(T') > w_\alpha(T)$, a contradiction with our choice.■

Fact 2. *w is a claw.*

Proof of Fact 2. Assume that w is not a claw. Then there are at least two vertices $u, v \in N_T(w)$ such that $d_T(u) = s \geq 2$, $d_T(v) = t \geq 2$. By Fact 1, u, v are claws. Denoted by H the non-trivial component of $T - \{u, v\}$. Then $T \cong Q_{s,t}$ (see Figure 5). Assume that $s \geq t$. Then $w_\alpha(Q_{s+1,t-1}) > w_\alpha(Q_{s,t})$ by Lemma 2.9. Since $\Delta \geq \lceil \frac{n}{2} \rceil$, $s + t - 1 \leq n - \Delta \leq \lfloor \frac{n}{2} \rfloor \leq \Delta$. Thus $Q_{s+1,t-1} \in \mathscr{T}_{n,\Delta}$, and hence we get a contradiction with our choice. ■

By Facts 1 and 2, the proof of the theorem is complete. ■

3. Lower Bound

Lemma 3.1. *Suppose that G is a graph and $u, v \in V(G)$ with $d_G(u) > d_G(v) \geq 2$. Let $uu_0, vv_0 \in E(G)$ with $v_0 \in V_0(G)$, $N_G(u_0) \setminus \{u\} = \{u_1, u_2, \ldots, u_{s-1}\}$ $(s \geq 2)$ and u_0 being not on the path connecting u to v. Set $G' = G - u_0u_1 - \cdots - u_0u_{s-1} + v_0u_1 + \cdots + v_0u_{s-1}$. Then, for $\alpha \neq 0$,*

$$w_\alpha(G') < w_\alpha(G).$$

Proof. Note that

$$\begin{aligned} w_\alpha(G') - w_\alpha(G) &= s^\alpha d_G^\alpha(v) + d_G^\alpha(u) - s^\alpha d_G^\alpha(u) - d_G^\alpha(v) \\ &= (d_G^\alpha(v) - d_G^\alpha(u))(s^\alpha - 1) < 0, \end{aligned}$$

and hence the lemma holds. ■

From Lemma 3.1, we immediately get the following result.

Theorem 3.2. *Let $T \in \mathscr{T}_{n,\Delta}$ and $3 \leq \lceil \frac{n}{2} \rceil \leq \Delta \leq n - 2$. Then*

$$w_\alpha(T) \geq (\Delta - 1)\Delta^\alpha + (n - \Delta - 1)(n - \Delta)^\alpha + \Delta^\alpha(n - \Delta)^\alpha \tag{3}$$

and equality holds if and only if $T \cong W_{n,\Delta}$ for $\alpha < 0$.

Proof. First we note that if $T \cong W_{n,\Delta}$, then the equality in (3) holds.

Now we prove that if $T \in \mathscr{T}_{n,\Delta}$, then (3) holds and the equality in (3) holds only if $T \cong W_{n,\Delta}$ for $3 \leq \lceil \frac{n}{2} \rceil \leq \Delta \leq n - 2$.

Let $T \in \mathscr{T}_{n,\Delta}$. Let $w \in V(T)$ with $d_T(w) = \Delta \geq 3$. Since $\Delta \geq \lceil \frac{n}{2} \rceil$, we have $N_T(w) \cap V_0(T) \neq \emptyset$. Let $u_0 \in V_0(T)$ with $wu_0 \in E(T)$.

We choose T such that $w_\alpha(T)$ is as small as possible. We first show two facts.

Fact 1. *w is a claw.*

Proof of Fact 1. Assume that w is not a claw. Let $v \in V_1(T) \setminus \{w\}$ with $vv_0 \in E(T)$, where $v_0 \in V_0(T)$. Then there is a vertex $u \in N_T(w) \setminus V_0(T)$ such that u is not on the only path connecting w and v. Denote $N_T(u) \setminus \{w\} = \{u_1, \ldots, u_s\}(s \geq 1)$. Since $\Delta \geq \lceil \frac{n}{2} \rceil$ and $v \neq w$, we have $d_T(v) \leq n - \Delta - 1 \leq \lfloor \frac{n}{2} \rfloor - 1 \leq \Delta - 1 < d_T(w)$. Set

$$T' = T - uu_1 - \cdots - uu_s + v_0u_1 + \cdots + v_0u_s.$$

Then $T' \in \mathscr{T}_{n,\Delta}$. By Lemma 3.1, we have $w_\alpha(T') \leq w_\alpha(T)$, a contradiction with our choice. ■

By Fact 1, we can let u be the unique vertex with $wu \in E(T)$ and $d_T(u) \geq 2$. Let T_u be the subtree containing u in $T - w$.

Fact 2. $T_u \cong K_{1,n-\Delta-1}$.

