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The Third Minimal Randić Index Tree with k Pendant Vertices

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Abstract

The Randić index of an organic molecule whose molecular graph is G is the sum of the weights $(d(u)d(v))^{-\frac{1}{2}}$ of all edges uv of G, where $d(u)$ denotes the degree of the vertex u of the molecular graph G. In this paper, we investigate some minimal Randić index properties and give the tree with the third minimal Randić index among the trees with n vertices and k pendant vertices.

1. Introduction

Mathematical descriptors of molecular structure, such as various topological indices, have been widely used in structure-property-activity studies (see [1, 2, 3]). Among the numerous topological indices considered in chemical graph theory, only a few have been found noteworthy in practical application (see [4]). One of these is the connectivity index or Randić index. The Randić index of an organic molecule whose molecular graph is G is defined (see [5, 6]) as

$$R(G) = \sum_{u,v}(d(u)d(v))^{-\frac{1}{2}},$$

where $d(u)$ denotes the degree of the vertex u of the molecular graph G, the summation goes over all pairs of adjacent vertices of G . In Randić's study of alkanes: he showed that if alkanes are ordered so that their $R(G)$-value decrease then the extent of their branching should increase (see [7]). There are many works to study the trees with extremal Randić index and the bounds in some graph sets (see [8]).

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In this paper, we are interested in the Randić indices for trees. First we provide a survey of some known results concerning our results. Let T be a tree of order n. Yu (see [9]) gave a sharp upper bound of

$$R(T) \leq \frac{n + 2\sqrt{2} - 3}{2}.$$

In[10], trees with large general Randić index are considered. For a tree T of order n with k pendant vertices, the sharp upper bound on Randić index in the case $3 \leq k \leq n-2, n \geq 3k-2$ was given by Zhang, Lu and Tian(see [11]). In order to illustrate some more results on the minimal Randić index, we need some notations as follows.

Let $K_{1,k}(p_1, p_2, \cdots, p_s)$, $(s \leq k)$ be a tree created from the star $K_{1,k}$ of $k+1$ vertices by attaching paths of lengths p_1, $p_2, \cdots, p_s$ to s pendant vertices of $K_{1,k}$, respectively(see Fig. 1(a)). Let $K^n_{s,k-s}$ be the tree created from a path of length $n-k-1$ by adding s pendant edges and $k-s$ pendant edges to two ends of the path, respectively (see Fig. 1(b)).Denote

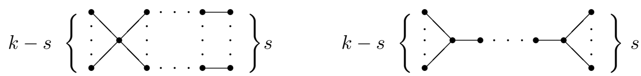
$S^n_{s,k-s} = \{K_{1,k}(p_1, p_2, \cdots, p_s) : \ p_i > 0, \ \sum_{i=1}^{s} p_i = n-k-1\}$,

$S_{n,k} = \bigcup_{s=1}^{k} S^n_{s,k-s}$,

$U_{n,k} = \{K^n_{s,k-s} : s = 2, \ldots, \lfloor \frac{k}{2} \rfloor\}$,

$\mathcal{T}_{n,k} = \{T : T$ is a tree with n vertices and k pendant vertices $\}$.

Clearly, $S_{n,k}, U_{n,k} \subseteq \mathcal{T}_{n,k}$.



(a) $K_{1,k}(p_1, p_2, \cdots, p_s) \in S^n_{s,k-s}$ (b) $K^n_{s,k-s} \in U_{n,k}$

Fig. 1

The trees with the minimum and the second minimum Randić index in $\mathcal{T}_{n,k}$ are characterized by Liu, Lu and Tian(see[12]) and Li et al.(see [8, 13]), respectively. A tree $T \in \mathcal{T}_{n,k}$ has the minimum Randić index if and only if $T \in S^n_{1,k-1}$ and its Randić index

$$R(T) = \frac{1}{2}(n-k) + \frac{1}{\sqrt{k}}(k + \frac{1}{\sqrt{2}} - 1) + \frac{1}{\sqrt{2}} - 1.$$

