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PI indices of tori $T_{p,q}[C_4, C_8]$ covered by C_4 and C_8[1]

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Abstract

The Padmakar-Ivan (PI) index of a graph $G = (V, E)$ is defined as $PI(G) = \sum_{e \in E}(n_u(e) + n_v(e))$, where $e = uv$, $n_u(e)$ is the number of edges of G lying closer to u than to v and $n_v(e)$ is the number of edges of G lying closer to v than to u. In this paper, a formula for calculating the PI index of a torus $T_{p,q}[C_4, C_8]$ is given.

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1 Introduction

Since the Wiener index was introduced by Wiener [1] in the study of paraffin boiling points, many topological indices have been designed [2]. Such a proliferation is still going on and is becoming counter productive. In 1990s, Gutman [3] and coworkers [4] introduced a generalization of the Wiener index (W) for cyclic graphs called Szeged index (Sz). The main advantage of the Szeged index is that it is a modification of W; otherwise, it coincides with the Wiener index. In [5,6] another topological index was introduced and it was named Padmakar-Ivan index, abbreviated as PI. This new topological index, PI, does not coincide with the Wiener index. Deng [9,10] gave the formulas for calculating the PI indices of $TUVC_6[2p, q]$ and catacondensed hexagonal systems and characterized the extremal catacondensed hexagonal systems with the minimum or maximum PI index. Ashrafi and Loghman [11] computed the PI index of zig-zag polyhex nanotubes.

The primary aim of this article is to introduce the method for calculation of PI index for a torus covering by C_4 and C_8. Our notations are mainly taken from [7,8]. Throughout this paper $G = T_{p,q}[C_4, C_8]$ denotes a torus covering by C_4 and C_8 with $2q$ rows and $2p$ columns in its cutting, see Figure 1.

2 The definition of PI index

Let G be a connected and undirected graph without multiple edges or loops. By $V(G)$ and $E(G)$ we denote the vertex and edge sets, respectively, of G.

If $G' = (V', E')$ is a subgraph of $G = (V, E)$ and contains all the edges of G that join two vertices in V', i.e., E' is the set of edges between vertices of V', then G' is an induced subgraph of G by V' and is denoted by $G[V']$.

Let $e = xy$ be an edge of G, X is the subset of vertices which are closer to x than y and Y is the subset of vertices which are closer to y than x in $V(G)$, i.e.,

$$X = \{v | v \in V(G), d_G(x, v) < d_G(y, v)\}$$
$$Y = \{v | v \in V(G), d_G(y, v) < d_G(x, v)\}$$

where $d_G(u, v)$ denotes the distance between vertices u and v of G. Let $G[X] = (X, E_1)$ and $G[Y] = (Y, E_2)$, $n_1(e) = |E_1|$, $n_2(e) = |E_2|$, here $n_1(e)$ is the number of edges nearer to x than y and $n_2(e)$ is the number of edges nearer to y than x.

