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The genetic reactions of cyclopropane. Part I

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1. The groups of cyclopropane

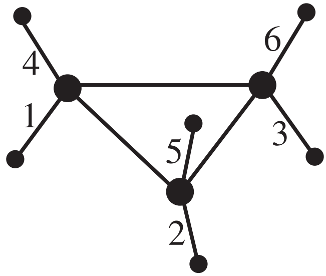
In this paper we shall use freely the terminology and notation from [2]. In accord to [4, V], or [1, Corollary 5.1.3], the group $G \leq S_6$ of univalent substitution isomerism of cyclopropane C_3H_6 coincides up to conjugacy with the group

$$\langle (123)(456), (14)(26)(35) \rangle,$$

which is isomorphic to the dihedral group of order 6. Since there are chiral pairs among the derivatives of cyclopropane, the group $G' \leq S_6$ of stereoisomerism of cyclopropane contains G and has order 12, so it coincides up to conjugacy with the group

$$\langle (123)(456), (14)(26)(35), (14)(25)(36) \rangle$$

that is isomorphic to the dihedral group of order 12 — see [4, V], or [1, Corollary 5.1.4]. The structural formula (graph) of cyclopropane



yields that its group $G'' \leq S_6$ of structural isomerism, up to conjugacy, coincides with the group

$$\langle (123)(456), (14)(26)(35), (14) \rangle$$

of order 48. Moreover, $G \leq G' \leq G''$.

2. The isomers of cyclopropane and their substitution reactions

Below, for any empirical formula $\lambda \in P_6$, we list the corresponding products of cyclopropane as well as the genetic reactions among them.

Case 1. $\lambda = (6)$.
We have

$$T_{(6);G} = T_{(6);G'} = T_{(6);G''} = \{a_{(6)}\},$$

where $a_{(6)}$ is the only G- and at the same time G'- and G''-orbit of the tabloid $A^{(6)} = (\{1,2,3,4,5,6\})$. The orbit $a_{(6)}$ represents the parent molecule of cyclopropane.

Case 2. $\lambda = (5,1)$.
The transitivity of the group G yields

$$T_{(5,1);G} = T_{(5,1);G'} = T_{(5,1);G''} = \{a_{(5,1)}\},$$

where $a_{(5,1)}$ is the only G- and at the same time G'- and G''-orbit of the tabloid $A^{(5,1)} = (\{1,2,3,4,5\},\{6\})$.
The only possible substitution reaction between the parent substance of cyclopropane and its mono-substitution derivative is designated $a_{(5,1)} < a_{(6)}$, because $R_{1,6}A^{(5,1)} = A^{(6)}$, and hence $A^{(5,1)} < A^{(6)}$. We remind that the operation $R_{1,6}$ applied on the tabloid $A^{(5,1)}$ means "replace the ligand of type x_2 in position 6 by a ligand of type x_1". The converse operation "replace the ligand of type x_1 in position 6 by a ligand of type x_2" represents the simple substitution reaction

$$A^{(6)} \longrightarrow A^{(5,1)}.$$

Case 3. $\lambda = (4,2)$.
We have

$$T_{(4,2);G} = \{a_{(4,2)}, b_{(4,2)}, c_{(4,2)}, e_{(4,2)}\},$$

where:
$a_{(4,2)}$ is the G-orbit of the tabloid $A^{(4,2)} = (\{1,2,3,4\},\{5,6\})$,
$b_{(4,2)}$ is the G-orbit of the tabloid $B^{(4,2)} = (\{1,2,4,5\},\{3,6\})$,
$c_{(4,2)}$ is the G-orbit of the tabloid $C^{(4,2)} = (\{1,2,4,6\},\{3,5\})$,
$e_{(4,2)}$ is the G-orbit of the tabloid $E^{(4,2)} = (\{1,3,4,5\},\{2,6\})$.
Below are all inequalities between the structural formulae of di-substitution homogeneous derivatives and the structural formula of the mono-substitution derivative of cyclopropane. We have

$$A^{(4,2)} < A^{(5,1)},\ B^{(4,2)} < A^{(5,1)},$$

$$(123)(456)C^{(4,2)} < A^{(5,1)},\ E^{(4,2)} < A^{(5,1)},$$

because

$$R_{1,5}A^{(4,2)} = R_{1,3}B^{(4,2)} = R_{1,1}(123)(456)C^{(4,2)} = R_{1,2}E^{(4,2)} = A^{(5,1)}.$$

Thus, we obtain the following substitution reactions

$$A^{(5,1)} \longrightarrow A^{(4,2)},\ A^{(5,1)} \longrightarrow B^{(4,2)},$$

$$A^{(5,1)} \longrightarrow (123)(456)C^{(4,2)},\ A^{(5,1)} \longrightarrow E^{(4,2)},$$

which mean "replace the ligand of type x_1 in position 5 of the tabloid $A^{(5,1)}$ by a ligand of type x_2", "replace the ligand of type x_1 in position 3 of the tabloid $A^{(5,1)}$ by a ligand of type x_2", "replace the ligand of type x_1 in position 1 of the tabloid $A^{(5,1)}$ by a ligand of type x_2", and, "replace the ligand of type x_1 in position 2 of the tabloid $A^{(5,1)}$ by a ligand of type x_2", respectively.
These simple substitution reactions are also designated by the inequalities

$$a_{(4,2)} < a_{(5,1)},\ b_{(4,2)} < a_{(5,1)},\ c_{(4,2)} < a_{(5,1)},\ e_{(4,2)} < a_{(5,1)}.$$

The set of G'-orbits in $T_{(4,2)}$ is

$$T_{(4,2);G'} = \{a_{(4,2)}, b_{(4,2)}, c_{(4,2)} \cup e_{(4,2)}\},$$

so the $(4,2)$-products that correspond to $c_{(4,2)}$ and $e_{(4,2)}$ are members of a chiral pair.
The set of G''-orbits in $T_{(4,2)}$ is

$$T_{(4,2);G''} = \{b_{(4,2)}, a_{(4,2)} \cup (c_{(4,2)} \cup e_{(4,2)})\},$$

hence the products that correspond to $a_{(4,2)}$, $c_{(4,2)}$, and $e_{(4,2)}$ are structurally identical, and any one of them is structurally isomeric with the product which corresponds to $b_{(4,2)}$.

Case 4. $\lambda = (4, 1^2)$.
In this case we have

$$T_{(4,1^2);G} = \{a_{(4,1^2)}, b_{(4,1^2)}, c_{(4,1^2)}, e_{(4,1^2)}, f_{(4,1^2)}\},$$

where:
$a_{(4,1^2)}$ is the G-orbit of the tabloid $A^{(4,1^2)} = (\{1,2,3,4\}, \{5\}, \{6\})$,
$b_{(4,1^2)}$ is the G-orbit of the tabloid $B^{(4,1^2)} = (\{1,2,3,4\}, \{6\}, \{5\})$,
$c_{(4,1^2)}$ is the G-orbit of the tabloid $C^{(4,1^2)} = (\{1,2,4,5\}, \{3\}, \{6\})$,
$e_{(4,1^2)}$ is the G-orbit of the tabloid $E^{(4,1^2)} = (\{1,2,4,6\}, \{3\}, \{5\})$,
$f_{(4,1^2)}$ is the G-orbit of the tabloid $F^{(4,1^2)} = (\{1,3,4,5\}, \{2\}, \{6\})$.
The following inequalities hold between the di-substitution homogeneous and the di-substitution heterogeneous derivatives of cyclopropane:

$$A^{(4,1^2)} < A^{(4,2)},\ B^{(4,1^2)} < A^{(4,2)},$$

$$C^{(4,1^2)} < B^{(4,2)},\ E^{(4,1^2)} < C^{(4,2)},\ F^{(4,1^2)} < E^{(4,2)}.$$

Indeed,

$$R_{2,6}A^{(4,1^2)} = R_{2,5}B^{(4,1^2)} = A^{(4,2)},$$

$$R_{2,6}C^{(4,1^2)} = B^{(4,2)},\ R_{2,5}E^{(4,1^2)} = C^{(4,2)},\ R_{2,6}F^{(4,1^2)} = E^{(4,2)}.$$

In this way we obtain the following substitution reactions

$$A^{(4,2)} \longrightarrow A^{(4,1^2)},\ A^{(4,2)} \longrightarrow B^{(4,1^2)},$$

$$B^{(4,2)} \longrightarrow C^{(4,1^2)},\ C^{(4,2)} \longrightarrow E^{(4,1^2)},\ E^{(4,2)} \longrightarrow F^{(4,1^2)},$$

which mean "replace the ligand of type x_2 in position 6 of the tabloid $A^{(4,2)}$ by a ligand of type x_3", "replace the ligand of type x_2 in position 5 of the tabloid $A^{(4,2)}$ by a ligand of type x_3", "replace the ligand of type x_2 in position 6 of the tabloid $B^{(4,2)}$ by a ligand of type x_3", "replace the ligand of type x_2 in position 5 of the tabloid $C^{(4,2)}$ by a ligand of type x_3", and, "replace the ligand of type x_2 in position 6 of the tabloid $E^{(4,2)}$ by a ligand of type x_3",respectively. In terms of inequalities these substitution reactions can be represented as follows:

$$a_{(4,1^2)} < a_{(4,2)},\ b_{(4,1^2)} < a_{(4,2)},$$

$$c_{(4,1^2)} < b_{(4,2)},\ e_{(4,1^2)} < c_{(4,2)},\ f_{(4,1^2)} < e_{(4,2)}.$$

Further, we obtain

$$T_{(4,1^2);G'} = \{a_{(4,1^2)} \cup b_{(4,1^2)}, c_{(4,1^2)}, e_{(4,1^2)} \cup f_{(4,1^2)}\},$$

and therefore the products that correspond to the members of any one of the sets $\{a_{(4,1^2)}, b_{(4,1^2)}\}$, and $\{e_{(4,1^2)}, f_{(4,1^2)}\}$ form a chiral pair, and the product that corresponds to $c_{(4,1^2)}$ is a dimer. Moreover,

$$T_{(4,1^2);G''} = \{(a_{(4,1^2)} \cup b_{(4,1^2)}) \cup (e_{(4,1^2)} \cup f_{(4,1^2)}), c_{(4,1^2)}\}.$$

Hence the four members of the above two chiral pairs are structurally identical, and each one of them is structurally isomeric to the product that corresponds to the dimer $c_{(4,1^2)}$.

Case 5. $\lambda = (3^2)$.
Now we have

$$T_{(3^2);G} = \{a_{(3^2)}, b_{(3^2)}, c_{(3^2)}, e_{(3^2)}\},$$

where:
$a_{(3^2)}$ is the G-orbit of the tabloid $A^{(3^2)} = (\{1,2,3\},\{4,5,6\})$,
$b_{(3^2)}$ is the G-orbit of the tabloid $B^{(3^2)} = (\{1,2,4\},\{3,5,6\})$,
$c_{(3^2)}$ is the G-orbit of the tabloid $C^{(3^2)} = (\{1,2,5\},\{3,4,6\})$,
$e_{(3^2)}$ is the G-orbit of the tabloid $E^{(3^2)} = (\{1,2,6\},\{3,4,5\})$.

We have the following inequalities between the tabloids of shape (3^2) and the tabloids of shape $(4,2)$:

$$A^{(3^2)} < A^{(4,2)},\ B^{(3^2)} < A^{(4,2)},\ (132)(465)C^{(3^2)} < A^{(4,2)},\ (123)(456)E^{(3^2)} < A^{(4,2)},$$

$$B^{(3^2)} < B^{(4,2)},\ C^{(3^2)} < B^{(4,2)},$$

$$B^{(3^2)} < C^{(4,2)},\ E^{(3^2)} < C^{(4,2)},$$

$$(132)(465)C^{(3^2)} < E^{(4,2)},\ (132)(465)E^{(3^2)} < E^{(4,2)},$$

because

$$R_{1,4}A^{(3^2)} = R_{1,3}B^{(3^2)} = R_{1,2}(132)(465)C^{(3^2)} = R_{1,1}(123)(456)E^{(3^2)} = A^{(4,2)},$$

$$R_{1,5}B^{(3^2)} = R_{1,4}C^{(3^2)} = B^{(4,2)},$$

$$R_{1,6}B^{(3^2)} = R_{1,4}E^{(3^2)} = C^{(4,2)},$$

$$R_{1,5}(132)(465)C^{(3^2)} = R_{1,4}(132)(465)E^{(3^2)} = E^{(4,2)}.$$

Thus, the substitution reactions among di-substitution homogeneous derivatives of cyclopropane and its tri-substitution homogeneous derivatives, can be represented as follows:

$$a_{(3^2)} < a_{(4,2)},\ b_{(3^2)} < a_{(4,2)},\ c_{(3^2)} < a_{(4,2)},\ e_{(3^2)} < a_{(4,2)},$$

$$b_{(3^2)} < b_{(4,2)},\ c_{(3^2)} < b_{(4,2)},$$

$$b_{(3^2)} < c_{(4,2)},\ e_{(3^2)} < c_{(4,2)},$$

$$c_{(3^2)} < e_{(4,2)},\ e_{(3^2)} < e_{(4,2)}.$$

The set of all G'-orbits is

$$T_{(3^2);G'} = \{a_{(3^2)}, b_{(3^2)} \cup c_{(3^2)}, e_{(3^2)}\},$$

so the products that correspond to the members of the set $\{b_{(3^2)}, c_{(3^2)}\}$ form a chiral pair,and the products that correspond to $a_{(3^2)}$ and $e_{(3^2)}$ are dimers.
The set of all G''-orbits is

$$T_{(3^2);G''} = \{a_{(3^2)} \cup e_{(3^2)}, (b_{(3^2)} \cup c_{(3^2)})\},$$

and this yields structural identity of the dimers which correspond to $a_{(3^2)}$ and $e_{(3^2)}$, and each one of them is structurally isomeric to any member of the above chiral pair.