And a tree $T \in \mathcal{T}_{n,k}$ has the second minimum Randić index if and only if $T \in S^n_{2,k-2}$ and its Randić index

$$R(T) = \frac{1}{2}(n-k) + \frac{1}{\sqrt{k}}(k+\sqrt{2}-2) + \sqrt{2} - \frac{3}{2}.$$

Furthermore, we investigate some minimal Randić index properties, and characterize the tree $K^n_{2,k-2}$ with the third minimal Randić index and its Randić index

$$R(T) = \frac{1}{2}(n-k) + \frac{1}{\sqrt{k-1}}(k+\frac{1}{\sqrt{2}}-2) + \frac{2}{\sqrt{3}} + \frac{1}{\sqrt{6}} - \frac{3}{2}.$$

2. Notations and Lemmas

Let $G(V,E)$ be a graph with vertex set V and edge set E. Suppose $x \in V(G)$, $S \subseteq V(G)$. Denote the neighborhood of x by $N_G(x)$, $N_G(S) = \bigcup_{v \in S} N_G(v)$ and $V_i(G) = \{v : v \in V(G), d(v) = \mid N_G(v) \mid = i\}$. The maximum degree of G is denoted by $\triangle(G)$. Let T be a tree. For $x, y \in V(T)$, we use $T - x$ or $T - xy$ to denote the graph which arises from the tree T by deleting the vertex $x \in V(T)$ or the edge $xy \in E(T)$. Similarly, $T + xy$ is a graph that arises from T by adding an edge $xy \notin E(T)$. A vertex $x \in V(T)$ is called a pendant vertex if $x \in V_1(T)$. An edge in $E(T)$ is called a pendant edge if one end of the edge is in $V_1(T)$. A path $P = v_0 v_1 \cdots v_s$ of T is called a chain of T if $s > 1$ and $d(v_1) = \cdots = d(v_{s-1}) = 2$. If $d(v_0) = 1$, $d(v_s) \geq 3$ or $d(v_s) = 1$, $d(v_0) \geq 3$, then P is called a *pendant chain* of T.

In order to compare the Randić index between trees, we need two functions with monotonous properties in the following lemma.

Lemma 1.

(1) Let $F(x,b) = f(x,b) - f(x+1,b)$ where $f(x,b) = \sqrt{x} + \frac{b}{\sqrt{x}}$. If $x > 0$ and $b < 0$, then $F(x,b)$ is a monotonously increasing function.

(2) Let $G(x) = \frac{3}{\sqrt{x}} - \frac{1}{\sqrt{x-1}}$. If $x \geq 3$, then $G(x)$ is a monotonously decreasing function.

Proof. By derivation to functions $F(x,b)$ and $G(x)$ in x, we obtain

$$\begin{aligned} F'_x(x,b) &= \frac{1}{2\sqrt{x}} - \frac{b}{2\sqrt{x^3}} - \frac{1}{2\sqrt{x+1}} + \frac{b}{2\sqrt{(x+1)^3}} \\ &= \frac{1}{2}(\frac{1}{\sqrt{x}} - \frac{1}{\sqrt{x+1}}) - \frac{b}{2}(\frac{1}{\sqrt{x^3}} - \frac{1}{\sqrt{(x+1)^3}}) \\ &> 0 \end{aligned}$$

when $x > 0$ and $b < 0$.

$$\begin{aligned} G'(x) &= -\frac{3}{2\sqrt{x^3}} + \frac{1}{2\sqrt{(x-1)^3}} \\ &= \frac{1}{2}\left(\frac{1}{\sqrt{(x-1)^3}} - \frac{1}{\sqrt{\frac{x^3}{9}}}\right) \\ &< 0 \end{aligned}$$

when $x \geq 3$.