Then the PI index of G is defined as

$$PI(G) = \sum_{e \in E(G)} [n_1(e) + n_2(e)]$$

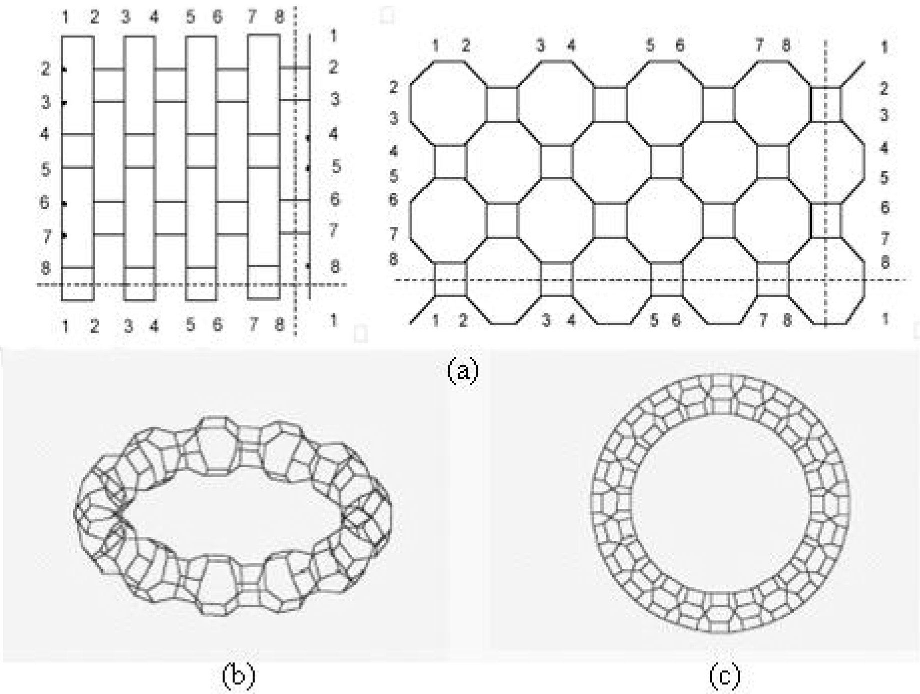

Figure 1. (a) a cutting of $G = T_{4,2}[C_4, C_8]$;
(b) side view; (c) top view.

In all cases of cyclic graphs, there are edges equidistant to the both ends of the edges. Such edges are not taken into account. Let $[X, Y]$ denote the subset of edges between X and Y, $n(e) = |[X, Y]|$. Then $n(e) = |E(G)| - (n_1(e) + n_2(e))$ is the number of edges equidistant to the both ends of e for a bipartite connected graph G (It includes the current edge e in $n(e)$). And

$$PI(G) = |E(G)|^2 - \sum_{e \in E(G)} n(e)$$

Therefore, for computing the PI index of a bipartite connected graph G, it is enough to calculate $n(e)$ for each $e \in E(G)$.

For the horizontal and vertical edges, we can observe the following results by the symmetry of the tori $T_{p,q}[C_4, C_8]$

Lemma 1. Let e be any horizontal edge in $G = T_{p,q}[C_4, C_8]$, then $n(e) = 4q$.

Proof. Let e be any horizontal edge between columns j and j+1 in $G = T_{p,q}[C_4, C_8]$, $1 \leq j \leq 2p$, where $2p + 1 \equiv 1(mod 2p)$. Then all the edges equidistant to the both ends of e are the edges between columns j and j+1 or between columns p+j and p+j+1. So, $n(e) = 4q$.

Lemma 2. Let e be any vertical edge in $G = T_{p,q}[C_4, C_8]$, then $n(e) = 4p$.

Proof. Let e be any vertical edge between rows i and i+1 in $G = T_{p,q}[C_4, C_8]$, $1 \leq i \leq 4q$, where $4q + 1 \equiv 1(mod 4q)$. Then all the edges equidistant to the both ends of e are the edges between rows i and i+1 or between rows 2q+i and 2q+i+1. So, $n(e) = 4p$.

To calculating $n(e)$ for the oblique edges e, we need only calculate $n(e)$ for $e = x_{11}x_{21}$ by the symmetry of $G = T_{p,q}[C_4, C_8]$.

3 The distances in $G = T_{p,q}[C_4, C_8]$

For $e = x_{11}x_{21}$, we will give a formula for calculating the distances from x_{11} (or x_{21}) in the following, and find the subset X of vertices of $V(G)$ which are closer to x_{11} than x_{21} and the subset Y of vertices which are closer to x_{21} than x_{11}.