Proof of Fact 2. Assume that $T_u \ncong K_{1,n-\Delta-1}$.Then there exists an edge $v'v$ such that $v'v$ is not a pendant edge. Then $d_T(v') = s \geq 2$, $d_T(v) = t \geq 2$. Choose $v'v$ such that $d_T(w, v)$ is as large as possible. Then v is a claw. Denote $N_T(v) \cap V_0(T) = \{v_1, v_2, \ldots, v_{t-1}\}$. Set $T' = T - vv_1 - vv_2 - \cdots - vv_{t-1} + v'v_1 + v'v_2 + \cdots + v'v_{t-1}$. Then

$$\begin{aligned} w_\alpha(T) - w_\alpha(T') &> (t-1)t^\alpha + s^\alpha t^\alpha - t(s+t-1)^\alpha \\ &= (t-1)[t^\alpha - (s+t-1)^\alpha] + (st)^\alpha - (s+t-1)^\alpha \\ &= -\alpha(t-1)(s-1)\xi^\alpha + \alpha(t-1)(s-1)\eta^\alpha \\ &= \alpha(t-1)(s-1)(\eta^\alpha - \xi^\alpha) > 0, \end{aligned}$$

where $\xi \in (t, s+t-1)$ and $\eta \in (s+t-1, st)$. ■

By Facts 1 and 2, the proof of the theorem is complete. ■

Theorem 3.3. *Let $T \in \mathscr{T}_{n,\Delta}$. Then*

$$w_\alpha(T) \geq (\Delta - 1 + 2^\alpha)\Delta^\alpha + 2^\alpha + (n - \Delta - 2)4^\alpha \tag{4}$$

and equality holds if and only if $T \cong Y_{n,\Delta}$ for $\alpha > 0$.

Proof. First we note that if $T \cong Y_{n,\Delta}$, then the equality in (4) holds.

Now we prove that if $T \in \mathscr{T}_{n,\Delta}$, then (4) holds and the equality in (4) holds only if $T \cong Y_{n,\Delta}$.

Let $T \in \mathscr{T}_{n,\Delta}$. We choose T such that $w_\alpha(T)$ is as small as possible. Let $w \in V(T)$ with $d_T(w) = \Delta \geq 3$. By an argument similar to the proof of Theorem 3.2, we have $|N_T(w) \cap V_0(T)| = \Delta - 1$, that is, w is a claw. Therefore, we can let u be the unique vertex with $wu \in E(T)$ and $d_T(u) \geq 2$. Let T_u be the subtree containing u in $T - w$.

Fact A. $T_u \cong P_{n-\Delta}$.

Proof of Fact A. Assume that $T_u \not\cong P_{n-\Delta}$. Then there is a vertex v such that $d_T(v) = s \geq 3$. Choose v such that $d_{T_u}(u, v)$ is as large as possible. Let $P_l^0 = v_0 v_1 \cdots v_l (v_l = v)$ is a pendant chain with end vertex v. Since T_u ia a tree, there is a unique path between u and v and only one of v's neighbors, say v', is on the path. Let $N_{T_u}(v) \setminus \{v', v_{l-1}\} = \{x_1, \ldots, x_{s-2}\}$. Then $d_{T_u}(x_i) = a_i \geq 1$ and $d_{T_u}(v') = b \geq 2$. Set $T' = T - vx_1 - \cdots - vx_{s-2} + v_0 x_1 + \cdots + v_0 x_{s-2}$. Then $T' \in \mathscr{T}_{n,\Delta}$. If $l = 1$, then

$$\begin{aligned}
w_\alpha(T) - w_\alpha(T') &= b^\alpha(s^\alpha - 2^\alpha) + \sum_{i=1}^{s-2} a_i^\alpha (s^\alpha - (s-1)^\alpha) + s^\alpha - (2s-2)^\alpha \\
&\geq 2^\alpha(s^\alpha - 2^\alpha) + (s-2)(s^\alpha - (s-1)^\alpha) + s^\alpha - (2s-2)^\alpha \\
&> 2^\alpha(s^\alpha - 2^\alpha) + s^\alpha - (2s-2)^\alpha \\
&= 2^\alpha(s^\alpha - (s-1)^\alpha) + s^\alpha - 4^\alpha.
\end{aligned}$$

Thus if $s \geq 4$, then $w_\alpha(T) - w_\alpha(T') > 0$; if $s = 3$, then $w_\alpha(T) - w_\alpha(T') > 6^\alpha + 3^\alpha - 2 \cdot 4^\alpha > 0$.

If $l \geq 2$, then

$$\begin{aligned}
& w_\alpha(T) - w_\alpha(T') \\
= \; & b^\alpha(s^\alpha - 2^\alpha) + \sum_{i=1}^{s-2} a_i^\alpha (s^\alpha - (s-1)^\alpha) + 2^\alpha(s^\alpha - 2^\alpha) + 2^\alpha(1 - (s-1)^\alpha) \\
\geq \; & 2^\alpha(s^\alpha - 2^\alpha) + (s-2)(s^\alpha - (s-1)^\alpha) + 2^\alpha(1 - 2^\alpha + s^\alpha - (s-1)^\alpha).
\end{aligned}$$

Thus if $s \geq 4$, then $w_\alpha(T) - w_\alpha(T') > 2^\alpha(4^\alpha - 2^{\alpha+1} + 1) = 2^\alpha(2^\alpha - 1)^2 > 0$; if $s = 3$, then $w_\alpha(T) - w_\alpha(T') \geq 2^\alpha(3^\alpha - 2^\alpha) + (3^\alpha - 2^\alpha) + 2^\alpha(1 - 2^\alpha + 3^\alpha - 2^\alpha) > 6^\alpha + 3^\alpha - 2 \cdot 4^\alpha > 0$.

Therefore, in either case, we get a tree $T' \in T_{n,\Delta}$ such that $w_\alpha(T) > w_\alpha(T')$, a contradiction with our choice. ■

By Fact A, the proof of the theorem is complete. ■

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