Therefore the functions $F(x,b) = f(x,b) - f(x+1,b)$ and $G(x)$ are monotonous increasing and monotonous decreasing respectively in x. ■

Lemma 2. Let $T \in \mathcal{T}_{n,k}$. If T has $s(\geq 2)$ pendant chains, then there exist $\overline{T} \in \mathcal{T}_{n,k}$ with $s-1$ pendant chains such that $R(\overline{T}) < R(T)$.

Proof. Assume that T has s pendant chains, and $P = v_0 v_1 \cdots v_h$, $P' = v_0' v_1' \cdots v_l'$ $(h, l \geq 2)$ are its two pendant chains with $d(v_0)$, $d(v_0') = 1$ and $d(v_h), d(v_l') \geq 3$. Let $\overline{T} = T - v_{h-1}v_{h-2} + v_0 v_0'$. Then $\overline{T} \in \mathcal{T}_{n,k}$ with $s-1$ pendant chains (see Fig.2).

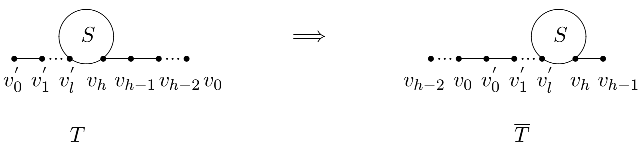


Fig. 2

It is not difficult to check that

$$\begin{aligned} R(T) - R(\overline{T}) &= \left(\frac{1}{\sqrt{d(v_h)}} - \frac{1}{\sqrt{2}}\right)\left(\frac{1}{\sqrt{2}} - 1\right) \\ &> 0, \end{aligned}$$

Therefore $R(\overline{T}) < R(T)$. ■

It is not difficult to check that $R(T_1) = R(T_2)$ for $T_1,\ T_2 \in S^n_{i,k-i}$, $i = 1,\ 2, \ldots, k$. The Randić index ordering of trees in $S_{n,k}$ is obtained immediately by Lemma 2.

Corollary. For any $T_1, T_2 \in S_{n,k}$, suppose $T_1 \in S^n_{i,k-i}$ and $T_2 \in S^n_{j,k-j}$, $i \leq j \leq k$.

(1) If $i = j$ then $R(T_1) = R(T_2)$;

(2) if $i < j$ the $R(T_1) < R(T_2)$.

In order to characterize the tree with the third minimum Randić index, we first characterize two extremal properties of trees in $\mathcal{T}_{n,k} \setminus (S^n_{1,k-1} \bigcup S^n_{2,k-2})$.

Lemma 3. Suppose $T \in \mathcal{T}_{n,k} \setminus (S^n_{1,k-1} \bigcup S^n_{2,k-2})$, $k \geq 4$. If $R(T) = min\{R(T) : T \in \mathcal{T}_{n,k} \setminus (S^n_{1,k-1} \bigcup S^n_{2,k-2})\}$, then $T \notin S_{n,k}$.

Proof. By contradiction. Choose a tree $T \in \mathcal{T}_{n,k}$ such that $R(T) = min\{R(T) : T \in \mathcal{T}_{n,k} \setminus (S^n_{1,k-1} \bigcup S^n_{2,k-2})\}$. If $T \in S_{n,k}$, then $T \in S^n_{3,k-3}$ by the choice of T and Corollary. It is easy to obtain that $R(T) = \frac{1}{2}(n-k) + \frac{1}{\sqrt{k}}(k + \frac{3}{\sqrt{2}} - 3) + \frac{3}{\sqrt{2}} - 2$. Clearly, $K^n_{2,k-2} \in \mathcal{T}_{n,k} \setminus (S^n_{1,k-1} \bigcup S^n_{2,k-2})$, and we have

$$\begin{aligned} R(T) \quad &- \quad R(K^n_{2,k-2}) \\ &= \frac{1}{\sqrt{k}}(k + \frac{3}{\sqrt{2}} - 3) - \frac{1}{\sqrt{k-1}}(k + \frac{1}{\sqrt{2}} - 2) + \frac{3}{\sqrt{2}} - \frac{2}{\sqrt{3}} - \frac{1}{\sqrt{6}} - \frac{1}{2} \\ &= \frac{2}{\sqrt{k}}(\frac{1}{\sqrt{2}} - 1) - F(k-1, \frac{1}{\sqrt{2}} - 1) + \frac{3}{\sqrt{2}} - \frac{2}{\sqrt{3}} - \frac{1}{\sqrt{6}} - \frac{1}{2}. \end{aligned}$$