We first consider four graphs G_1, G_2, G_3 and G_4, where G_1 is obtaining from $G = T_{p,q}[C_4, C_8]$ by deleting the horizontal edges between columns 1 and 2p and the vertical edges between rows 1 and 4q (see Figure 2), G_2 is obtaining from $G = T_{p,q}[C_4, C_8]$ by deleting the horizontal edges between columns 1 and 2 and the vertical edges between rows 1 and 4q, G_3 is obtaining from $G = T_{p,q}[C_4, C_8]$ by deleting the horizontal edges between columns 1 and 2p and the vertical edges between rows 3 and 4, G_4 is obtaining from $G = T_{p,q}[C_4, C_8]$ by deleting the horizontal edges between columns 1 and 2 and the vertical edges between rows 3 and 4. And the distances from x_{11} (or x_{21}) in G is the minimum of the ones in G_1, G_2, G_3 and G_4.

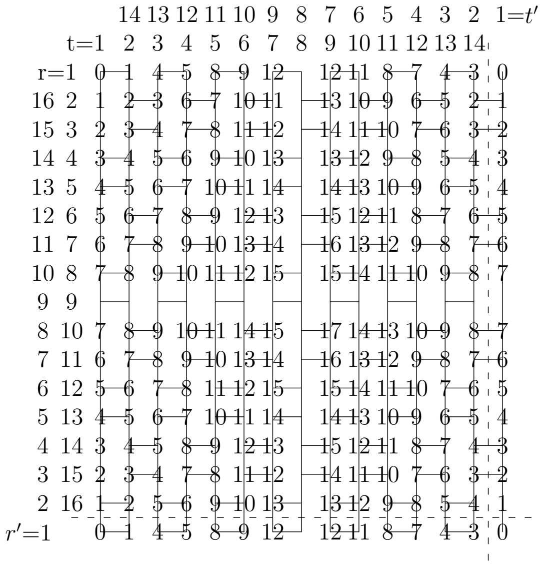

Figure 2. Some distances from the vertex x_{11} in G_1, G_2, G_3 and G_4, where p=7 and q=4.

Table 1. The values of $d_1(x_{11}, x_{rt}) - t$.

	1	2	3	4	5	6	7	8	9	10	11	12
1	-1	-1	1	1	3	3	5	5	7	7	9	9
2	0	0	0	2	2	4	4	6	6	8	8	10
3	1	1	1	3	3	5	5	7	7	9	9	11
4	2	2	2	2	4	4	6	6	8	8	10	10
5	3	3	3	3	5	5	7	7	9	9	11	11
6	4	4	4	4	4	6	6	8	8	10	10	12
7	5	5	5	5	5	7	7	9	9	11	11	13
8	6	6	6	6	6	6	8	8	10	10	12	12
9	7	7	7	7	7	7	9	9	11	11	13	13

Now, we calculate the distances from x_{11} in G_1 as showing in Figure 2. And Table 1 lists the values of $d_1(x_{11}, x_{rt}) - t$, where $d_1(x_{11}, x_{rt})$ is the distance between x_{11} and x_{rt} in G_1.

From Table 1, we can see that

$$d_1(x_{11}, x_{rt}) - t = \begin{cases} r-2, & 1 \leq t \leq [\frac{r}{2}] + 2; \\ 2[\frac{2t+r+1}{4}] - 3, & t \geq [\frac{r}{2}] + 3 \text{ and r is odd}; \\ 2[\frac{2t+r-2}{4}] - 2, & t \geq [\frac{r}{2}] + 3 \text{ and r is even}. \end{cases}$$

where $[x]$ denotes the maximum integer not larger than x over all the paper. So, we have

Lemma 3. $d_1(x_{11}, x_{rt}) = t + \begin{cases} r-2, & 1 \le t \le [\frac{r}{2}]+2; \\ 2[\frac{2t+r+1}{4}]-3, & t \ge [\frac{r}{2}]+3 \text{ and r is odd;} \\ 2[\frac{2t+r-2}{4}]-2, & t \ge [\frac{r}{2}]+3 \text{ and r is even.} \end{cases}$

Lemma 3 can be easily proved by the inductive method on t, we omit here.