Case 6. $\lambda = (3,2,1)$.
We have

$$T_{(3,2,1);G} =$$

$$\{a_{(3,2,1)}, b_{(3,2,1)}, c_{(3,2,1)}, e_{(3,2,1)}, f_{(3,2,1)}, h_{(3,2,1)}, k_{(3,2,1)}, \ell_{(3,2,1)}, m_{(3,2,1)}, p_{(3,2,1)}\}$$

where:

$a_{(3,2,1)}$ is the G-orbit of the tabloid $A^{(3,2,1)} = (\{1,2,3\},\{4,5\},\{6\})$,
$b_{(3,2,1)}$ is the G-orbit of the tabloid $B^{(3,2,1)} = (\{1,2,4\},\{3,5\},\{6\})$,
$c_{(3,2,1)}$ is the G-orbit of the tabloid $C^{(3,2,1)} = (\{1,2,4\},\{3,6\},\{5\})$,
$e_{(3,2,1)}$ is the G-orbit of the tabloid $E^{(3,2,1)} = (\{1,2,4\},\{5,6\},\{3\})$,
$f_{(3,2,1)}$ is the G-orbit of the tabloid $F^{(3,2,1)} = (\{1,2,5\},\{3,4\},\{6\})$,
$h_{(3,2,1)}$ is the G-orbit of the tabloid $H^{(3,2,1)} = (\{1,2,5\},\{3,6\},\{4\})$,
$k_{(3,2,1)}$ is the G-orbit of the tabloid $K^{(3,2,1)} = (\{1,2,5\},\{4,6\},\{3\})$,
$\ell_{(3,2,1)}$ is the G-orbit of the tabloid $L^{(3,2,1)} = (\{1,2,6\},\{3,4\},\{5\})$,
$m_{(3,2,1)}$ is the G-orbit of the tabloid $M^{(3,2,1)} = (\{1,2,6\},\{3,5\},\{4\})$,
$p_{(3,2,1)}$ is the G-orbit of the tabloid $P^{(3,2,1)} = (\{1,2,6\},\{4,5\},\{3\})$.
The inequalities between the tabloids of shape $(3,2,1)$ and the tabloids of shape $(4,1^2)$ and (3^2), respectively, are as follows:

$$A^{(3,2,1)} < A^{(4,1^2)},\ B^{(3,2,1)} < A^{(4,1^2)},$$

$$(132)(465)H^{(3,2,1)} < A^{(4,1^2)},\ (123)(456)L^{(3,2,1)} < A^{(4,1^2)},$$

$$(132)(465)A^{(3,2,1)} < B^{(4,1^2)},\ C^{(3,2,1)} < B^{(4,1^2)},$$

$$(132)(465)F^{(3,2,1)} < B^{(4,1^2)}, (123)(456)M^{(3,2,1)} < B^{(4,1^2)},$$

$$B^{(3,2,1)} < C^{(4,1^2)},\ (15)(24)(36)E^{(3,2,1)} < C^{(4,1^2)},$$

$$F^{(3,2,1)} < C^{(4,1^2)},\ (15)(24)(36)K^{(3,2,1)} < C^{(4,1^2)},$$

$$C^{(3,2,1)} < E^{(4,1^2)},\ (14)(26)(35)E^{(3,2,1)} < E^{(4,1^2)},$$

$$L^{(3,2,1)} < E^{(4,1^2)},\ (14)(26)(35)P^{(3,2,1)} < E^{(4,1^2)},$$

$$(132)(465)H^{(3,2,1)} < F^{(4,1^2)},\ (15)(24)(36)K^{(3,2,1)} < F^{(4,1^2)},$$

$$(132)(465)M^{(3,2,1)} < F^{(4,1^2)},\ (15)(24)(36)P^{(3,2,1)} < F^{(4,1^2)},$$

and

$$A^{(3,2,1)} < A^{(3^2)},$$

$$B^{(3,2,1)} < B^{(3^2)},\ C^{(3,2,1)} < B^{(3^2)},\ E^{(3,2,1)} < B^{(3^2)},$$

$$F^{(3,2,1)} < C^{(3^2)},\ H^{(3,2,1)} < C^{(3^2)},\ K^{(3,2,1)} < C^{(3^2)},$$

$$L^{(3,2,1)} < E^{(3^2)},\ M^{(3,2,1)} < E^{(3^2)},\ P^{(3,2,1)} < E^{(3^2)},$$

because

$$R_{1,4}A^{(3,2,1)} = R_{1,3}B^{(3,2,1)} = R_{1,2}(132)(465)H^{(3,2,1)} = R_{1,1}(123)(456)L^{(3,2,1)} = A^{(4,1^2)},$$

$$R_{1,4}(132)(465)A^{(3,2,1)} = R_{1,3}C^{(3,2,1)} = R_{1,2}(132)(465)F^{(3,2,1)} =$$

$$R_{1,1}(123)(456)(36)M^{(3,2,1)} = B^{(4,1^2)},$$

$$R_{1,5}B^{(3,2,1)} = R_{1,1}(15)(24)(36)E^{(3,2,1)} = R_{1,4}F^{(3,2,1)} =$$

$$R_{1,2}(15)(24)(36)K^{(3,2,1)} = C^{(4,1^2)},$$

$$R_{1,6}C^{(3,2,1)} = R_{1,2}(14)(26)(35)E^{(3,2,1)} = R_{1,4}L^{(3,2,1)} =$$

$$R_{1,1}(14)(26)(35)P^{(3,2,1)} = E^{(4,1^2)},$$

$$R_{1,5}(132)(465)H^{(3,2,1)} = R_{1,3}(15)(24)(36)K^{(3,2,1)} = R_{1,4}(132)(465)M^{(3,2,1)} =$$

$$R_{1,1}(15)(24)(36)P^{(3,2,1)} = F^{(4,1^2)},$$

and

$$R_{2,6}A^{(3,2,1)} = A^{(3^2)},$$

$$R_{2,6}B^{(3,2,1)} = R_{2,5}C^{(3,2,1)} = R_{2,3}E^{(3,2,1)} = B^{(3^2)},$$

$$R_{2,6}F^{(3,2,1)} = R_{2,4}H^{(3,2,1)} = R_{2,3}K^{(3,2,1)} = C^{(3^2)},$$

$$R_{2,5}L^{(3,2,1)} = R_{2,4}M^{(3,2,1)} = R_{2,3}P^{(3,2,1)} = E^{(3^2)}.$$

Therefore all simple substitution reactions between $(3,2,1)$-derivatives of cyclopropane and its $(4,1^2)$-derivatives and (3^2)-derivatives, respectively, are:

$$a_{(3,2,1)} < a_{(4,1^2)},\ b_{(3,2,1)} < a_{(4,1^2)},$$

$$h_{(3,2,1)} < a_{(4,1^2)},\ \ell_{(3,2,1)} < a_{(4,1^2)},$$

$$a_{(3,2,1)} < b_{(4,1^2)},\ c_{(3,2,1)} < b_{(4,1^2)},$$

$$f_{(3,2,1)} < b_{(4,1^2)},\ m_{(3,2,1)} < b_{(4,1^2)},$$

$$b_{(3,2,1)} < c_{(4,1^2)},\ e_{(3,2,1)} < c_{(4,1^2)},$$

$$f_{(3,2,1)} < c_{(4,1^2)},\ k_{(3,2,1)} < c_{(4,1^2)},$$

$$c_{(3,2,1)} < e_{(4,1^2)},\ e_{(3,2,1)} < e_{(4,1^2)},$$

$$\ell_{(3,2,1)} < e_{(4,1^2)},\ p_{(3,2,1)} < e_{(4,1^2)},$$

$$h_{(3,2,1)} < f_{(4,1^2)},\ k_{(3,2,1)} < f_{(4,1^2)},$$

$$m_{(3,2,1)} < f_{(4,1^2)},\ p_{(3,2,1)} < f_{(4,1^2)},$$

and

$$a_{(3,2,1)} < a_{(3^2)},$$

$$b_{(3,2,1)} < b_{(3^2)},\ c_{(3,2,1)} < b_{(3^2)},\ e_{(3,2,1)} < b_{(3^2)},$$

$$f_{(3,2,1)} < c_{(3^2)},\ h_{(3,2,1)} < c_{(3^2)},\ k_{(3,2,1)} < c_{(3^2)},$$

$$\ell_{(3,2,1)} < e_{(3^2)},\ m_{(3,2,1)} < e_{(3^2)},\ p_{(3,2,1)} < e_{(3^2)}.$$

The set of G'-orbits in $T_{(3,2,1)}$ is

$$T_{(3,2,1);G'} =$$

$$\{a_{(3,2,1)}, b_{(3,2,1)} \cup f_{(3,2,1)}, c_{(3,2,1)} \cup h_{(3,2,1)}, e_{(3,2,1)} \cup k_{(3,2,1)}, \ell_{(3,2,1)} \cup m_{(3,2,1)}, p_{(3,2,1)}\}.$$

In particular, the products that correspond to the members of the two-element sets

$$\{b_{(3,2,1)}, f_{(3,2,1)}\},\ \{c_{(3,2,1)}, h_{(3,2,1)}\},\ \{e_{(3,2,1)}, k_{(3,2,1)}\},\ \{\ell_{(3,2,1)}, m_{(3,2,1)}\},$$

form chiral pairs and the products that correspond to $a_{(3,2,1)}$ and $p_{(3,2,1)}$ are dimers. Further, the set of G''-orbits in $T_{(3,2,1)}$ is

$$T_{(3,2,1);G''} = \{a_{(3,2,1)} \cup (\ell_{(3,2,1)} \cup m_{(3,2,1)}) \cup p_{(3,2,1)},$$
$$(b_{(3,2,1)} \cup f_{(3,2,1)}) \cup (e_{(3,2,1)} \cup k_{(3,2,1)}), (c_{(3,2,1)} \cup h_{(3,2,1)})\}.$$

Hence the members of different sets below are structural isomers:

$$\{a_{(3,2,1)}, \ell_{(3,2,1)}, m_{(3,2,1)}, p_{(3,2,1)}\}, \{b_{(3,2,1)}, f_{(3,2,1)}, e_{(3,2,1)}, k_{(3,2,1)}\}, \{c_{(3,2,1)}, h_{(3,2,1)}\}.$$

Case 7. $\lambda = (3, 1^3)$.
Now, we have

$$T_{(3,1^3);G} = \{a_{(3,1^3)}, \bar{a}_{(3,1^3)}, b_{(3,1^3)}, \bar{b}_{(3,1^3)}, c_{(3,1^3)}, \bar{c}_{(3,1^3)}, e_{(3,1^3)}, \bar{e}_{(3,1^3)}, f_{(3,1^3)}, \bar{f}_{(3,1^3)},$$
$$h_{(3,1^3)}, \bar{h}_{(3,1^3)}, k_{(3,1^3)}, \bar{k}_{(3,1^3)}, \ell_{(3,1^3)}, \bar{\ell}_{(3,1^3)}, m_{(3,1^3)}, \bar{m}_{(3,1^3)}, p_{(3,1^3)}, \bar{p}_{(3,1^3)}\},$$

where:
$a_{(3,1^3)}$ is the G-orbit of the tabloid $A^{(3,1^3)} = (\{1,2,3\},\{4\},\{5\},\{6\})$,
$\bar{a}_{(3,1^3)}$ is the G-orbit of the tabloid $\bar{A}^{(3,1^3)} = (\{1,2,3\},\{4\},\{6\},\{5\})$,
$b_{(3,1^3)}$ is the G-orbit of the tabloid $B^{(3,1^3)} = (\{1,2,4\},\{3\},\{5\},\{6\})$,
$\bar{b}_{(3,1^3)}$ is the G-orbit of the tabloid $\bar{B}^{(3,1^3)} = (\{1,2,4\},\{3\},\{6\},\{5\})$,
$c_{(3,1^3)}$ is the G-orbit of the tabloid $C^{(3,1^3)} = (\{1,2,4\},\{5\},\{3\},\{6\})$,
$\bar{c}_{(3,1^3)}$ is the G-orbit of the tabloid $\bar{C}^{(3,1^3)} = (\{1,2,4\},\{5\},\{6\},\{3\})$,
$e_{(3,1^3)}$ is the G-orbit of the tabloid $E^{(3,1^3)} = (\{1,2,4\},\{6\},\{3\},\{5\})$,
$\bar{e}_{(3,1^3)}$ is the G-orbit of the tabloid $\bar{E}^{(3,1^3)} = (\{1,2,4\},\{6\},\{5\},\{3\})$,
$f_{(3,1^3)}$ is the G-orbit of the tabloid $F^{(3,1^3)} = (\{1,2,5\},\{3\},\{4\},\{6\})$,
$\bar{f}_{(3,1^3)}$ is the G-orbit of the tabloid $\bar{F}^{(3,1^3)} = (\{1,2,5\},\{3\},\{6\},\{4\})$,
$h_{(3,1^3)}$ is the G-orbit of the tabloid $H^{(3,1^3)} = (\{1,2,5\},\{4\},\{3\},\{6\})$,
$\bar{h}_{(3,1^3)}$ is the G-orbit of the tabloid $\bar{H}^{(3,1^3)} = (\{1,2,5\},\{4\},\{6\},\{3\})$,
$k_{(3,1^3)}$ is the G-orbit of the tabloid $K^{(3,1^3)} = (\{1,2,5\},\{6\},\{3\},\{4\})$,
$\bar{k}_{(3,1^3)}$ is the G-orbit of the tabloid $\bar{K}^{(3,1^3)} = (\{1,2,5\},\{6\},\{4\},\{3\})$,
$\ell_{(3,1^3)}$ is the G-orbit of the tabloid $L^{(3,1^3)} = (\{1,2,6\},\{3\},\{4\},\{5\})$,
$\bar{\ell}_{(3,1^3)}$ is the G-orbit of the tabloid $\bar{L}^{(3,1^3)} = (\{1,2,6\},\{3\},\{5\},\{4\})$,
$m_{(3,1^3)}$ is the G-orbit of the tabloid $M^{(3,1^3)} = (\{1,2,6\},\{4\},\{3\},\{5\})$,
$\bar{m}_{(3,1^3)}$ is the G-orbit of the tabloid $\bar{M}^{(3,1^3)} = (\{1,2,6\},\{4\},\{5\},\{3\})$,
$p_{(3,1^3)}$ is the G-orbit of the tabloid $P^{(3,1^3)} = (\{1,2,6\},\{5\},\{3\},\{4\})$,
$\bar{p}_{(3,1^3)}$ is the G-orbit of the tabloid $\bar{P}^{(3,1^3)} = (\{1,2,6\},\{5\},\{4\},\{3\})$.