Since $F(k-1, \frac{1}{\sqrt{2}} - 1)$ is a monotonously increasing in k by Lemma 1(1). Moreover, $\frac{2}{\sqrt{k}}(\frac{1}{\sqrt{2}} - 1)$ is monotonously increasing in k. Thus

$$R(T) - R(K^n_{2,k-2}) \geq \begin{cases} \frac{2}{\sqrt{4}}(\frac{1}{\sqrt{2}} - 1) - F(4, \frac{1}{\sqrt{2}} - 1) + \frac{3}{\sqrt{2}} - \frac{2}{\sqrt{3}} - \frac{1}{\sqrt{6}} - \frac{1}{2}, & \text{if } 4 \leq k \leq 5 \\ \frac{2}{\sqrt{6}}(\frac{1}{\sqrt{2}} - 1) - F(7, \frac{1}{\sqrt{2}} - 1) + \frac{3}{\sqrt{2}} - \frac{2}{\sqrt{3}} - \frac{1}{\sqrt{6}} - \frac{1}{2}, & \text{if } 6 \leq k \leq 8 \\ \frac{2}{\sqrt{9}}(\frac{1}{\sqrt{2}} - 1) - F(13, \frac{1}{\sqrt{2}} - 1) + \frac{3}{\sqrt{2}} - \frac{2}{\sqrt{3}} - \frac{1}{\sqrt{6}} - \frac{1}{2}, & \text{if } 9 \leq k \leq 14 \\ \frac{2}{\sqrt{15}}(\frac{1}{\sqrt{2}} - 1) - F(28, \frac{1}{\sqrt{2}} - 1) + \frac{3}{\sqrt{2}} - \frac{2}{\sqrt{3}} - \frac{1}{\sqrt{6}} - \frac{1}{2}, & \text{if } 15 \leq k \leq 29 \\ \frac{2}{\sqrt{30}}(\frac{1}{\sqrt{2}} - 1) - F(99, \frac{1}{\sqrt{2}} - 1) + \frac{3}{\sqrt{2}} - \frac{2}{\sqrt{3}} - \frac{1}{\sqrt{6}} - \frac{1}{2}, & \text{if } 30 \leq k \leq 100 \end{cases}$$

> 0 for $k \leq 100$.

For $k > 100$, we have

$$\begin{aligned}
R(T) \quad &- \quad R(K^n_{2,k-2}) \\
&= \frac{1}{\sqrt{k}}(k+\frac{3}{\sqrt{2}}-3) - \frac{1}{\sqrt{k-1}}(k+\frac{1}{\sqrt{2}}-2) + \frac{3}{\sqrt{2}} - \frac{2}{\sqrt{3}} - \frac{1}{\sqrt{6}} - \frac{1}{2} \\
&= (\sqrt{k}-\sqrt{k-1}) + (\frac{1}{\sqrt{2}}-1)(\frac{3}{\sqrt{k}} - \frac{1}{\sqrt{k-1}}) + \frac{3}{\sqrt{2}} - \frac{2}{\sqrt{3}} - \frac{1}{\sqrt{6}} - \frac{1}{2} \\
&> (\frac{1}{\sqrt{2}}-1)(\frac{3}{\sqrt{k}} - \frac{1}{\sqrt{k-1}}) + \frac{3}{\sqrt{2}} - \frac{2}{\sqrt{3}} - \frac{1}{\sqrt{6}} - \frac{1}{2} \\
&= (\frac{1}{\sqrt{2}}-1)G(k) + \frac{3}{\sqrt{2}} - \frac{2}{\sqrt{3}} - \frac{1}{\sqrt{6}} - \frac{1}{2} \\
&\geq (\frac{1}{\sqrt{2}}-1)G(101) + \frac{3}{\sqrt{2}} - \frac{2}{\sqrt{3}} - \frac{1}{\sqrt{6}} - \frac{1}{2} \\
&> 0
\end{aligned}$$