Table 2. The values of $d_2(x_{11}, x_{rt'}) - t'$.

	1	2	3	4	5	6	7	8	9	10	11	12
1	-1	1	1	3	3	5	5	7	7	9	9	11
2	0	0	2	2	4	4	6	6	8	8	10	10
3	1	1	3	3	5	5	7	7	9	9	11	11
4	2	2	2	4	4	6	6	8	8	10	10	12
5	3	3	3	5	5	7	7	9	9	11	11	13
6	4	4	4	4	6	6	8	8	10	10	12	12
7	5	5	5	5	7	7	9	9	11	11	13	13
8	6	6	6	6	6	8	8	10	10	12	12	14
9	7	7	7	7	7	9	9	11	11	13	13	15

Similarly, we calculate the distances from x_{11} in G_2 as showing in Figure 2. And Table 2 lists the values of $d_2(x_{11}, x_{rt'}) - t'$, where $d_2(x_{11}, x_{rt'})$ is the distance between x_{11} and $x_{rt'}$ in G_2 and

$$t' = \begin{cases} 1, & t = 1 \\ 2p+2-t, & t \ge 2 \end{cases}$$

From Table 2, we can see that

$$d_2(x_{11}, x_{rt'}) - t' = \begin{cases} r-2, & 1 \le t' \le [\frac{r}{2}]+1; \\ 2[\frac{2t'+r}{4}]-1, & t' \ge [\frac{r}{2}]+2 \text{ and r is odd;} \\ 2[\frac{2t'+r}{4}]-2, & t' \ge [\frac{r}{2}]+2 \text{ and r is even.} \end{cases}$$

So, we have

Lemma 4. $d_2(x_{11}, x_{rt'}) = t' + \begin{cases} r-2, & 1 \le t' \le [\frac{r}{2}]+1; \\ 2[\frac{2t'+r}{4}]-1, & t' \ge [\frac{r}{2}]+2 \text{ and r is odd;} \\ 2[\frac{2t'+r}{4}]-2, & t' \ge [\frac{r}{2}]+2 \text{ and r is even} \end{cases}$

Table 3. The values of $d_3(x_{11}, x_{r't}) - t$.

	1	2	3	4	5	6	7	8	9	10	11	12
1	-1	-1	1	1	3	3	5	5	7	7	9	9
2	0	0	2	2	4	4	6	6	8	8	10	10
3	1	1	1	3	3	5	5	7	7	9	9	11
4	2	2	2	4	4	6	6	8	8	10	10	12
5	3	3	3	3	5	5	7	7	9	9	11	11
6	4	4	4	4	6	6	8	8	10	10	12	12
7	5	5	5	5	5	7	7	9	9	11	11	13
8	6	6	6	6	6	8	8	10	10	12	12	14
9	7	7	7	7	7	7	9	9	11	11	13	13

From Table 3, we can see the values of $d_3(x_{11}, x_{r't}) - t$

$$d_3(x_{11}, x_{r't}) - t = \begin{cases} r' - 2, & 1 \leq t \leq [\frac{r'-1}{2}] + 2; \\ 2[\frac{2t+r'+1}{4}] - 3, & t \geq [\frac{r'-1}{2}] + 3 \text{ and } r' \text{ is odd;} \\ 2[\frac{2t+r'}{4}] - 2, & t \geq [\frac{r'-1}{2}] + 3 \text{ and } r' \text{ is even.} \end{cases}$$

where $d_3(x_{11}, x_{r't})$ is the distance between x_{11} and $x_{r't}$ in G_3, and

$$r' = \begin{cases} 1, & r = 1 \\ 4q + 2 - r, & r \geq 2 \end{cases}$$

.