The inequalities between the tabloids of shape $(3,1^3)$ and the tabloids of shape $(3,2,1)$ are as follows:

$$A^{(3,1^3)} < A^{(3,2,1)},\ \bar{A}^{(3,1^3)} < A^{(3,2,1)},$$

$$B^{(3,1^3)} < B^{(3,2,1)},\ \bar{B}^{(3,1^3)} < B^{(3,2,1)},\ C^{(3,1^3)} < B^{(3,2,1)},\ \bar{C}^{(3,1^3)} < B^{(3,2,1)},$$
$$B^{(3,1^3)} < C^{(3,2,1)},\ \bar{B}^{(3,1^3)} < C^{(3,2,1)},\ E^{(3,1^3)} < C^{(3,2,1)},\ \bar{E}^{(3,1^3)} < C^{(3,2,1)},$$
$$C^{(3,1^3)} < E^{(3,2,1)},\ \bar{C}^{(3,1^3)} < E^{(3,2,1)},\ E^{(3,1^3)} < E^{(3,2,1)},\ \bar{E}^{(3,1^3)} < E^{(3,2,1)},$$
$$F^{(3,1^3)} < F^{(3,2,1)},\ \bar{F}^{(3,1^3)} < F^{(3,2,1)},\ H^{(3,1^3)} < F^{(3,2,1)},\ \bar{H}^{(3,1^3)} < F^{(3,2,1)},$$
$$F^{(3,1^3)} < H^{(3,2,1)},\ \bar{F}^{(3,1^3)} < H^{(3,2,1)},\ K^{(3,1^3)} < H^{(3,2,1)},\ \bar{K}^{(3,1^3)} < H^{(3,2,1)},$$
$$H^{(3,1^3)} < K^{(3,2,1)},\ \bar{H}^{(3,1^3)} < K^{(3,2,1)},\ K^{(3,1^3)} < K^{(3,2,1)},\ \bar{K}^{(3,1^3)} < K^{(3,2,1)},$$
$$L^{(3,1^3)} < L^{(3,2,1)},\ \bar{L}^{(3,1^3)} < L^{(3,2,1)},\ M^{(3,1^3)} < L^{(3,2,1)},\ \bar{M}^{(3,1^3)} < L^{(3,2,1)},$$
$$L^{(3,1^3)} < M^{(3,2,1)},\ \bar{L}^{(3,1^3)} < M^{(3,2,1)},\ P^{(3,1^3)} < M^{(3,2,1)},\ \bar{P}^{(3,1^3)} < M^{(3,2,1)},$$
$$M^{(3,1^3)} < P^{(3,2,1)},\ \bar{M}^{(3,1^3)} < P^{(3,2,1)},\ P^{(3,1^3)} < P^{(3,2,1)},\ \bar{P}^{(3,1^3)} < P^{(3,2,1)},$$

because

$$R_{2,5}R_{3,6}A^{(3,1^3)} = R_{2,5}\bar{A}^{(3,1^3)} = A^{(3,2,1)},$$
$$R_{2,5}R_{3,6}B^{(3,1^3)} = R_{2,5}\bar{B}^{(3,1^3)} = R_{2,3}R_{3,6}C^{(3,1^3)} = R_{2,3}\bar{C}^{(3,1^3)} = B^{(3,2,1)},$$
$$R_{2,6}B^{(3,1^3)} = R_{2,6}R_{3,5}\bar{B}^{(3,1^3)} = R_{2,3}R_{3,5}E^{(3,1^3)} = R_{2,3}\bar{E}^{(3,1^3)} = C^{(3,2,1)},$$
$$R_{2,6}C^{(3,1^3)} = R_{2,6}R_{3,3}\bar{C}^{(3,1^3)} = R_{2,5}E^{(3,1^3)} = R_{2,5}R_{3,3}\bar{E}^{(3,1^3)} = E^{(3,2,1)},$$
$$R_{2,4}R_{3,6}F^{(3,1^3)} = R_{2,4}\bar{F}^{(3,1^3)} = R_{2,3}R_{3,6}H^{(3,1^3)} = R_{2,3}\bar{H}^{(3,1^3)} = F^{(3,2,1)},$$
$$R_{2,6}F^{(3,1^3)} = R_{2,6}R_{3,4}\bar{F}^{(3,1^3)} = R_{2,3}R_{3,4}K^{(3,1^3)} = R_{2,3}\bar{K}^{(3,1^3)} = H^{(3,2,1)},$$
$$R_{2,6}H^{(3,1^3)} = R_{2,6}R_{3,3}\bar{H}^{(3,1^3)} = R_{2,4}K^{(3,1^3)} = R_{2,4}R_{3,3}\bar{K}^{(3,1^3)} = K^{(3,2,1)},$$
$$R_{2,4}R_{3,5}L^{(3,1^3)} = R_{2,4}\bar{L}^{(3,1^3)} = R_{2,3}R_{3,5}M^{(3,1^3)} = R_{2,3}\bar{M}^{(3,1^3)} = L^{(3,2,1)},$$
$$R_{2,5}L^{(3,1^3)} = R_{2,5}R_{3,4}\bar{L}^{(3,1^3)} = R_{2,3}R_{3,4}P^{(3,1^3)} = R_{2,3}\bar{P}^{(3,1^3)} = M^{(3,2,1)},$$
$$R_{2,5}M^{(3,1^3)} = R_{2,5}R_{3,3}\bar{M}^{(3,1^3)} = R_{2,4}P^{(3,1^3)} = R_{2,4}R_{3,3}\bar{P}^{(3,1^3)} = P^{(3,2,1)}.$$

Thus, all substitution reactions among $(3,1^3)$-derivatives and $(3,2,1)$-derivatives of cyclopropane are designated by the following inequalities:

$$a_{(3,1^3)} < a_{(3,2,1)},\ \bar{a}_{(3,1^3)} < a_{(3,2,1)},$$

$$b_{(3,1^3)} < b_{(3,2,1)},\ \bar{b}_{(3,1^3)} < b_{(3,2,1)},\ c_{(3,1^3)} < b_{(3,2,1)},\ \bar{c}_{(3,1^3)} < b_{(3,2,1)},$$
$$b_{(3,1^3)} < c_{(3,2,1)},\ \bar{b}_{(3,1^3)} < c_{(3,2,1)},\ e_{(3,1^3)} < c_{(3,2,1)},\ \bar{e}_{(3,1^3)} < c_{(3,2,1)},$$
$$c_{(3,1^3)} < e_{(3,2,1)},\ \bar{c}_{(3,1^3)} < c_{(3,2,1)},\ e_{(3,1^3)} < e_{(3,2,1)},\ \bar{e}_{(3,1^3)} < e_{(3,2,1)},$$

$$f_{(3,1^3)} < f_{(3,2,1)},\ \bar{f}_{(3,1^3)} < f_{(3,2,1)},\ h_{(3,1^3)} < f_{(3,2,1)},\ \bar{h}_{(3,1^3)} < f_{(3,2,1)},$$
$$f_{(3,1^3)} < h_{(3,2,1)},\ \bar{f}_{(3,1^3)} < h_{(3,2,1)},\ k_{(3,1^3)} < h_{(3,2,1)},\ \bar{k}_{(3,1^3)} < h_{(3,2,1)},$$
$$h_{(3,1^3)} < k_{(3,2,1)},\ \bar{h}_{(3,1^3)} < k_{(3,2,1)},\ k_{(3,1^3)} < k_{(3,2,1)},\ \bar{k}_{(3,1^3)} < k_{(3,2,1)},$$
$$\ell_{(3,1^3)} < \ell_{(3,2,1)},\ \bar{\ell}_{(3,1^3)} < \ell_{(3,2,1)},\ m_{(3,1^3)} < \ell_{(3,2,1)},\ \bar{m}_{(3,1^3)} < \ell_{(3,2,1)},$$
$$\ell_{(3,1^3)} < m_{(3,2,1)},\ \bar{\ell}_{(3,1^3)} < m_{(3,2,1)},\ p_{(3,1^3)} < m_{(3,2,1)},\ \bar{p}_{(3,1^3)} < m_{(3,2,1)},$$
$$m_{(3,1^3)} < p_{(3,2,1)},\ \bar{m}_{(3,1^3)} < p_{(3,2,1)},\ p_{(3,1^3)} < p_{(3,2,1)},\ \bar{p}_{(3,1^3)} < p_{(3,2,1)}.$$

We have

$$T_{(3,1^3);G'} =$$
$$\{a_{(3,1^3)} \cup \bar{a}_{(3,1^3)}, b_{(3,1^3)} \cup f_{(3,1^3)}, \bar{b}_{(3,1^3)} \cup \bar{f}_{(3,1^3)}, c_{(3,1^3)} \cup h_{(3,1^3)}, \bar{c}_{(3,1^3)} \cup \bar{h}_{(3,1^3)},$$
$$e_{(3,1^3)} \cup k_{(3,1^3)}, \bar{e}_{(3,1^3)} \cup \bar{k}_{(3,1^3)}, \ell_{(3,1^3)} \cup \bar{\ell}_{(3,1^3)}, m_{(3,1^3)} \cup p_{(3,1^3)}, \bar{m}_{(3,1^3)} \cup \bar{p}_{(3,1^3)}\}.$$

Therefore the members of any one of the following two-element sets are chiral pairs:

$$\{a_{(3,1^3)}, \bar{a}_{(3,1^3)}\},\ \{b_{(3,1^3)}, f_{(3,1^3)}\},\ \{\bar{b}_{(3,1^3)}, \bar{f}_{(3,1^3)}\}, \{\bar{e}_{(3,1^3)}, \bar{k}_{(3,1^3)}\},\ \{c_{(3,1^3)}, h_{(3,1^3)}\},$$
$$\{\bar{c}_{(3,1^3)}, \bar{h}_{(3,1^3)}\},\ \{e_{(3,1^3)}, k_{(3,1^3)}\},\ \{\bar{e}_{(3,1^3)}, \bar{k}_{(3,1^3)}\},\ \{\ell_{(3,1^3)}, \bar{\ell}_{(3,1^3)}\},$$
$$\{m_{(3,1^3)}, p_{(3,1^3)}\},\ \{\bar{m}_{(3,1^3)}, \bar{p}_{(3,1^3)}\}.$$

Further,

$$T_{(3,1^3);G''} =$$
$$\{(a_{(3,1^3)} \cup \bar{a}_{(3,1^3)}) \cup (\ell_{(3,1^3)} \cup \bar{\ell}_{(3,1^3)}) \cup (m_{(3,1^3)} \cup p_{(3,1^3)}) \cup (\bar{m}_{(3,1^3)} \cup \bar{p}_{(3,1^3)}),$$
$$(b_{(3,1^3)} \cup f_{(3,1^3)}) \cup (\bar{e}_{(3,1^3)} \cup \bar{k}_{(3,1^3)}), (\bar{b}_{(3,1^3)} \cup \bar{f}_{(3,1^3)}) \cup (e_{(3,1^3)} \cup k_{(3,1^3)}),$$
$$(c_{(3,1^3)} \cup h_{(3,1^3)}) \cup (\bar{c}_{(3,1^3)} \cup \bar{h}_{(3,1^3)})\}.$$

Thus, the members of any one set from the list below are structurally identical as long as the members of different sets are structural isomers:

$$\{a_{(3,1^3)}, \bar{a}_{(3,1^3)}, \ell_{(3,1^3)}, \bar{\ell}_{(3,1^3)}, m_{(3,1^3)}, p_{(3,1^3)}, \bar{m}_{(3,1^3)}, \bar{p}_{(3,1^3)}\},$$
$$\{b_{(3,1^3)}, f_{(3,1^3)}, \bar{e}_{(3,1^3)}, \bar{k}_{(3,1^3)}\},\ \{\bar{b}_{(3,1^3)}, \bar{f}_{(3,1^3)}, e_{(3,1^3)}, k_{(3,1^3)}\},$$
$$\{c_{(3,1^3)}, h_{(3,1^3)}, \bar{c}_{(3,1^3)}, \bar{h}_{(3,1^3)}\}.$$

Case 8. $\lambda = (2^3)$.
We have

$$T_{(2^3);G} = \{a_{(2^3)}, \bar{a}_{(2^3)}, b_{(2^3)}, \bar{b}_{(2^3)}, c_{(2^3)}, \bar{c}_{(2^3)}, e_{(2^3)}, \bar{e}_{(2^3)}, f_{(2^3)}, \bar{f}_{(2^3)},$$
$$h_{(2^3)}, \bar{h}_{(2^3)}, k_{(2^3)}, \bar{k}_{(2^3)}, \ell_{(2^3)}, \bar{\ell}_{(2^3)}, m_{(2^3)}, \bar{m}_{(2^3)}\},$$

where:
$a_{(2^3)}$ is the G-orbit of the tabloid $A^{(2^3)} = (\{1,2\},\{3,4\},\{5,6\})$,
$\bar{a}_{(2^3)}$ is the G-orbit of the tabloid $\bar{A}^{(2^3)} = (\{1,2\},\{3,5\},\{4,6\})$,

$b_{(2^3)}$ is the G-orbit of the tabloid $B^{(2^3)} = (\{1,2\},\{3,6\},\{4,5\})$,
$\bar{b}_{(2^3)}$ is the G-orbit of the tabloid $\bar{B}^{(2^3)} = (\{1,2\},\{4,5\},\{3,6\})$,
$c_{(2^3)}$ is the G-orbit of the tabloid $C^{(2^3)} = (\{1,2\},\{4,6\},\{3,5\})$,
$\bar{c}_{(2^3)}$ is the G-orbit of the tabloid $\bar{C}^{(2^3)} = (\{1,2\},\{5,6\},\{3,4\})$,
$e_{(2^3)}$ is the G-orbit of the tabloid $E^{(2^3)} = (\{1,4\},\{2,3\},\{5,6\})$,
$\bar{e}_{(2^3)}$ is the G-orbit of the tabloid $\bar{E}^{(2^3)} = (\{1,4\},\{2,5\},\{3,6\})$,
$f_{(2^3)}$ is the G-orbit of the tabloid $F^{(2^3)} = (\{1,4\},\{2,3\},\{5,6\})$,
$\bar{f}_{(2^3)}$ is the G-orbit of the tabloid $\bar{F}^{(2^3)} = (\{1,4\},\{3,5\},\{2,6\})$;,
$h_{(2^3)}$ is the G-orbit of the tabloid $H^{(2^3)} = (\{1,5\},\{2,3\},\{4,6\})$,
$\bar{h}_{(2^3)}$ is the G-orbit of the tabloid $\bar{H}^{(2^3)} = (\{1,5\},\{2,4\},\{3,6\})$,
$k_{(2^3)}$ is the G-orbit of the tabloid $K^{(2^3)} = (\{1,5\},\{2,6\},\{3,4\})$,
$\bar{k}_{(2^3)}$ is the G-orbit of the tabloid $\bar{K}^{(2^3)} = (\{1,5\},\{3,6\},\{2,4\})$,
$\ell_{(2^3)}$ is the G-orbit of the tabloid $L^{(2^3)} = (\{1,6\},\{2,3\},\{4,5\})$,
$\bar{\ell}_{(2^3)}$ is the G-orbit of the tabloid $\bar{L}^{(2^3)} = (\{1,6\},\{2,4\},\{3,5\})$,
$m_{(2^3)}$ is the G-orbit of the tabloid $M^{(2^3)} = (\{1,6\},\{2,5\},\{3,4\})$,
$\bar{m}_{(2^3)}$ is the G-orbit of the tabloid $\bar{M}^{(2^3)} = (\{1,6\},\{3,4\},\{2,5\})$.
All inequalities between the tabloids of shape (2^3) and those of shape $(3,2,1)$ are as follows:

$$A^{(2^3)} < A^{(3,2,1)},\ \bar{A}^{(2^3)} < A^{(3,2,1)},\ (123)(456)B^{(2^3)} < A^{(3,2,1)},$$

$$\bar{B}^{(2^3)} < A^{(3,2,1)},\ (123)(456)C^{(2^3)} < A^{(3,2,1)},\ (132)(465)\bar{C}^{(2^3)} < A^{(3,2,1)},$$