By Lemma 1(2), $G(k)$ is monotonous decreasing in k. This contradicts to $R(T) = min\{R(T) : T \in \mathcal{T}_{n,k} \setminus (S^n_{1,k-1} \bigcup S^n_{2,k-2})\}$. Consequently, $T \notin S_{n,k}$. ■

Lemma 4. Suppose $T \in \mathcal{T}_{n,k} \setminus (S^n_{1,k-1} \bigcup S^n_{2,k-2})$, $k \geq 4$. If $R(T) = min\{R(T) : T \in \mathcal{T}_{n,k} \setminus (S^n_{1,k-1} \bigcup S^n_{2,k-2})\}$, then T contains no any pendant chains.

Proof. By contradiction. Assume that $T \in \mathcal{T}_{n,k} \setminus (S^n_{1,k-1} \bigcup S^n_{2,k-2})$ with $R(T) = min\{R(T) : T \in \mathcal{T}_{n,k} \setminus (S^n_{1,k-1} \bigcup S^n_{2,k-2})\}$ and T has a pendant chain $P = v_0 v_1 \cdots v_s$ with $d(v_0) = 1$. There are at least two vertices of degrees greater than 2 in the tree T by Lemma 3. Therefore there exists an edge or a chain $P' = v'_0 v'_1 \cdots v'_l$ $(l \geq 1)$ with $d(v'_0),\ d(v'_l) \geq 3$. Let $\overline{T}$ be obtained from $T - \{v_0, v_1, \cdots, v_{s-2}\}$ by using the path $P'' = v'_0 v'_1 \cdots v'_{l+s-1}$ of length $l+s-1$ instead of the path $P' = v'_0 v'_1 \cdots v'_l$ (see Fig. 3). Then $\overline{T} \in \mathcal{T}_{n,k} \setminus (S^n_{1,k-1} \bigcup S^n_{2,k-2})$.

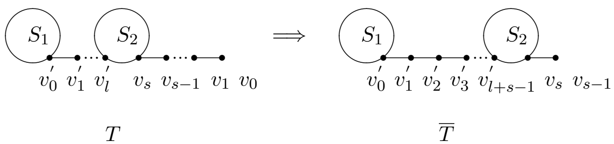


Fig. 3

When $l = 1$,

$$\begin{aligned} R(T) - R(\overline{T}) &= \frac{1}{\sqrt{d(v_0')d(v_l')}} + \frac{1}{\sqrt{2d(v_s)}} + \frac{1}{\sqrt{2}} - \frac{1}{\sqrt{2d(v_0')}} - \frac{1}{\sqrt{2d(v_l')}} - \frac{1}{\sqrt{d(v_s)}} \\ &= (1 - \frac{1}{\sqrt{2}})(\frac{1}{\sqrt{2}} - \frac{1}{\sqrt{d(v_s)}}) + (\frac{1}{\sqrt{2}} - \frac{1}{\sqrt{d(v_0')}})(\frac{1}{\sqrt{2}} - \frac{1}{\sqrt{d(v_l')}}) \\ &> 0. \end{aligned}$$