So, we have

Lemma 5. $d_3(x_{11}, x_{r't}) = t + \begin{cases} r' - 2, & 1 \leq t \leq [\frac{r'-1}{2}] + 2; \\ 2[\frac{2t+r'+1}{4}] - 3, & t \geq [\frac{r'-1}{2}] + 3 \text{ and } r' \text{ is odd;} \\ 2[\frac{2t+r'}{4}] - 2, & t \geq [\frac{r'-1}{2}] + 3 \text{ and } r' \text{ is even.} \end{cases}$

Table 4. The values of $d_4(x_{11}, x_{r't'}) - t'$.

	1	2	3	4	5	6	7	8	9	10	11	12
1	-1	1	1	3	3	5	5	7	7	9	9	11
2	0	2	2	4	4	6	6	8	8	10	10	12
3	1	1	3	3	5	5	7	7	9	9	11	11
4	2	2	4	4	6	6	8	8	10	10	12	12
5	3	3	3	5	5	7	7	9	9	11	11	13
6	4	4	4	6	6	8	8	10	10	12	12	14
7	5	5	5	5	7	7	9	9	11	11	13	13
8	6	6	6	6	8	8	10	10	12	12	14	14
9	7	7	7	7	7	9	9	11	11	13	13	15

From Table 4, we can see the values of $d_4(x_{11}, x_{r't'}) - t'$

$$d_4(x_{11}, x_{r't'}) - t' = \begin{cases} r'-2, & 1 \le t' \le [\frac{r'-1}{2}]+1; \\ 2[\frac{2t'+r'}{4}]-1, & t' \ge [\frac{r'-1}{2}]+2 \text{ and } r' \text{ is odd;} \\ 2[\frac{2t'+r'-2}{4}], & t' \ge [\frac{r'-1}{2}]+2 \text{ and } r' \text{ is even.} \end{cases}$$

where $d_4(x_{11}, x_{r't'})$ is the distance between x_{11} and $x_{r't'}$ in G_4.
So, we have

Lemma 6. $d_4(x_{11}, x_{r't'}) = t' + \begin{cases} r'-2, & 1 \le t' \le [\frac{r'-1}{2}]+1; \\ 2[\frac{2t'+r'}{4}]-1, & t' \ge [\frac{r'-1}{2}]+2 \text{ and } r' \text{ is odd;} \\ 2[\frac{2t'+r'-2}{4}], & t' \ge [\frac{r'-1}{2}]+2 \text{ and } r' \text{ is even.} \end{cases}$

Since the vertices x_{rt} in G_1, $x_{rt'}$ in G_2, $x_{r't}$ in G_3 and $x_{r't'}$ in G_4 are identical, we have

Lemma 7. (i)If $t = 1$, then $d_1(x_{11}, x_{rt}) = d_2(x_{11}, x_{rt'})$ and $d_3(x_{11}, x_{r't}) = d_4(x_{11}, x_{r't'})$;

(ii) If $2 \le t \le p+1$, then $d_1(x_{11}, x_{rt}) \le d_2(x_{11}, x_{rt'})$ and $d_3(x_{11}, x_{r't}) \le d_4(x_{11}, x_{r't'})$;

(iii) If $p+2 \le t \le 2p$, then $d_1(x_{11}, x_{rt}) \ge d_2(x_{11}, x_{rt'})$ and $d_3(x_{11}, x_{r't}) \ge d_4(x_{11}, x_{r't'})$;

(iv)If $r = 1$, then $d_1(x_{11}, x_{rt}) = d_3(x_{11}, x_{r't})$ and $d_2(x_{11}, x_{rt'}) = d_4(x_{11}, x_{r't'})$;

(v) If $2 \le r \le 2q+1$, then $d_1(x_{11}, x_{rt}) \le d_3(x_{11}, x_{r't})$ and $d_2(x_{11}, x_{rt'}) \le d_4(x_{11}, x_{r't'})$;

(vi) If $2q+2 \le r \le 4q$, then $d_1(x_{11}, x_{rt}) \ge d_3(x_{11}, x_{r't})$ and $d_2(x_{11}, x_{rt'}) \ge d_4(x_{11}, x_{r't'})$.