$$A^{(2^3)} < B^{(3,2,1)},\ \bar{A}^{(2^3)} < B^{(3,2,1)},\ \bar{B}^{(2^3)} < B^{(3,2,1)},$$

$$E^{(2^3)} < B^{(3,2,1)},\ \bar{E}^{(2^3)} < B^{(3,2,1)},\ \bar{F}^{(2^3)} < B^{(3,2,1)},$$

$$(123)(456)L^{(2^3)} < B^{(3,2,1)},\ (123)(456)\bar{L}^{(2^3)} < B^{(3,2,1)},\ (123)(456)\bar{M}^{(2^3)} < B^{(3,2,1)},$$

$$A^{(2^3)} < C^{(3,2,1)},\ B^{(2^3)} < C^{(3,2,1)},\ C^{(2^3)} < C^{(3,2,1)},$$

$$E^{(2^3)} < C^{(3,2,1)},\ (14)(26)(35)\bar{E}^{(2^3)} < C^{(3,2,1)},\ F^{(2^3)} < C^{(3,2,1)},$$

$$(123)(456)L^{(2^3)} < C^{(3,2,1)},\ (14)(26)(35)\bar{L}^{(2^3)} < C^{(3,2,1)},\ (123)(456)M^{(2^3)} < C^{(3,2,1)},$$

$$\bar{B}^{(2^3)} < E^{(3,2,1)},\ C^{(2^3)} < E^{(3,2,1)},\ \bar{C}^{(2^3)} < E^{(3,2,1)},$$

$$(14)(26)(35)E^{(2^3)} < E^{(3,2,1)},\ \bar{E}^{(2^3)} < E^{(3,2,1)},\ F^{(2^3)} < E^{(3,2,1)},$$

$$(14)(26)(35)L^{(2^3)} < E^{(3,2,1)},\ (14)(26)(35)\bar{L}^{(2^3)} < E^{(3,2,1)},\ (123)(456)\bar{M}^{(2^3)} < E^{(3,2,1)},$$

$$A^{(2^3)} < F^{(3,2,1)},\ \bar{A}^{(2^3)} < F^{(3,2,1)},\ \bar{B}^{(2^3)} < F^{(3,2,1)},$$

$$(123)(456)E^{(2^3)} < F^{(3,2,1)},\ (15)(24)(36)\bar{E}^{(2^3)} < F^{(3,2,1)},\ (123)(456)F^{(2^3)} < F^{(3,2,1)},$$

$$H^{(2^3)} < F^{(3,2,1)}, \bar{H}^{(2^3)} < F^{(3,2,1)},\ (15)(24)(36)K^{(2^3)} < F^{(3,2,1)},$$

$$\bar{A}^{(2^3)} < H^{(3,2,1)},\ B^{(2^3)} < H^{(3,2,1)},\ \bar{C}^{(2^3)} < H^{(3,2,1)},\ (123)(456)E^{(2^3)} < H^{(3,2,1)},$$

$$(123)(456)\bar{E}^{(2^3)} < H^{(3,2,1)},\ (123)(456)\bar{F}^{(2^3)} < H^{(3,2,1)},$$
$$H^{(2^3)} < H^{(3,2,1)},\ K^{(2^3)} < H^{(3,2,1)},\ \bar{K}^{(2^3)} < H^{(3,2,1)},$$
$$\bar{B}^{(2^3)} < K^{(3,2,1)},\ C^{(2^3)} < K^{(3,2,1)},\ \bar{C}^{(2^3)} < K^{(3,2,1)},\ (15)(24)(36)E^{(2^3)} < K^{(3,2,1)},$$
$$(15)(24)(36)\bar{E}^{(2^3)} < K^{(3,2,1)},\ (123)(456)\bar{F}^{(2^3)} < K^{(3,2,1)},$$
$$(15)(24)(36)H^{(2^3)} < K^{(3,2,1)},\ \bar{H}^{(2^3)} < K^{(3,2,1)},\ K^{(2^3)} < K^{(3,2,1)},$$
$$A^{(2^3)} < L^{(3,2,1)},\ B^{(2^3)} < L^{(3,2,1)},\ C^{(2^3)} < L^{(3,2,1)},\ (123)(456)H^{(2^3)} < L^{(3,2,1)},$$
$$(123)(456)K^{(2^3)} < L^{(3,2,1)},\ (123)(456)\bar{K}^{(2^3)} < L^{(3,2,1)},$$
$$L^{(2^3)} < L^{(3,2,1)},\ \bar{L}^{(2^3)} < L^{(3,2,1)},\ \bar{M}^{(2^3)} < L^{(3,2,1)},$$
$$\bar{A}^{(2^3)} < M^{(3,2,1)},\ B^{(2^3)} < M^{(3,2,1)},\ \bar{C}^{(2^3)} < M^{(3,2,1)},\ (123)(456)H^{(2^3)} < M^{(3,2,1)},$$
$$(123)(456)\bar{H}^{(2^3)} < M^{(3,2,1)},\ (16)(25)(34)K^{(2^3)} < M^{(3,2,1)},$$
$$L^{(2^3)} < M^{(3,2,1)},\ (16)(25)(34)\bar{L}^{(2^3)} < M^{(3,2,1)},\ M^{(2^3)} < M^{(3,2,1)},$$
$$\bar{B}^{(2^3)} < P^{(3,2,1)},\ C^{(2^3)} < P^{(3,2,1)},\ \bar{C}^{(2^3)} < P^{(3,2,1)},\ (16)(25)(34)H^{(2^3)} < P^{(3,2,1)},$$
$$(16)(25)(34)K^{(2^3)} < P^{(3,2,1)},\ (123)(456)\bar{K}^{(2^3)} < P^{(3,2,1)},$$
$$(16)(25)(34)L^{(2^3)} < P^{(3,2,1)},\ \bar{L}^{(2^3)} < P^{(3,2,1)},\ M^{(2^3)} < P^{(3,2,1)},$$

because

$$R_{1,3}R_{2,5}A^{(2^3)} = R_{1,3}R_{2,4}\bar{A}^{(2^3)} = R_{1,1}R_{2,5}(123)(456)B^{(2^3)} =$$
$$R_{1,3}\bar{B}^{(2^3)} = R_{1,1}(123)(456)C^{(2^3)} = R_{1,2}(132)(465)\bar{C}^{(2^3)} = A^{(3,2,1)},$$
$$R_{1,4}R_{2,5}A^{(2^3)} = R_{1,4}\bar{A}^{(2^3)} = R_{1,4}R_{2,3}\bar{B}^{(2^3)} =$$
$$R_{1,2}R_{2,5}E^{(2^3)} = R_{1,2}R_{2,3}\bar{E}^{(2^3)} = R_{1,2}\bar{F}^{(2^3)} =$$
$$R_{1,1}R_{2,5}(123)(456)L^{(2^3)} = R_{1,1}(123)(456)\bar{L}^{(2^3)} = R_{1,1}R_{2,3}(123)(456)\bar{M}^{(2^3)} = B^{(3,2,1)},$$
$$R_{1,4}R_{2,6}A^{(2^3)} = R_{1,4}B^{(2^3)} = R_{1,4}R_{2,3}C^{(2^3)} =$$
$$R_{1,2}R_{2,6}E^{(2^3)} = R_{1,2}(14)(26)(35)\bar{E}^{(2^3)} = R_{1,2}R_{2,3}F^{(2^3)} =$$
$$R_{1,1}R_{2,6}(123)(456)L^{(2^3)} = R_{1,1}R_{2,3}(14)(26)(35)\bar{L}^{(2^3)} =$$
$$R_{1,1}(123)(456)M^{(2^3)} = C^{(3,2,1)},$$
$$R_{1,4}R_{2,6}\bar{B}^{(2^3)} = R_{1,4}R_{2,5}C^{(2^3)} = R_{1,4}\bar{C}^{(2^3)} =$$
$$R_{1,2}(14)(26)(35)E^{(2^3)} = R_{1,2}R_{2,6}\bar{E}^{(2^3)} = R_{1,2}R_{2,5}F^{(2^3)} =$$
$$R_{1,1}(14)(26)(35)L^{(2^3)} = R_{1,1}R_{2,5}(14)(26)(35)\bar{L}^{(2^3)} =$$
$$R_{1,1}R_{2,6}(123)(456)\bar{M}^{(2^3)} = E^{(3,2,1)},$$

$$R_{1,5}A^{(2^3)} = R_{1,5}R_{2,4}\bar{A}^{(2^3)} = R_{1,5}R_{2,3}\bar{B}^{(2^3)} =$$
$$R_{1,1}R_{2,4}(123)(456)E^{(2^3)} = R_{1,1}R_{2,3}(15)(24)(36)\bar{E}^{(2^3)} = R_{1,1}(123)(456)F^{(2^3)} =$$
$$R_{1,2}R_{2,4}H^{(2^3)} = R_{1,2}R_{2,3}\bar{H}^{(2^3)} = R_{1,2}(15)(24)(36)K^{(2^3)} = F^{(3,2,1)},$$
$$R_{1,5}R_{2,6}\bar{A}^{(2^3)} = R_{1,5}B^{(2^3)} = R_{1,5}R_{2,3}\bar{C}^{(2^3)} =$$
$$R_{1,1}R_{2,6}(123)(456)E^{(2^3)} = R_{1,1}(123)(456)\bar{E}^{(2^3)} = R_{1,1}R_{2,3}(123)(456)\bar{F}^{(2^3)} =$$
$$R_{1,2}R_{2,6}H^{(2^3)} = R_{1,2}R_{2,3}K^{(2^3)} = R_{1,2}\bar{K}^{(2^3)} = H^{(3,2,1)},$$
$$R_{1,5}R_{2,6}\bar{B}^{(2^3)} = R_{1,5}C^{(2^3)} = R_{1,5}R_{2,4}\bar{C}^{(2^3)} =$$
$$R_{1,1}(15)(24)(36)E^{(2^3)} = R_{1,1}R_{2,6}(15)(24)(36)\bar{E}^{(2^3)} = R_{1,1}R_{2,4}(123)(456)\bar{F}^{(2^3)} =$$
$$R_{1,2}(15)(24)(36)H^{(2^3)} = R_{1,2}R_{2,6}\bar{H}^{(2^3)} = R_{1,2}R_{2,4}K^{(2^3)} = K^{(3,2,1)},$$
$$R_{1,6}A^{(2^3)} = R_{1,6}R_{2,4}B^{(2^3)} = R_{1,6}R_{2,3}C^{(2^3)} =$$
$$R_{1,1}R_{2,4}(123)(456)H^{(2^3)} = R_{1,1}(123)(456)K^{(2^3)} = R_{1,1}R_{2,3}(123)(456)\bar{K}^{(2^3)} =$$
$$R_{1,2}R_{2,4}L^{(2^3)} = R_{1,2}R_{2,3}\bar{L}^{(2^3)} = R_{1,2}\bar{M}^{(2^3)} = L^{(3,2,1)},$$
$$R_{1,6}\bar{A}^{(2^3)} = R_{1,6}R_{2,5}B^{(2^3)} = R_{1,6}R_{2,3}\bar{C}^{(2^3)} =$$
$$R_{1,1}R_{2,5}(123)(456)H^{(2^3)} = R_{1,1}(123)(456)\bar{H}^{(2^3)} = R_{1,1}R_{2,3}(16)(25)(34)K^{(2^3)} =$$
$$R_{1,2}R_{2,5}L^{(2^3)} = R_{1,2}(16)(25)(34)\bar{L}^{(2^3)} = R_{1,2}R_{2,3}M^{(2^3)} = M^{(3,2,1)},$$
$$R_{1,6}\bar{B}^{(2^3)} = R_{1,6}R_{2,5}C^{(2^3)} = R_{1,6}R_{2,4}\bar{C}^{(2^3)} =$$
$$R_{1,1}(16)(25)(34)H^{(2^3)} = R_{1,1}R_{2,4}(16)(25)(34)K^{(2^3)} = R_{1,1}R_{2,5}(123)(456)\bar{K}^{(2^3)} =$$
$$R_{1,2}(16)(25)(34)L^{(2^3)} = R_{1,2}R_{2,5}\bar{L}^{(2^3)} = R_{1,2}R_{2,4}M^{(2^3)} = P^{(3,2,1)}.$$

Therefore the substitution reactions among (2^3)-products and $(3,2,1)$-products of cyclopropane are as follows:

$$a_{(2^3)} < a_{(3,2,1)},\ \bar{a}_{(2^3)} < a_{(3,2,1)},\ b_{(2^3)} < a_{(3,2,1)},$$
$$\bar{b}_{(2^3)} < a_{(3,2,1)},\ c_{(2^3)} < a_{(3,2,1)},\ \bar{c}_{(2^3)} < a_{(3,2,1)},$$
$$a_{(2^3)} < b_{(3,2,1)},\ \bar{a}_{(2^3)} < b_{(3,2,1)},\ \bar{b}_{(2^3)} < b_{(3,2,1)},$$
$$e_{(2^3)} < b_{(3,2,1)},\ \bar{e}_{(2^3)} < b_{(3,2,1)},\ \bar{f}_{(2^3)} < b_{(3,2,1)},$$
$$\ell_{(2^3)} < b_{(3,2,1)},\ \bar{\ell}_{(2^3)} < b_{(3,2,1)},\ \bar{m}_{(2^3)} < b_{(3,2,1)},$$
$$a_{(2^3)} < c_{(3,2,1)},\ b_{(2^3)} < c_{(3,2,1)},\ c_{(2^3)} < c_{(3,2,1)},$$
$$e_{(2^3)} < c_{(3,2,1)},\ \bar{e}_{(2^3)} < c_{(3,2,1)},\ f_{(2^3)} < c_{(3,2,1)},$$
$$\ell_{(2^3)} < c_{(3,2,1)},\ \bar{\ell}_{(2^3)} < c_{(3,2,1)},\ m_{(2^3)} < c_{(3,2,1)},$$