And when $l \geq 2$,

$$\begin{aligned} R(T) - R(\overline{T}) &= \frac{1}{\sqrt{2}} + \frac{1}{\sqrt{2d(v_s)}} - \frac{1}{\sqrt{4}} - \frac{1}{\sqrt{d(v_s)}} \\ &= (1 - \frac{1}{\sqrt{2}})(\frac{1}{\sqrt{2}} - \frac{1}{\sqrt{d(v_s)}}) \\ &> 0. \end{aligned}$$

This is a contradiction to $R(T) = min\{R(T) : T \in \mathcal{T}_{n,k} \setminus (S^n_{1,k-1} \bigcup S^n_{2,k-2})\}$. ■

Lemma 5 [8,13]. Suppose $K^n_{s,k-s}$, $K^n_{t,k-t} \in U_{n,k}$. If $s < t < k - t$, then $R(K^n_{s,k-s}) < R(K^n_{t,k-t})$.

3. Extremal Property of $K^n_{2,k-2}$

Clearly, to determine that $K^n_{2,k-2}$ has the property of the third minimum Randić index in $\mathcal{T}_{n,k}$ $(3 \leq k \leq n-3)$ is equivalent to determine it has the minimum Randić index in $\mathcal{T}_{n,k} \setminus (S^n_{1,k-1} \bigcup S^n_{2,k-2})$. Suppose $T \in \mathcal{T}_{n,k}$. Note that if $k = 2$, then T is a path, and hence $R(T) = \frac{n+2\sqrt{2}-3}{2}$; if $k = n-1$, then T is a star, and hence $R(T) = \sqrt{n-1}$. Moreover, $T \in S^5_{1,2}$ in the case $k = 3$ and $n = 5$; $T \in S^6_{1,2} \bigcup S^6_{2,1}$ in the case $k = 3$ and $n = 6$; $R(T) = \frac{1+4\sqrt{3}}{\sqrt{9}}$ or $T \in S^6_{1,3}$ in the case $k = 4$ and $n = 6$. If $k = 3$ and $n \geq 7$, then we have $\mathcal{T}_{n,3} = S^n_{1,2} \bigcup S^n_{2,1} \bigcup S^n_{3,0}$. Thus, for any $T_i \in S^n_{i,k-i}$ $(1 \leq i \leq 3)$, we get $R(T_1) < R(T_2) < R(T_3)$ by Corollary. If $k = n-2$ and $n \geq 7$, then we have $\mathcal{T}_{n,n-2} = U_{n,n-2}$. Thus $R(K^n_{2,n-4}) < R(K^n_{3,n-5}) < \cdots < R(K^n_{\lfloor \frac{n-2}{2} \rfloor, \lceil \frac{n-2}{2} \rceil})$ by Lemma 5. Therefore we just need to show the final case $4 \leq k \leq n-3$ and $n \geq 7$.

Theorem. Suppose $T \in \mathcal{T}_{n,k}$, $4 \leq k \leq n-3$, $n \geq 7$. If $R(T)$ is the third minimal Randić index in $\mathcal{T}_{n,k}$, then $T \cong K^n_{2,k-2}$ and

$$R(T) = \frac{1}{2}(n-k) + \frac{1}{\sqrt{k-1}}(k + \frac{1}{\sqrt{2}} - 2) + \frac{2}{\sqrt{3}} + \frac{1}{\sqrt{6}} - \frac{3}{2}.$$

Proof. For convenience, denote

$$\varphi(n,k) = \frac{1}{2}(n-k) + \frac{1}{\sqrt{k-1}}(k + \frac{1}{\sqrt{2}} - 2) + \frac{2}{\sqrt{3}} + \frac{1}{\sqrt{6}} - \frac{3}{2}.$$

It is easy to obtained that

$$\varphi(n-1,k-1) - \varphi(n,k) = F(k-2, \frac{1}{\sqrt{2}} - 1)$$

and

$$R(K^n_{2,k-2}) = \frac{1}{2}(n-k) + \frac{1}{\sqrt{k-1}}(k + \frac{1}{\sqrt{2}} - 2) + \frac{2}{\sqrt{3}} + \frac{1}{\sqrt{6}} - \frac{3}{2} = \varphi(n,k).$$

Choose a tree $T \in \mathcal{T}_{n,k} \setminus (S^n_{1,k-1} \bigcup S^n_{2,k-2})$ such that $R(T) = min\{R(T) : T \in \mathcal{T}_{n,k} \setminus (S^n_{1,k-1} \bigcup S^n_{2,k-2})\}$. By Lemma 3 and Lemma 4, $T \notin S_{n,k}$ and the tree T contains no pendant chain.