Proof. (i) It is immediate from Lemmas 3 $\sim$ 6.

(ii) $2 \le t \le p+1$.

Case 1. $t \ge [\frac{r}{2}]+3$. Then $[\frac{r}{2}]+3 \le t \le p+1$ and $[\frac{r}{2}] \le p-2$,
$t' = 2p+2-t \ge p+1 \ge [\frac{r}{2}]+2$.

(a) If r is even, then by Lemmas 3 and 4

$$\begin{aligned} d_2(x_{11}, x_{rt'}) - d_1(x_{11}, x_{rt}) &= (t' + 2[\tfrac{2t'+r}{4}] - 2) - (t + 2[\tfrac{2t+r-2}{4}] - 2) \\ &= 4p + 4 - 2t + 2([\tfrac{-2t+r}{4}] - [\tfrac{2t+r-2}{4}]) \\ &\ge 4p + 4 - 4t \quad (\text{since } [\tfrac{-2t+r}{4}] - [\tfrac{2t+r-2}{4}] \ge -t) \\ &\ge 0. \end{aligned}$$

(b) If r is odd, then by Lemmas 3 and 4

$$\begin{aligned} d_2(x_{11}, x_{rt'}) - d_1(x_{11}, x_{rt}) &= (t' + 2[\tfrac{2t'+r}{4}] - 1) - (t + 2[\tfrac{2t+r+1}{4}] - 3) \\ &= 4p + 6 - 2t + 2([\tfrac{-2t+r}{4}] - [\tfrac{2t+r+1}{4}]) \\ &\ge 4p + 6 - 4t > 0. \end{aligned}$$

Case 2. $2 \le t \le [\frac{r}{2}]+2$.

(a) If $t' \le [\frac{r}{2}]+1$, then by Lemmas 3 and 4

$$\begin{aligned} d_2(x_{11}, x_{rt'}) - d_1(x_{11}, x_{rt}) &= (r + t' - 2) - (r + t - 2) \\ &= t' - t = 2p + 2 - 2t \ge 0. \end{aligned}$$

(b) If $t' \geq [\frac{r}{2}]+2$, i.e., $2p+2-t \geq [\frac{r}{2}]+2$, then $t+[\frac{r}{2}] \leq 2p$.
When r is odd, $[\frac{r}{2}] = \frac{r-1}{2}$. And $2t+r \leq 4p+1$, $r-2t \geq r-2[\frac{r}{2}]-4 = -3$.
By Lemmas 2 and 3, we have

$$\begin{aligned} d_2(x_{11}, x_{rt'}) - d_1(x_{11}, x_{rt}) &= (t' + 2[\tfrac{2t'+r}{4}] - 1) - (r+t-2) \\ &= 4p+5+2[\tfrac{r-2t}{4}] - (2t+r) \\ &\geq 4p+5+2(-1) - (4p+1) > 0. \end{aligned}$$

When r is even, $[\frac{r}{2}] = \frac{r}{2}$. And $2t+r \leq 4p$, $r-2t \geq r-2[\frac{r}{2}]-4 = -4$.
By Lemmas 2 and 3, we have

$$\begin{aligned} d_2(x_{11}, x_{rt'}) - d_1(x_{11}, x_{rt}) &= (t' + 2[\tfrac{2t'+r}{4}] - 2) - (r+t-2) \\ &= 4p+4+2[\tfrac{r-2t}{4}] - (2t+r) \\ &\geq 4p+5+2(-1) - 4p > 0. \end{aligned}$$

Instead of r above by r', we can obtain the proof of $d_3(x_{11}, x_{r't}) \leq d_4(x_{11}, x_{r't'})$.

(iii) $p+2 \leq t \leq 2p$. Then $2 \leq t' = 2p+2-t \leq p$.

Case 1. $t' \geq [\frac{r}{2}]+2$.