$$\bar{b}_{(2^3)} < e_{(3,2,1)},\ c_{(2^3)} < e_{(3,2,1)},\ \bar{c}_{(2^3)} < e_{(3,2,1)},$$
$$e_{(2^3)} < e_{(3,2,1)},\ \bar{e}_{(2^3)} < e_{(3,2,1)},\ f_{(2^3)} < e_{(3,2,1)},$$
$$\ell_{(2^3)} < e_{(3,2,1)},\ \bar{\ell}_{(2^3)} < e_{(3,2,1)},\ \bar{m}_{(2^3)} < e_{(3,2,1)},$$
$$a_{(2^3)} < f_{(3,2,1)},\ \bar{a}_{(2^3)} < f_{(3,2,1)},\ \bar{b}_{(2^3)} < f_{(3,2,1)},$$
$$e_{(2^3)} < f_{(3,2,1)},\ \bar{e}_{(2^3)} < f_{(3,2,1)},\ f_{(2^3)} < f_{(3,2,1)},$$
$$h_{(2^3)} < f_{(3,2,1)}, \bar{h}_{(2^3)} < f_{(3,2,1)},\ k_{(2^3)} < f_{(3,2,1)},$$
$$\bar{a}_{(2^3)} < h_{(3,2,1)},\ b_{(2^3)} < h_{(3,2,1)},\ \bar{c}_{(2^3)} < h_{(3,2,1)},$$
$$e_{(2^3)} < h_{(3,2,1)},\ \bar{e}_{(2^3)} < h_{(3,2,1)},\ \bar{f}_{(2^3)} < h_{(3,2,1)},$$
$$h_{(2^3)} < h_{(3,2,1)},\ k_{(2^3)} < h_{(3,2,1)},\ \bar{k}_{(2^3)} < h_{(3,2,1)},$$
$$\bar{b}_{(2^3)} < k_{(3,2,1)},\ c_{(2^3)} < k_{(3,2,1)},\ \bar{c}_{(2^3)} < k_{(3,2,1)},$$
$$e_{(2^3)} < k_{(3,2,1)},\ \bar{e}_{(2^3)} < k_{(3,2,1)},\ \bar{f}_{(2^3)} < k_{(3,2,1)},$$
$$h_{(2^3)} < k_{(3,2,1)},\ \bar{h}_{(2^3)} < k_{(3,2,1)},\ k_{(2^3)} < k_{(3,2,1)},$$
$$a_{(2^3)} < \ell_{(3,2,1)},\ b_{(2^3)} < \ell_{(3,2,1)},\ c_{(2^3)} < \ell_{(3,2,1)},$$
$$h_{(2^3)} < \ell_{(3,2,1)},\ k_{(2^3)} < \ell_{(3,2,1)},\ \bar{k}_{(2^3)} < \ell_{(3,2,1)},$$
$$\ell_{(2^3)} < \ell_{(3,2,1)},\ \bar{\ell}_{(2^3)} < \ell_{(3,2,1)},\ \bar{m}_{(2^3)} < \ell_{(3,2,1)},$$
$$\bar{a}_{(2^3)} < m_{(3,2,1)},\ b_{(2^3)} < m_{(3,2,1)},\ \bar{c}_{(2^3)} < m_{(3,2,1)},$$
$$h_{(2^3)} < m_{(3,2,1)},\ \bar{h}_{(2^3)} < m_{(3,2,1)},\ k_{(2^3)} < m_{(3,2,1)},$$
$$\ell_{(2^3)} < m_{(3,2,1)},\ \bar{\ell}_{(2^3)} < m_{(3,2,1)},\ m_{(2^3)} < m_{(3,2,1)},$$
$$\bar{b}_{(2^3)} < p_{(3,2,1)},\ c_{(2^3)} < p_{(3,2,1)},\ \bar{c}_{(2^3)} < p_{(3,2,1)},$$
$$h_{(2^3)} < p_{(3,2,1)},\ k_{(2^3)} < p_{(3,2,1)},\ \bar{k}_{(2^3)} < p_{(3,2,1)},$$
$$\ell_{(2^3)} < p_{(3,2,1)},\ \bar{\ell}_{(2^3)} < p_{(3,2,1)},\ m_{(2^3)} < p_{(3,2,1)}.$$

The set of all G'-orbits is

$$T_{(2^3);G'} =$$
$$\{a_{(2^3)} \cup \bar{a}_{(2^3)}, b_{(2^3)}, \bar{b}_{(2^3)}, c_{(2^3)} \cup \bar{c}_{(2^3)}, e_{(2^3)}, \bar{e}_{(2^3)}, f_{(2^3)} \cup \bar{f}_{(2^3)},$$
$$h_{(2^3)} \cup \ell_{(2^3)}, \bar{h}_{(2^3)} \cup \bar{m}_{(2^3)}, k_{(2^3)} \cup \bar{\ell}_{(2^3)}, \bar{k}_{(2^3)} \cup m_{(2^3)}\}.$$

Hence, the members of any one of the two-element sets

$$\{a_{(2^3)}, \bar{a}_{(2^3)}\},\ \{c_{(2^3)}, \bar{c}_{(2^3)}\},\ \{f_{(2^3)}, \bar{f}_{(2^3)}\},$$
$$\{h_{(2^3)}, \ell_{(2^3)}\},\ \{\bar{h}_{(2^3)}, \bar{m}_{(2^3)}\},\ \{k_{(2^3)}, \bar{\ell}_{(2^3)}\},\ \{\bar{k}_{(2^3)}, m_{(2^3)}\}$$

form a chiral pair and $b_{(2^3)}$, $\bar{b}_{(2^3)}$, $e_{(2^3)}$, $\bar{e}_{(2^3)}$, represent dimers.
Moreover, the set of all G''-orbits is

$$T_{(2^3);G''} =$$

$$\{(a_{(2^3)} \cup \bar{a}_{(2^3)}) \cup (c_{(2^3)} \cup \bar{c}_{(2^3)}) \cup (h_{(2^3)} \cup \ell_{(2^3)}) \cup (k_{(2^3)} \cup \bar{\ell}_{(2^3)}),$$
$$b_{(2^3)} \cup (\bar{k}_{(2^3)} \cup m_{(2^3)}), \bar{b}_{(2^3)} \cup (\bar{h}_{(2^3)} \cup \bar{m}_{(2^3)}), e_{(2^3)} \cup (f_{(2^3)} \cup \bar{f}_{(2^3)}), \bar{e}_{(2^3)}\}.$$

Thus, any one from the sets below gathers all formulae that represent structurally identical derivatives, so members of different sets represent structural isomers.

$$\{a_{(2^3)}, \bar{a}_{(2^3)}, c_{(2^3)}, \bar{c}_{(2^3)}, h_{(2^3)}, \ell_{(2^3)}, k_{(2^3)}, \bar{\ell}_{(2^3)}\}$$

$$\{b_{(2^3)}, \bar{k}_{(2^3)}, m_{(2^3)}\},\ \{\bar{b}_{(2^3)}, \bar{h}_{(2^3)}, \bar{m}_{(2^3)}\},\ \{e_{(2^3)}, f_{(2^3)}, \bar{f}_{(2^3)}\},\ \{\bar{e}_{(2^3)}\}.$$

3. Identification of the derivatives

Now, we will describe the Lunn-Senior's automorphism groups $Aut_0''(T_{D;G})$ for $D = D_k$, $k = 1, \ldots, 6$, where

$$D_1 = \{(6), (5,1), (4,2)\},\ D_2 = D_1 \cup \{(4,1^2)\},$$

$$D_3 = D_2 \cup \{(3^2)\},\ D_4 = D_3 \cup \{(3,2,1)\},$$
$$D_5 = D_4 \cup \{(3,1^3)\},\ D_6 = D_5 \cup \{(2^3)\}.$$

The elements of the $Aut_0''(T_{D;G})$-orbits in the set $T_{D;G}$ will represent the products of cyclopropane that can not be distinguished via substitution reactions among the elements of $T_{D;G}$.
We remind that we will use without referring all terminology and notation from [2], especially those from the beginning of section 3. For convenience of the reader, we state explicitly once again the conditions of the main [2, Lemma 3.1], as well as the assumptions and notation introduced before [2, Lemmas 3.3 - 3.7]. We note that the correct version of [2, Corollary 3.2] can be found in [3].
Let $U, V, \bar{V} \subset T_{d;G}$ be unions of G''-orbits, such that $U \subset V$, $V \backslash U \subset \bar{V}$, and the difference $\bar{V} \backslash U$ consists of minimal elements of the partially ordered set $V \cup \bar{V}$. Assume that $\bar{V}$ is a barrier of $V \backslash U$ in V, and the automorphism group $Aut_0''(U)$ is a commutative 2-group. Set $H = \{\beta \in Aut_0''(U) \mid \beta(C_>(\bar{V}; a)) = C_>(\bar{V}; a), a \in V \backslash U\}$. Moreover, for any pair $X, V \subset T_{d;G}$ of sets that are unions of G''-orbits with $X \subset V$ we denote by $I_{V,X}$ the image of the restriction homomorphism

$$\varrho_{V,X} \colon Aut_0''(V) \to Aut_0''(X).$$

Lemma 3.1. *Let the difference $V \backslash U$ be a G''-orbit that consists of several chiral pairs $\{A, A^1\}$, $\{B, B^1\}, \ldots$, and eventually, of several dimers. Suppose that: $C_>(\bar{V}; A) = C_>(\bar{V}; A^1) = P$, $C_>(\bar{V}; B) = C_>(\bar{V}; B^1) = Q, \ldots$, the cones P, $Q, \ldots$ are pairwise different, the cones of the dimers are pairwise different, and that*

$$I_{V,U} = H.$$

Then there exists a decomposition

$$Aut_0''(V) = H \times \langle s \rangle \times \langle t \rangle \times \cdots,$$

where $s = (A, A^1)$, $t = (B, B^1), \ldots$, *the restriction homomorphism* $\varrho_{V,U}$ *has kernel* $\langle s \rangle \times \langle t \rangle \times \cdots$, *and* $Aut''_0(V)$ *is a commutative* 2*-group.*

PROOF: [3, Corollary 3.2, (i)] yields $H \cup Hs \cup Ht \cup \ldots \subset Aut''_0(V)$. Now, let $\alpha \in Aut''_0(V)$. Then $\varrho_{V,U}(\alpha) \in H$, and according to [2, Lemma 3.1, (iv)], and to the fact that α maps any chiral pair onto a chiral pair, we have $\alpha(\{A, A^1\}) = \{A, A^1\}$, $\alpha(\{B, B^1\}) = \{B, B^1\}, \ldots$, and conclude that α leaves all dimers invariant. Therefore $\alpha \in H \cup Hs \cup Ht \cup \ldots \cup Hst \cup \ldots$. Now, we note that every pair among the automorphisms s, $t, \ldots$ commute, $s^2 = id$, $t^2 = id, \ldots$, and each one of them commutes with the elements of H. Hence the subgroup $\langle H, s, t, \ldots \rangle$ of $Aut''_0(V)$ is a commutative 2-group. In particular, $\langle H, s, t, \ldots \rangle = H \cup Hs \cup Ht \cup \ldots \cup Hst \cup \ldots$, and the proof is finished.

LEMMA 3.2. *Let the difference* $V \backslash U$ *be a* G''*-orbit that consists of two chiral pairs* $\{A, A^1\}$, $\{B, B^1\}$. *Suppose that* $C_>(\bar{V}; A) = C_>(\bar{V}; A^1) = P$, *there exists a decomposition*

$$I_{V,U} = H \times \langle w \rangle,$$

where $w(P) = P$ *and* $w(Q) = Q^1$, *for* $Q = C_>(\bar{V}; B)$, $Q^1 = C_>(\bar{V}; B^1)$. *Then there exists a decomposition*

$$Aut''_0(V) = H \times \langle s \rangle \times \langle wt \rangle,$$

where $s = (A, A^1)$, $t = (B, B^1)$, *the restriction homomorphism* $\varrho_{V,U}$ *has kernel* $\langle s \rangle$, *and* $Aut''_0(V)$ *is a commutative* 2*-group.*

PROOF: Since $w \notin H$, we obtain that the cones Q, Q_1, P, are pairwise different, and then [3, Corollary 3.2, (i)] yields $H \cup Hs \cup Hwt \cup Hwst \subset Aut''_0(V)$. Now, let $\alpha \in Aut''_0(V)$ with $\varrho_{V,U}(\alpha) \in H$ (respectively, $\varrho_{V,U}(\alpha) \in Hw$). Then in accordance with [2, Lemma 3.1, (iv)], $\alpha(\{A, A^1\}) = \{A, A^1\}$, $\alpha(B) = B$, $\alpha(B^1) = B^1$ (respectively, $\alpha(\{A, A^1\}) = \{A, A^1\}$, $\alpha(B) = B^1$, $\alpha(B^1) = B$), hence $\varrho_{V,V\backslash U}(\alpha) = id$, or $\varrho_{V,V\backslash U}(\alpha) = s$ (respectively, $\varrho_{V,V\backslash U}(\alpha) = t$, or $\varrho_{V,V\backslash U}(\alpha) = st$). Thus, we have $\alpha \in H \cup Hs$ (respectively, $\alpha \in Hwst \cup Hwt$). Since the automorphisms s and wt commute, $s^2 = id$, $(wt)^2 = id$, and each one of them commutes with the elements of H, then the subgroup $\langle H, s, wt \rangle$ of $Aut''_0(V)$ is a commutative 2-group. In particular, $\langle H, s, wt \rangle = H \cup Hs \cup Hwt \cup Hwst$, and the proof is done.

LEMMA 3.3. *Let the difference* $V \backslash U$ *be a* G''*-orbit that consists of a chiral pair* $\{A, A^1\}$ *and of several dimers. Suppose that the cones of the dimers are pairwise different,* $Aut''_0(U)$*-invariant, and that there exists a decomposition*

$$I_{V,U} = H \times \langle w \rangle$$

with $w(P) = P^1$, *where* $P = C_>(\bar{V}; A)$, $P^1 = C_>(\bar{V}; A^1)$. *Then there exists a decomposition*

$$Aut''_0(V) = H \times \langle ws \rangle,$$

where $s = (A, A^1)$, *the restriction homomorphism* $\varrho_{V,U}$ *is injective, and* $Aut''_0(V)$ *is a commutative* 2*-group.*

PROOF: Straightforward generalization of [2, Lemma 3.7].

LEMMA 3.4. *Let the difference $V \backslash U$ be a G''-orbit that consists of two types of chiral pairs: several chiral pairs $\{A, A^1\}$, $\{B, B^1\}, \ldots,$ with $C_>(\bar{V}; A) = C_>(\bar{V}; A^1) = P$, $C_>(\bar{V}; B) = C_>(\bar{V}; B^1) = Q, \ldots,$ two chiral pairs $\{C, C^1\}$, $\{E, E^1\}$, with $C_>(\bar{V}; C) = C_>(\bar{V}; E) = R$, $C_>(\bar{V}; C^1) = C_>(\bar{V}; E^1) = R^1$, and of several dimers. Let us suppose that the cones P, $Q, \ldots,$ are pairwise different, and the cones $\tilde{P}$, $\tilde{Q}, \ldots,$ of the dimers are pairwise different. If*

$$I_{V,U} = H \times \langle w \rangle,$$

where $w(P) = P$, $w(Q) = Q, \ldots,$ $w(\tilde{P}) = \tilde{P}$, $w(\tilde{Q}) = \tilde{Q}, \ldots,$ and $w(R) = R^1$, then there exists a decomposition

$$Aut_0''(V) = H \times \langle z \rangle \times \langle s \rangle \times \langle t \rangle \times \cdots \times \langle wx \rangle,$$

where $s = (A, A^1)$, $t = (B, B^1), \ldots,$ $z = (C, E)(C^1, E^1)$, $x = (C, C^1)(E, E^1)$, the restriction homomorphism $\varrho_{V,U}$ has kernel $\langle z \rangle \times \langle s \rangle \times \langle t \rangle \times \cdots$, and $Aut_0''(V)$ is a commutative 2-group.