We now prove the conclusion by induction on k. When $k = 4$, we have $\Delta(T) = 3$. Otherwise $T \in S_{n,4}$, a contradiction. Furthermore, $|V_3(T)| = 2$ and $V_3(T) \subseteq N_T(V_1)$. Thus $T \cong K^n_{2,2}$. Assume that $k \geq 5$ and the result holds for $k-1$. Next, choose a vertex $u \in N_T(V_1)$ such that $d(u)$ is the maximum and $3 \leq d(u) \leq k-1$. Let $N_T(u) \cap V_1(T) = \{v_1, \cdots, v_r\} (r \geq 1)$, $N_T(u) \backslash V_1(T) = \{x_1, \cdots, x_{t-r}\}$, and $d(x_j) = d_j$ $(1 \leq j \leq t-r)$. Then $t - r \geq 1$ $(T \ncong K_{1,n-1})$ and $d_j \geq 2$ $(1 \leq j \leq t-r)$. Let $\overline{T} = T - v_1$. Thus $\overline{T} = T - v_1 \in \mathcal{T}_{n-1,k-1} \setminus (S^{n-1}_{1,k-2} \bigcup S^{n-1}_{2,k-3})$ and $R(\overline{T}) \geq \varphi(n-1,k-1)$ by the hypothesis of induction.Therefore

$$\begin{aligned}
R(T) &= R(\overline{T}) + \frac{r}{\sqrt{t}} - \frac{r-1}{\sqrt{t-1}} + \sum_{i=1}^{t-r} \frac{1}{\sqrt{d_i}}(\frac{1}{\sqrt{t}} - \frac{1}{\sqrt{t-1}}) \\
&\geq R(\overline{T}) + \frac{r}{\sqrt{t}} - \frac{r-1}{\sqrt{t-1}} + \frac{1}{\sqrt{2}}(t-r)(\frac{1}{\sqrt{t}} - \frac{1}{\sqrt{t-1}}) \\
&\geq \varphi(n-1,k-1) + \frac{r}{\sqrt{t}} - \frac{r-1}{\sqrt{t-1}} + \frac{1}{\sqrt{2}}(t-r)(\frac{1}{\sqrt{t}} - \frac{1}{\sqrt{t-1}}) \\
&= \varphi(n,k) + F(k-2, \frac{1}{\sqrt{2}} - 1) - F(t-1, \frac{1}{\sqrt{2}} - 1) + (\frac{1}{\sqrt{2}} - 1)(t-r-1)(\frac{1}{\sqrt{t}} - \frac{1}{\sqrt{t-1}}) \\
&\geq \varphi(n,k) + F(k-2, \frac{1}{\sqrt{2}} - 1) - F(t-1, \frac{1}{\sqrt{2}} - 1) \\
&\geq \varphi(n,k),
\end{aligned}$$

since $k - 1 \geq t$ and $F(x, \frac{1}{\sqrt{2}} - 1)$ is monotonously increasing according to Lemma 1(1). $R(T) = \varphi(n,k)$ if and only if all inequalities above must be equalities. Thus we have $R(\overline{T}) = \varphi(n -$

$1, k-1), k-1 = t, t-r = 1$ and $d_1 = 2$. By the induction hypothesis, $\overline{T} \cong K^{n-1}_{2,k-3}$. Therefore $T \cong K^n_{2,k-2}$ and the proof is completed.

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