PROOF: The relation $w \notin H$ implies $R \neq R^1$, and then each of the cones R and R^1 is different from any of the cones P, $Q, \ldots,$ $\tilde{P}$, $\tilde{Q}, \ldots.$ [3, Corollary 3.2, (i)] yields that H, Hs, $Ht, \ldots,$ Hz, Hwx are subsets of $Aut_0''(V)$. Now, let $\alpha \in Aut_0''(V)$, and let $\beta = \varrho_{V,U}(\alpha)$, $\alpha_0 = \varrho_{V,V\backslash U}(\alpha)$. Suppose that $\beta \in H$. In accordance with [3, Corollary 3.2, (i)] we obtain $C_>(\bar{V}; a) = C_>(\bar{V}; \alpha_0(a))$ for all $a \in V \backslash U$. Hence $\alpha_0(\{A, A^1\}) = \{A, A^1\}$, $\alpha_0(\{B, B^1\}) = \{B, B^1\}, \ldots,$ $\alpha_0(\{C, E\}) = \{C, E\}$, $\alpha_0(\{C^1, E^1\}) = \{C^1, E^1\}$, and α_0 leaves the dimers invariant. Moreover, α_0 maps any chiral pair onto a chiral pair, therefore α_0 on the set $\{C, E, C^1, E^1\}$ is either id, or z. Thus, $\alpha \in K \cup Kz$, where $K = H \times \langle s \rangle \times \langle t \rangle \times \cdots$. Now, suppose that $\beta \in Hw$. Then $\beta(R) = R^1$, and β leaves the cones P, $Q, \ldots,$ and the cones $\tilde{P}$, $\tilde{Q}, \ldots,$ invariant. According to [3, Corollary 3.2, (i)], we have $\alpha_0(\{A, A^1\}) = \{A, A^1\}$, $\alpha_0(\{B, B^1\}) = \{B, B^1\}, \ldots,$ $\alpha_0(\{C, E\}) = \{C^1, E^1\}$, $\alpha_0(\{C^1, E^1\}) = \{C, E\}$, and α_0 leaves the dimers invariant. Since α_0 maps any chiral pair onto a chiral pair, we obtain that α_0 on the set $\{C, E, C^1, E^1\}$ is either x, or $y = zx$. Now, we have $\alpha \in Kwx \cup Kwy$. Since x, y, z, commute among themselves, and each of them commutes with K, the proof is completed.

LEMMA 3.5. *Let the difference $V \backslash U$ be a G''-orbit that consists of several chiral pairs $\{A, A^1\}$, $\{B, B^1\}, \ldots,$ $\{C, C^1\}$, $\{E, E^1\}$, with $C_>(\bar{V}; C) = C_>(\bar{V}; E) = R$, $C_>(\bar{V}; C^1) = C_>(\bar{V}; E^1) = R^1$, and of several dimers. Suppose that the cones of the members of the chiral pairs $\{A, A^1\}$, $\{B, B^1\}, \ldots,$ the cones R, R^1, and the cones of the dimers, are all pairwise different, and suppose that*

$$I_{V,U} = H \times \langle w \rangle,$$

where w permutes the cones of the members of each chiral pair and leaves the cones of the dimers invariant. Then there exists a decomposition

$$Aut_0''(V) = H \times \langle z \rangle \times \langle wx \rangle,$$

where $z = (C, E)(C^1, E^1)$, $x = (A, A^1)(B, B^1) \ldots (C, C^1)(E, E^1)$, the restriction homomorphism $\varrho_{V,U}$ has kernel $\langle z \rangle$, and $Aut_0''(V)$ is a commutative 2-group.

PROOF: [3, Corollary 3.2, (i)] yields that H, Hz, Hwx, Hwy, where $y = zx$, are subsets of $Aut_0''(V)$. Now, let $\alpha \in Aut_0''(V)$, and let $\beta = \varrho_{V,U}(\alpha)$, $\alpha_0 = \varrho_{V,V\backslash U}(\alpha)$.

Suppose that $\beta \in H$. In accordance with [3, Corollary 3.2, (i)], the bijection α_0 leaves the members of the chiral pairs $\{A, A^1\}$, $\{B, B^1\}, \ldots$, as well as the dimers invariant, and $\alpha_0(\{C, E\}) = \{C, E\}$, $\alpha_0(\{C^1, E^1\}) = \{C^1, E^1\}$. Moreover, α_0 maps any chiral pair onto a chiral pair, therefore α_0 on the set $\{C, E, C^1, E^1\}$ is either id, or z. Thus, $\alpha \in H \cup Hz$. Now, suppose that $\beta \in Hw$. Again, according to [3, Corollary 3.2, (i)], we have $\alpha_0(A) = A^1$, $\alpha_0(A^1) = A$, $\alpha_0(B) = B^1$, $\alpha_0(B^1) = B, \ldots$, $\alpha_0(\{C, E\}) = \{C^1, E^1\}$, $\alpha_0(\{C^1, E^1\}) = \{C, E\}$, and α_0 leaves the dimers invariant. Since α_0 maps any chiral pair onto a chiral pair, we obtain that α_0 on the set $\{C, E, C^1, E^1\}$ is either wx, or wy. Now, we have $\alpha \in Hx \cup Hy$. Since x, y, z, commute, and since each of them commutes with H, the proof is done.

THEOREM 3.6. *One has:*

$$\text{(i) } Aut''_0(T_{D_1;G}) = \langle (c_{(4,2)}, e_{(4,2)}) \rangle \simeq C_2;$$

$$\text{(ii) } Aut''_0(T_{D_2;G}) = \langle (a_{(4,1^2)}, b_{(4,1^2)}), (c_{(4,2)}, e_{(4,2)})(e_{(4,1^2)}, f_{(4,1^2)}) \rangle \simeq C_2 \times C_2;$$

$$\text{(iii) } Aut''_0(T_{D_3;G}) = \langle (a_{(4,1^2)}, b_{(4,1^2)}), (c_{(4,2)}, e_{(4,2)})(e_{(4,1^2)}, f_{(4,1^2)})(b_{(3^2)}, c_{(3^2)}) \rangle$$
$$\simeq C_2 \times C_2;$$

$$\text{(iv) } Aut''_0(T_{D_4;G}) = \langle (c_{(4,2)}, e_{(4,2)})(a_{(4,1^2)}, b_{(4,1^2)})(e_{(4,1^2)}, f_{(4,1^2)})(b_{(3^2)}, c_{(3^2)})$$
$$(c_{(3,2,1)}, h_{(3,2,1)})(b_{(3,2,1)}, f_{(3,2,1)})(e_{(3,2,1)}, k_{(3,2,1)})(\ell_{(3,2,1)}, m_{(3,2,1)}) \rangle \simeq C_2;$$

$$\text{(v) } Aut''_0(T_{D_5;G}) = \langle (c_{(3,1^3)}, \bar{c}_{(3,1^3)})(h_{(3,1^3)}, \bar{h}_{(3,1^3)}), (m_{(3,1^3)}, \bar{m}_{(3,1^3)})(p_{(3,1^3)}, \bar{p}_{(3,1^3)}),$$
$$(a_{(3,1^3)}, \bar{a}_{(3,1^3)}), (\ell_{(3,1^3)}, \bar{\ell}_{(3,1^3)}), (c_{(4,2)}, e_{(4,2)})(a_{(4,1^2)}, b_{(4,1^2)})(e_{(4,1^2)}, f_{(4,1^2)})$$
$$(b_{(3^2)}, c_{(3^2)})(c_{(3,2,1)}, h_{(3,2,1)})(b_{(3,2,1)}, f_{(3,2,1)})(e_{(3,2,1)}, k_{(3,2,1)})(\ell_{(3,2,1)}, m_{(3,2,1)})$$
$$(b_{(3,1^3)}, f_{(3,1^3)})(\bar{e}_{(3,1^3)}, \bar{k}_{(3,1^3)})(\bar{b}_{(3,1^3)}, \bar{f}_{(3,1^3)})(e_{(3,1^3)}, k_{(3,1^3)})(c_{(3,1^3)}, h_{(3,1^3)})$$
$$(\bar{c}_{(3,1^3)}, \bar{h}_{(3,1^3)})(m_{(3,1^3)}, p_{(3,1^3)})(\bar{m}_{(3,1^3)}, \bar{p}_{(3,1^3)}) \rangle \simeq C_2 \times C_2 \times C_2 \times C_2 \times C_2;$$

$$\text{(vi) } Aut''_0(T_{D_6;G}) = \langle (c_{(3,1^3)}, \bar{c}_{(3,1^3)})(h_{(3,1^3)}, \bar{h}_{(3,1^3)}), (m_{(3,1^3)}, \bar{m}_{(3,1^3)})(p_{(3,1^3)}, \bar{p}_{(3,1^3)}),$$
$$(a_{(3,1^3)}, \bar{a}_{(3,1^3)}), (\ell_{(3,1^3)}, \bar{\ell}_{(3,1^3)}), (h_{(2^3)}, k_{(2^3)})(\ell_{(2^3)}, \bar{\ell}_{(2^3)}),$$
$$(c_{(4,2)}, e_{(4,2)})(a_{(4,1^2)}, b_{(4,1^2)})(e_{(4,1^2)}, f_{(4,1^2)})(b_{(3^2)}, c_{(3^2)})(c_{(3,2,1)}, h_{(3,2,1)})$$
$$(b_{(3,2,1)}, f_{(3,2,1)})(e_{(3,2,1)}, k_{(3,2,1)})(\ell_{(3,2,1)}, m_{(3,2,1)})(b_{(3,1^3)}, f_{(3,1^3)})(\bar{e}_{(3,1^3)}, \bar{k}_{(3,1^3)})$$
$$(\bar{b}_{(3,1^3)}, \bar{f}_{(3,1^3)})(e_{(3,1^3)}, k_{(3,1^3)})(c_{(3,1^3)}, h_{(3,1^3)})(\bar{c}_{(3,1^3)}, \bar{h}_{(3,1^3)})(m_{(3,1^3)}, p_{(3,1^3)})$$
$$(\bar{m}_{(3,1^3)}, \bar{p}_{(3,1^3)})(h_{(2^3)}, \ell_{(2^3)})(k_{(2^3)}, \bar{\ell}_{(2^3)})(\bar{k}_{(2^3)}, m_{(2^3)})(\bar{h}_{(2^3)}, \bar{m}_{(2^3)})(f_{(2^3)}, \bar{f}_{(2^3)}) \rangle$$
$$\simeq C_2 \times C_2 \times C_2 \times C_2 \times C_2 \times C_2.$$

PROOF: (i) Let us set $D'_0 = \{(6), (5,1)\}$. Section 2, Cases 2, 3, yield that $Aut''_0(T_{D_0;G})$ is the trivial group. The structure of the $(4,2)$-level as well as the inequalities among tabloids that correspond to the dominance order inequality $(4,2) < (5,1)$ are presented in Section 2, Case 3. We set $U = T_{D_0;G}$, $U_1^{(1)} = T_{D_0;G} \cup \{c_{(4,2)}, e_{(4,2)}, a_{(4,2)}\}$, $\bar{V} = T_{D'_1;G}$, where $D'_1 = \{((5,1), (4,2)\}$, and note that $\bar{V}$ is a barrier of $T_{(4,2);G}$ in $T_{D_1;G}$. We have

$$C_>(\bar{V}; c_{(4,2)}) = C_>(\bar{V}; e_{(4,2)}) = C_>(\bar{V}; a_{(4,2)}) = C_>(\bar{V}; b_{(4,2)}) = \{a_{(5,1)}\}.$$

Since the conditions of Lemma 3.1 are satisfied for $V = U_1^{(1)}$, we obtain $Aut_0''(U_1^{(1)}) = \langle (c_{(4,2)}, e_{(4,2)}) \rangle$. By adding the dimer $b_{(4,2)}$ to the set $U_1^{(1)}$, we get $T_{D_1;G}$, and [2, Lemma 3.3] implies (i).
(ii) We can find in Section 2, Case 4, the structure of $(4,1^2)$-level, as well as the inequalities among tabloids that correspond to the inequality $(4,1^2) < (4,2)$. We set $U = T_{D_1;G}$, $U_2^{(1)} = U \cup \{a_{(4,1^2)}, b_{(4,1^2)}, e_{(4,1^2)}, f_{(4,1^2)}\}$, and $\bar{V} = T_{D_2';G}$, where $D_2' = \{((4,2),(4,1^2)\}$. The set $\bar{V}$ is a barrier of $T_{(4,1^2);G}$ in $T_{D_2;G}$. We have

$$P = C_>(\bar{V}; a_{(4,1^2)}) = C_>(\bar{V}; b_{(4,1^2)}) = \{a_{(4,2)}\},$$

$$Q = C_>(\bar{V}; e_{(4,1^2)}) = \{c_{(4,2)}\},\ Q^1 = C_>(\bar{V}; f_{(4,1^2)}) = \{e_{(4,2)}\}.$$

There exists a decomposition $Aut_0''(U) = H \times \langle w \rangle$, where H is the trivial subgroup, $w = (c_{(4,2)}, e_{(4,2)})$, and, moreover, $w(P) = P$, $w(Q) = Q^1$. Now, in accord with [3, Corollary 3.2, (i)], we have $H \cup Hwt \subset Aut_0''(U_2^{(1)})$, where $t = (e_{(4,1^2)}, f_{(4,1^2)})$, so, in particular, the restriction homomorphism $\varrho_{U_2^{(1)},U}$ is surjective. Therefore Lemma 3.2 for $s = (a_{(4,1^2)}, b_{(4,1^2)})$ yields

$$Aut_0''(U_2^{(1)}) = \langle (a_{(4,1^2)}, b_{(4,1^2)}), (c_{(4,2)}, e_{(4,2)})(e_{(4,1^2)}, f_{(4,1^2)}) \rangle.$$

The cone $C_>(\bar{V}; c_{(4,1^2)})$ of the dimer $c_{(4,1^2)}$ is $Aut_0''(U_2^{(1)})$-invariant, and by adding this dimer to the set $U_2^{(1)}$ we get $T_{D_2;G}$. Now, [2, Lemma 3.3] finishes the proof of (ii).
(iii) The inequalities among tabloids, which correspond the dominance order inequality $(3^2) < (4,2)$, as well as the description of the (3^2)-level are presented in Section 2, Case 5. We set $\bar{V} = T_{D_3';G}$, where $D_3' = \{((4,2),(4,1^2),(3^2)\}$ and note that $\bar{V}$ is a barrier of $T_{(3^2);G}$ in $T_{D_3;G}$. First, we add the two dimers to $T_{D_2;G}$ and get $U_3^{(1)} = T_{D_2;G} \cup \{a_{(3^2)}, e_{(3^2)}\}$. The cones $C_>(\bar{V}; a_{(3^2)}) = \{a_{(4,2)}\}$, $C_>(\bar{V}; e_{(3^2)}) = \{a_{(4,2)}, c_{(4,2)}, e_{(4,2)}\}$ of the dimers are $Aut_0''(T_{D_2;G})$-invariant, therefore [2, Lemma 3.3] yields

$$Aut_0''(U_3^{(1)}) = \langle (a_{(4,1^2)}, b_{(4,1^2)}), (c_{(4,2)}, e_{(4,2)})(e_{(4,1^2)}, f_{(4,1^2)}) \rangle.$$

Next, we supplement the set $U_3^{(1)}$ with the chiral pair $\{b_{(3^2)}, c_{(3^2)}\}$ and obtain $T_{D_3;G}$. We have $P = C_>(\bar{V}; b_{(3^2)}) = \{a_{(4,2)}, b_{(4,2)}, c_{(4,2)}\}$, $P^1 = C_>(\bar{V}; c_{(3^2)}) = \{a_{(4,2)}, b_{(4,2)}, e_{(4,2)}\}$, and the group $Aut_0''(U_3^{(1)})$ can be decomposed as $Aut_0''(U_3^{(1)}) = H \times \langle w \rangle$, where as usual $H = \langle (a_{(4,1^2)}, b_{(4,1^2)}) \rangle$ is the group of automorphisms that leave the cones P and P^1 invariant, and the automorphism $w = (c_{(4,2)}, e_{(4,2)})(e_{(4,1^2)}, f_{(4,1^2)})$ permutes P and P^1. Now, [2, Lemma 3.4, (i)] implies (iii).
(iv) In Section 2, Case 6, we have a description of the $(3,2,1)$-level, and the inequalities among tabloids that correspond to the inequalities $(3,2,1) < (4,1^2)$ and $(3,2,1) < (3^2)$ in the dominance order. We set $\bar{V} = T_{D_4';G}$, where $D_4' = \{(4,1^2),(3^2),(3,2,1)\}$ and note that $\bar{V}$ is a barrier of $T_{(3,2,1);G}$ in $T_{D_4;G}$. Let us first add the chiral pair that taken alone is a G''-orbit: $U_4^{(1)} = T_{D_3;G} \cup \{c_{(3,2,1)}, h_{(3,2,1)}\}$. We have $P = C_>(\bar{V}; c_{(3,2,1)}) = \{b_{(4,1^2)}, e_{(4,1^2)}, b_{(3^2)}\}$, $P^1 = C_>(\bar{V}; h_{(3,2,1)}) = \{a_{(4,1^2)}, f_{(4,1^2)}, c_{(3^2)}\}$. Among the four elements of the automorphism group $Aut_0''(T_{D_3;G})$ only two induce a permutation of the cones P and P^1: id and $w = (a_{(4,1^2)}, b_{(4,1^2)})(c_{(4,2)}, e_{(4,2)})(e_{(4,1^2)}, f_{(4,1^2)})(b_{(3^2)}, c_{(3^2)})$.

Then [3, Corollary 3.2, (i), (ii)] yields $H \leq Aut_0''(U_4^{(1)})$, and $Hwr \leq Aut_0''(U_4^{(1)})$, where $H = \langle id \rangle$, and $r = (c_{(3,2,1)}, h_{(3,2,1)})$. Thus, $I_{U_4^{(1)}, T_{D_3;G}} = H \times \langle w \rangle$. Now, [2, Lemma 3.4, (i)] implies $Aut_0''(U_4^{(1)}) = H \times \langle wr \rangle$. Next we set $U_4^{(2)} = U_4^{(1)} \cup \{b_{(3,2,1)}, f_{(3,2,1)}, e_{(3,2,1)}, k_{(3,2,1)}\}$. We denote for short the corresponding cones as follows:

$$Q = C_>(\bar{V}; b_{(3,2,1)}) = \{a_{(4,1^2)}, c_{(4,1^2)}, b_{(3^2)}\},$$

$$Q^1 = C_>(\bar{V}; f_{(3,2,1)}) = \{b_{(4,1^2)}, c_{(4,1^2)}, c_{(3^2)}\},$$

$$R = C_>(\bar{V}; e_{(3,2,1)}) = \{c_{(4,1^2)}, e_{(4,1^2)}, b_{(3^2)}\},$$

$$R^1 = C_>(\bar{V}; k_{(3,2,1)}) = \{c_{(4,1^2)}, f_{(4,1^2)}, c_{(3^2)}\}.$$

We have $u(Q) = Q^1$, $u(R) = R^1$, for $u = wr$. Again, [3, Corollary 3.2, (i), (ii)] implies $H \leq Aut_0''(U_4^{(2)})$, and $Hus \leq Aut_0''(U_4^{(2)})$, where $s = (b_{(3,2,1)}, f_{(3,2,1)})(e_{(3,2,1)}, k_{(3,2,1)})$, so the restriction homomorphism $\varrho_{U_4^{(2)}, U_4^{(1)}}$ is surjective, and in accordance with [2, Lemma 3.6, (i)] we obtain $Aut_0''(U_4^{(2)}) = H \times \langle us \rangle$. Further, we supplement $U_4^{(2)}$ with the last G''-orbit consisting of a chiral pair and two dimers:

$$T_{D_4;G} = U_4^{(2)} \cup \{\ell_{(3,2,1)}, m_{(3,2,1)}, a_{(3,2,1)}, p_{(3,2,1)}\}.$$

The corresponding cones are

$$X = C_>(\bar{V}; \ell_{(3,2,1)}) = \{a_{(4,1^2)}, e_{(4,1^2)}, e_{(3^2)}\},$$

$$X^1 = C_>(\bar{V}; m_{(3,2,1)}) = \{b_{(4,1^2)}, f_{(4,1^2)}, e_{(3^2)}\},$$

$$\tilde{P} = C_>(\bar{V}; a_{(3,2,1)}) = \{a_{(4,1^2)}, b_{(4,1^2)}, a_{(3^2)}\},$$

$$\tilde{Q} = C_>(\bar{V}; p_{(3,2,1)}) = \{e_{(4,1^2)}, f_{(4,1^2)}, e_{(3^2)}\}.$$

We have $Aut_0''(U_4^{(2)}) = H \times \langle v \rangle$, where $v = us$, and $v(X) = X^1$, $v(\tilde{P}) = \tilde{P}$, $v(\tilde{Q}) = \tilde{Q}$. Moreover, if $t = (\ell_{(3,2,1)}, m_{(3,2,1)})$, then $H \leq Aut_0''(T_{D_4;G})$, and $Hvt \subset Aut_0''(T_{D_4;G})$, because of [3, Corollary 3.2, (i), (ii)]. Therefore the corresponding restriction homomorphism is surjective, and Lemma 3.3 yields part (iv).

(v) The $(3,1^3)$-level is described in Section 2, Case 7, where all inequalities between tabloids, that correspond to the inequality $(3,1^3) < (3,2,1)$, are presented. Let $\bar{V} = T_{D_5';G}$, where $D_5' = \{((3,2,1),(3,1^3)\}$. The set $\bar{V}$ is a barrier of $T_{(3,1^3);G}$ in $T_{D_5;G}$. First, we supplement consecutively the set $T_{D_4;G}$ with two G''-orbits that consist of two chiral pairs each: $U_5^{(1)} = T_{D_4;G} \cup \{b_{(3,1^3)}, f_{(3,1^3)}, \bar{e}_{(3,1^3)}, \bar{k}_{(3,1^3)}\}$, and $U_5^{(2)} = U_5^{(1)} \cup \{\bar{b}_{(3,1^3)}, \bar{f}_{(3,1^3)}, e_{(3,1^3)}, k_{(3,1^3)}\}$. The chiral involution u from $Aut_0''(T_{D_4;G})$ permutes the cones

$$C_>(\bar{V}; b_{(3,1^3)}) = C_>(\bar{V}; \bar{b}_{(3,1^3)}) = \{b_{(3,2,1)}, c_{(3,2,1)}\},$$

$$C_>(\bar{V}; f_{(3,1^3)}) = C_>(\bar{V}; \bar{f}_{(3,1^3)}) = \{f_{(3,2,1)}, h_{(3,2,1)}\},$$

and the cones

$$C_>(\bar{V}; e_{(3,1^3)}) = C_>(\bar{V}; \bar{e}_{(3,1^3)}) = \{c_{(3,2,1)}, e_{(3,2,1)}\},$$

$$C_>(\bar{V}; k_{(3,1^3)}) = C_>(\bar{V}; \bar{k}_{(3,1^3)}) = \{h_{(3,2,1)}, k_{(3,2,1)}\}.$$

Now, the decomposition $Aut_0''(T_{D_4;G}) = H \times \langle u \rangle$, where $H = \{id\}$, the surjectivity of the corresponding restriction homomorphism, as well as [2, Lemma 3.6, (i)], yield consecutively $Aut_0''(U_5^{(1)}) = H \times \langle v \rangle$, for $v = u(b_{(3,1^3)}, f_{(3,1^3)})(\bar{e}_{(3,1^3)}, \bar{k}_{(3,1^3)})$, and $Aut_0''(U_5^{(2)}) = H \times \langle w \rangle$, for $w = v(\bar{b}_{(3,1^3)}, \bar{f}_{(3,1^3)})(e_{(3,1^3)}, k_{(3,1^3)})$. Further, let us set $U_5^{(3)} = U_5^{(2)} \cup \{c_{(3,1^3)}, h_{(3,1^3)}, \bar{c}_{(3,1^3)}, \bar{h}_{(3,1^3)}\}$. We have

$$C_>(\bar{V}; c_{(3,1^3)}) = C_>(\bar{V}; \bar{c}_{(3,1^3)}) = \{b_{(3,2,1)}, e_{(3,2,1)}\},$$

$$C_>(\bar{V}; h_{(3,1^3)}) = C_>(\bar{V}; \bar{h}_{(3,1^3)}) = \{f_{(3,2,1)}, k_{(3,2,1)}\},$$

and the automorphism w permutes these two cones. According to [2, Lemma 3.4, (ii)], we get $Aut_0''(U_5^{(3)}) = H \times \langle z \rangle \times \langle wx \rangle$, where $z = (c_{(3,1^3)}, \bar{c}_{(3,1^3)})(h_{(3,1^3)}, \bar{h}_{(3,1^3)})$, and $x = (c_{(3,1^3)}, h_{(3,1^3)})(\bar{c}_{(3,1^3)}, \bar{h}_{(3,1^3)})$. Finally, we add the G''-orbit consisting of four chiral pairs and get $T_{D_5;G} = U_5^{(3)} \cup \{a_{(3,1^3)}, \bar{a}_{(3,1^3)}, \ell_{(3,1^3)}, \bar{\ell}_{(3,1^3)}, m_{(3,1^3)}, p_{(3,1^3)}, \bar{m}_{(3,1^3)}, \bar{p}_{(3,1^3)}\}$. The corresponding cones are

$$P = C_>(\bar{V}; a_{(3,1^3)}) = C_>(\bar{V}; \bar{a}_{(3,1^3)}) = \{a_{(3,2,1)}\},$$

$$Q = C_>(\bar{V}; \ell_{(3,1^3)}) = C_>(\bar{V}; \bar{\ell}_{(3,1^3)}) = \{\ell_{(3,2,1)}, m_{(3,2,1)}\},$$

$$R = C_>(\bar{V}; m_{(3,1^3)}) = C_>(\bar{V}; \bar{m}_{(3,1^3)}) = \{\ell_{(3,2,1)}, p_{(3,2,1)}\},$$

$$R^1 = C_>(\bar{V}; p_{(3,1^3)}) = C_>(\bar{V}; \bar{p}_{(3,1^3)}) = \{m_{(3,2,1)}, p_{(3,2,1)}\}.$$

We have the decomposition $Aut_0''(U_5^{(3)}) = H \times \langle y \rangle$, where $H = \langle z \rangle$, $y = wx$. The group H consists of all automorphisms that leave the cones P, Q, R, R^1, invariant, and $y(P) = P$, $y(Q) = Q$, $y(R) = R^1$. Thus the corresponding restriction homomorphism is surjective and Lemma 3.4 yields part (v).
(vi) In Section 2, Case 8, we describe the (2^3)-level. All inequalities between tabloids, that correspond to the inequality $(2^3) < (3,2,1)$, are presented there. Let $\bar{V} = T_{D_6';G}$, where $D_5' = \{((3,2,1),(2^3)\}$. The set $\bar{V}$ is a barrier of $T_{(2^3);G}$ in $T_{D_6;G}$. Let us first add to the set $T_{D_5;G}$ the G''-orbit that contains four chiral pairs: $U_6^{(1)} = T_{D_5;G}) \cup \{a_{(2^3)}, \bar{a}_{(2^3)}, c_{(2^3)}, \bar{c}_{(2^3)}, h_{(2^3)}, \ell_{(2^3)}, k_{(2^3)}, \bar{\ell}_{(2^3)}\}$. Their cones are

$$C_>(\bar{V}; a_{(2^3)}) = \{a_{(3,2,1)}, b_{(3,2,1)}, c_{(3,2,1)}, f_{(3,2,1)}, \ell_{(3,2,1)}\},$$

$$C_>(\bar{V}; \bar{a}_{(2^3)}) = \{a_{(3,2,1)}, b_{(3,2,1)}, f_{(3,2,1)}, h_{(3,2,1)}, m_{(3,2,1)}\},$$

$$C_>(\bar{V}; c_{(2^3)}) = \{a_{(3,2,1)}, c_{(3,2,1)}, e_{(3,2,1)}, k_{(3,2,1)}, \ell_{(3,2,1)}, p_{(3,2,1)}\},$$

$$C_>(\bar{V}; \bar{c}_{(2^3)}) = \{a_{(3,2,1)}, e_{(3,2,1)}, h_{(3,2,1)}, k_{(3,2,1)}, m_{(3,2,1)}, p_{(3,2,1)}\},$$

$$C_>(\bar{V}; h_{(2^3)}) = C_>(\bar{V}; k_{(2^3)}) = \{f_{(3,2,1)}, h_{(3,2,1)}, k_{(3,2,1)}, \ell_{(3,2,1)}, m_{(3,2,1)}, p_{(3,2,1)}\},$$

$$C_>(\bar{V}; \ell_{(2^3)}) = C_>(\bar{V}; \bar{\ell}_{(2^3)}) = \{b_{(3,2,1)}, c_{(3,2,1)}, e_{(3,2,1)}, \ell_{(3,2,1)}, m_{(3,2,1)}, p_{(3,2,1)}\}.$$

The group $Aut_0''(T_{D_5;G})$ can be decomposed as $Aut_0''(T_{D_5;G}) = H \times \langle w \rangle$, where w is its last generator, as written in (v), and H is generated by all the rest. The automorphism

w permutes the cones of the members of each of the added four chiral pairs. Therefore, the corresponding restriction homomorphism is surjective, and now Lemma 3.5 implies

$$\begin{gathered}Aut''_0(U_6^{(1)}) = \langle (c_{(3,1^3)}, \bar{c}_{(3,1^3)})(h_{(3,1^3)}, \bar{h}_{(3,1^3)}), (m_{(3,1^3)}, \bar{m}_{(3,1^3)})(p_{(3,1^3)}, \bar{p}_{(3,1^3)}),\\ (a_{(3,1^3)}, \bar{a}_{(3,1^3)}), (\ell_{(3,1^3)}, \bar{\ell}_{(3,1^3)}), (h_{(2^3)}, k_{(2^3)})(\ell_{(2^3)}, \bar{\ell}_{(2^3)}),\\ (c_{(4,2)}, e_{(4,2)})(a_{(4,1^2)}, b_{(4,1^2)})(e_{(4,1^2)}, f_{(4,1^2)})(b_{(3^2)}, c_{(3^2)})(c_{(3,2,1)}, h_{(3,2,1)})\\ (b_{(3,2,1)}, f_{(3,2,1)})(e_{(3,2,1)}, k_{(3,2,1)})(\ell_{(3,2,1)}, m_{(3,2,1)})(b_{(3,1^3)}, f_{(3,1^3)})(\bar{e}_{(3,1^3)}, \bar{k}_{(3,1^3)})\\ (\bar{b}_{(3,1^3)}, \bar{f}_{(3,1^3)})(e_{(3,1^3)}, k_{(3,1^3)})(c_{(3,1^3)}, h_{(3,1^3)})(\bar{c}_{(3,1^3)}, \bar{h}_{(3,1^3)})(m_{(3,1^3)}, p_{(3,1^3)})\\ (\bar{m}_{(3,1^3)}, \bar{p}_{(3,1^3)})(h_{(2^3)}, \ell_{(2^3)})(k_{(2^3)}, \bar{\ell}_{(2^3)})\rangle.\end{gathered}$$

Now, we add to $U_6^{(1)}$ consecutively the three G''-orbits consisting of one chiral pair and one dimer: $U_6^{(2)} = U_6^{(1)} \cup \{\bar{k}_{(2^3)}, m_{(2^3)}, b_{(2^3)}\}$, $U_6^{(3)} = U_6^{(2)} \cup \{\bar{h}_{(2^3)}, \bar{m}_{(2^3)}, \bar{b}_{(2^3)}\}$, $U_6^{(4)} = U_6^{(3)} \cup \{f_{(2^3)}, \bar{f}_{(2^3)}, e_{(2^3)}\}$. Here are the corresponding cones:

$$\begin{gathered}C_>(\bar{V}; \bar{k}_{(2^3)}) = \{h_{(3,2,1)}, \ell_{(3,2,1)}, p_{(3,2,1)}\},\\ C_>(\bar{V}; m_{(2^3)}) = \{c_{(3,2,1)}, m_{(3,2,1)}, p_{(3,2,1)}\},\\ C_>(\bar{V}; b_{(2^3)}) = \{a_{(3,2,1)}, c_{(3,2,1)}, h_{(3,2,1)}, \ell_{(3,2,1)}, m_{(3,2,1)}\},\\ C_>(\bar{V}; \bar{h}_{(2^3)}) = \{f_{(3,2,1)}, k_{(3,2,1)}, m_{(3,2,1)}\},\\ C_>(\bar{V}; \bar{m}_{(2^3)}) = \{b_{(3,2,1)}, e_{(3,2,1)}, \ell_{(3,2,1)}\},\\ C_>(\bar{V}; \bar{b}_{(2^3)}) = \{a_{(3,2,1)}, b_{(3,2,1)}, e_{(3,2,1)}, f_{(3,2,1)}, k_{(3,2,1)}, p_{(3,2,1)}\},\\ C_>(\bar{V}; f_{(2^3)}) = \{c_{(3,2,1)}, e_{(3,2,1)}, f_{(3,2,1)}\},\\ C_>(\bar{V}; \bar{f}_{(2^3)}) = \{b_{(3,2,1)}, h_{(3,2,1)}, k_{(3,2,1)}\},\\ C_>(\bar{V}; e_{(2^3)}) = \{b_{(3,2,1)}, c_{(3,2,1)}, e_{(3,2,1)}, f_{(3,2,1)}, h_{(3,2,1)}, k_{(3,2,1)}\}.\end{gathered}$$

For any one of these G''-orbits, the last generator of the group $Aut''_0(U_6^{(1)})$, or its extension, permutes the cones of the members of the chiral pair and leaves the cone of the dimer invariant. Applying Lemma 3.3 three times, we obtain

$$\begin{gathered}Aut''_0(U_6^{(4)}) = \langle (c_{(3,1^3)}, \bar{c}_{(3,1^3)})(h_{(3,1^3)}, \bar{h}_{(3,1^3)}), (m_{(3,1^3)}, \bar{m}_{(3,1^3)})(p_{(3,1^3)}, \bar{p}_{(3,1^3)}),\\ (a_{(3,1^3)}, \bar{a}_{(3,1^3)}), (\ell_{(3,1^3)}, \bar{\ell}_{(3,1^3)}), (h_{(2^3)}, k_{(2^3)})(\ell_{(2^3)}, \bar{\ell}_{(2^3)}),\\ (c_{(4,2)}, e_{(4,2)})(a_{(4,1^2)}, b_{(4,1^2)})(e_{(4,1^2)}, f_{(4,1^2)})(b_{(3^2)}, c_{(3^2)})(c_{(3,2,1)}, h_{(3,2,1)})\\ (b_{(3,2,1)}, f_{(3,2,1)})(e_{(3,2,1)}, k_{(3,2,1)})(\ell_{(3,2,1)}, m_{(3,2,1)})(b_{(3,1^3)}, f_{(3,1^3)})(\bar{e}_{(3,1^3)}, \bar{k}_{(3,1^3)})\\ (\bar{b}_{(3,1^3)}, \bar{f}_{(3,1^3)})(e_{(3,1^3)}, k_{(3,1^3)})(c_{(3,1^3)}, h_{(3,1^3)})(\bar{c}_{(3,1^3)}, \bar{h}_{(3,1^3)})(m_{(3,1^3)}, p_{(3,1^3)})\\ (\bar{m}_{(3,1^3)}, \bar{p}_{(3,1^3)})(h_{(2^3)}, \ell_{(2^3)})(k_{(2^3)}, \bar{\ell}_{(2^3)})(\bar{k}_{(2^3)}, m_{(2^3)})(\bar{h}_{(2^3)}, \bar{m}_{(2^3)})(f_{(2^3)}, \bar{f}_{(2^3)})\rangle.\end{gathered}$$

We have $C_>(\bar{V}; \bar{e}_{(2^3)}) = C_>(\bar{V}; e_{(2^3)})$, so the cone of the dimer $\bar{e}_{(2^3)}$ is $Aut''_0(U_6^{(4)})$-invariant, and, in compliance with [2, Lemma 3.3], we get part (vi).

Corollary 3.7. *The chiral pairs $\{a_{(4,1^2)}, b_{(4,1^2)}\}$, $\{e_{(4,1^2)}, f_{(4,1^2)}\}$ can be distinguished via substitution reactions among the elements of $T_{D_2;G}$.*

Proof: Lunn-Senior's group $Aut''_0(T_{D_2;G})$ does not contain automorphism that maps the members of one of the chiral pairs onto the members of the other.

Remark 3.8. The members of the chiral pair $\{a_{(4,1^2)}, b_{(4,1^2)}\}$ can be obtained via substitution reactions by one and the same dimer $a_{(4,2)}$, whereas each of the members of the chiral pair $\{c_{(4,2)}, e_{(4,2)}\}$ produces via substitution reaction exactly one member of the chiral pair $\{e_{(4,1^2)}, f_{(4,1^2)}\}$, and the two members of the latter can be obtained in this way.

Corollary 3.9. *The two dimers $a_{(3^2)}$ and $e_{(3^2)}$, can be distinguished via substitution reactions among the elements of $T_{D_3;G}$.*

Proof: The group $Aut''_0(T_{D_3;G})$ does not contain automorphism that maps one of the dimers onto the other.

Remark 3.10. The dimer $a_{(3^2)}$ can be produced via substitution reactions by exactly one $(4,2)$-product whereas the dimer $e_{(3^2)}$ can be produced by three $(4,2)$-products.

Since the group $Aut''_0(T_{D_4;G})$ contains only the identity and the chiral automorphism, we obtain the following two corollaries:

Corollary 3.11. *The two dimers $a_{(3,2,1)}$ and $p_{(3,2.1)}$, can be distinguished via substitution reactions among the elements of $T_{D_4;G}$.*

Remark 3.12. The dimer $a_{(3,2,1)}$ can be produced via substitution reactions by the dimer $a_{(3^2)}$ which has the property that it can produce exactly one $(3,2,1)$-product. On the other hand, the dimer $p_{(3,2.1)}$ can be produced by the dimer $e_{(3^2)}$ which has the property that it can produce three $(3,2,1)$-products.

Corollary 3.13. *Any two chiral pairs from*

$$\{b_{(3,2,1)}, f_{(3,2,1)}\},\ \{c_{(3,2,1)}, h_{(3,2,1)}\},\ \{e_{(3,2,1)}, k_{(3,2,1)}\},$$

can be distinguished via substitution reactions among the elements of $T_{D_4;G}$.

Remark 3.14. Any member of the chiral pair $\{b_{(3,2,1)}, f_{(3,2,1)}\}$ can be produced via substitution reactions by one member of the chiral pair $\{a_{(4,1^2)}, b_{(4,1^2)}\}$, any member of the chiral pair $\{c_{(3,2,1)}, h_{(3,2,1)}\}$ can be produced via substitution reactions by one member of each chiral pair $\{a_{(4,1^2)}, b_{(4,1^2)}\}$, $\{e_{(4,1^2)}, f_{(4,1^2)}\}$, and any member of the chiral pair $\{e_{(3,2,1)}, k_{(3,2,1)}\}$ can be produced via substitution reactions by one member of the chiral pair $\{e_{(4,1^2)}, f_{(4,1^2)}\}$. In the end, it is enough to note that Corollary 3.7 holds.

Corollary 3.15. *The chiral pairs $\{c_{(3,1^3)}, h_{(3,1^3)}\}$, $\{\bar{c}_{(3,1^3)}, \bar{h}_{(3,1^3)}\}$ can not be distinguished via substitution reactions among the elements of $T_{D_6;G}$.*

Corollary 3.16. *The chiral pairs $\{m_{(3,1^3)}, p_{(3,1^3)}\}$, $\{\bar{m}_{(3,1^3)}, \bar{p}_{(3,1^3)}\}$ can not be distinguished via substitution reactions among the elements of $T_{D_6;G}$.*

Proofs: It is enough to note that Lunn-Senior's group $Aut''_0(T_{D_6;G})$ contains the automorphisms $(c_{(3,1^3)}, \bar{c}_{(3,1^3)})(h_{(3,1^3)}, \bar{h}_{(3,1^3)})$, and $(m_{(3,1^3)}, \bar{m}_{(3,1^3)})(p_{(3,1^3)}, \bar{p}_{(3,1^3)})$, respectively.

Corollary 3.17. *(i) The chiral pairs*

$$\{a_{(3,1^3)}, \bar{a}_{(3,1^3)}\},\ \{\ell_{(3,1^3)}, \bar{\ell}_{(3,1^3)}\}, \tag{3.18}$$

can be distinguished via substitution reactions among the elements of $T_{D_5;G}$;

(ii) any chiral pair from (3.18) and any one from the chiral pairs

$$\{m_{(3,1^3)}, p_{(3,1^3)}\},\ \{\bar{m}_{(3,1^3)}, \bar{p}_{(3,1^3)}\},$$

can be distinguished via substitution reactions among the elements of $T_{D_5;G}$.

Proof: In both parts (i) and (ii) Lunn-Senior's group $Aut_0''(T_{D_5;G})$ does not contain an automorphism that works.

Remark 3.19. The members of the chiral pair $\{a_{(3,1^3)}, \bar{a}_{(3,1^3)}\}$ can be produced via substitution reactions by exactly one $(3,2,1)$-product – the dimer $a_{(3,2,1)}$, any member of the chiral pair $\{\ell_{(3,1^3)}, \bar{\ell}_{(3,1^3)}\}$ can be produced via substitution reactions only by both members of the chiral pair $\{\ell_{(3,2,1)}, m_{(3,2,1)}\}$, and any member of the chiral pair $\{m_{(3,1^3)}, p_{(3,1^3)}\}$ (respectively, $\{\bar{m}_{(3,1^3)}, \bar{p}_{(3,1^3)}\}$) can be produced via substitution reactions by one member of the chiral pair $\{\ell_{(3,2,1)}, m_{(3,2,1)}\}$, and by the dimer $p_{(3,2,1)}$.

The two corollaries below can be proved in the same way.

Corollary 3.20. *The two chiral pairs* $\{b_{(3,1^3)}, f_{(3,1^3)}\}$, $\{\bar{e}_{(3,1^3)}, \bar{k}_{(3,1^3)}\}$ *can be distinguished via substitution reactions among the elements of* $T_{D_5;G}$.

Corollary 3.21. *The two chiral pairs* $\{\bar{b}_{(3,1^3)}, \bar{f}_{(3,1^3)}\}$, $\{e_{(3,1^3)}, k_{(3,1^3)}\}$ *can be distinguished via substitution reactions among the elements of* $T_{D_5;G}$.

Remark 3.22. The members of the chiral pair $\{b_{(3,2,1)}, f_{(3,2,1)}\}$ (respectively, the chiral pair $\{e_{(3,2,1)}, k_{(3,2,1)}\}$) produce the members of the chiral pair $\{b_{(3,1^3)}, f_{(3,1^3)}\}$ as well as the members of the chiral pair $\{\bar{b}_{(3,1^3)}, \bar{f}_{(3,1^3)}\}$ (respectively, $\{\bar{e}_{(3,1^3)}, \bar{k}_{(3,1^3)}\}$ as well as $\{e_{(3,1^3)}, k_{(3,1^3)}\}$). Moreover, the members of $\{b_{(3,2,1)}, f_{(3,2,1)}\}$ do not produce neither the members of $\{\bar{e}_{(3,1^3)}, \bar{k}_{(3,1^3)}\}$ nor the members of $\{e_{(3,1^3)}, k_{(3,1^3)}\}$, and similarly for $\{e_{(3,2,1)}, k_{(3,2,1)}\}$. In the end we note that in accord to Corollary 3.13 the two chiral pairs $\{b_{(3,2,1)}, f_{(3,2,1)}\}$ and $\{e_{(3,2,1)}, k_{(3,2,1)}\}$ are distinguishable via substitution reactions among the elements of $T_{D_4;G}$, and hence among the elements of $T_{D_5;G}$.

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