(a) If r is even, then by Lemmas 3 and 4

$$\begin{aligned} d_1(x_{11}, x_{rt}) - d_2(x_{11}, x_{rt'}) &= (t-2+2[\tfrac{2t+r-2}{4}]) - (t'-2+2[\tfrac{2t'+r}{4}]) \\ &= (2p-t'+2([\tfrac{4p+2-2t'+r}{4}]) - (t'-2+2[\tfrac{2t'+r}{4}]) \\ &= 4p+2-2t'+2([\tfrac{r-2t'+2}{4}] - [\tfrac{2t'+r}{4}]) \\ &\geq 4p+2-2t'+2(\tfrac{r-2t'-1}{4} - \tfrac{2t'+r}{4}) \\ &> 4p+2-2t'+2(t'-1) \geq 0. \end{aligned}$$

(b) If r is odd, then by Lemmas 3 and 4

$$\begin{aligned} d_1(x_{11}, x_{rt}) - d_2(x_{11}, x_{rt'}) &= (t-3+2[\tfrac{2t+r+1}{4}]) - (t'-1+2[\tfrac{2t'+r}{4}]) \\ &= 4p+2-2t'+2([\tfrac{r-2t'-1}{4}] - [\tfrac{2t'+r}{4}]) \\ &\geq 4p-4t'+2 > 0. \end{aligned}$$

Case 2. $2 \leq t' \leq [\frac{r}{2}]+1$.

(a) If $t \leq [\frac{r}{2}]+2$, then by Lemmas 3 and 4

$$\begin{aligned} d_1(x_{11}, x_{rt}) - d_2(x_{11}, x_{rt'}) &= (r+t-2) - (r+t'-2) \\ &= t-t' = 2p+2-2t' > 0. \end{aligned}$$

(b) If $t \geq [\frac{r}{2}]+3$, then by Lemmas 3 and 4

$$\begin{aligned} d_1(x_{11}, x_{rt}) - d_2(x_{11}, x_{rt'}) &= (t-2+2[\tfrac{2t+r-2}{4}]) - (r+t'-2) \\ &\geq (t-2+2[\tfrac{2r+4}{4}]) - (r+t'-2) \\ &= (t+r) - (r+t'-2) = 2p+4-2t' > 0 \end{aligned}$$

when r is even; and

$$\begin{aligned} d_1(x_{11}, x_{rt}) - d_2(x_{11}, x_{rt'}) &= (t-3+2[\tfrac{2t+r+1}{4}]) - (r+t'-2) \\ &\geq (t-3+2[\tfrac{2r+6}{4}]) - (r+t'-2) \\ &\geq (t-3+2(\tfrac{2r+3}{4})) - (r+t'-2) \\ &> t-t' = 2p+2-2t' > 0 \end{aligned}$$

when r is odd.

Instead of r above by r', we can obtain the proof of $d_3(x_{11}, x_{r't}) \geq d_4(x_{11}, x_{r't'})$.

Using the same methods as (i),(ii) and (iii), we can prove (iv),(v) and (vi). We omit these here.

Now by Lemma 7, we can directly give a formula of calculating the distances from x_{11} in $G = T_{p,q}[C_4, C_8]$.

Theorem 1. (i) $d(x_{11}, x_{rt}) = d_1(x_{11}, x_{rt})$ if $1 \le t \le p+1$ and $1 \le r \le 2q+1$;
(ii) $d(x_{11}, x_{rt}) = d_2(x_{11}, x_{rt})$ if $p+2 \le t \le 2p$ and $1 \le r \le 2q+1$;
(iii) $d(x_{11}, x_{rt}) = d_3(x_{11}, x_{rt})$ if $1 \le t \le p+1$ and $2q+2 \le r \le 4q$;
(iv) $d(x_{11}, x_{rt}) = d_4(x_{11}, x_{rt})$ if $p+2 \le t \le 2p$ and $2q+2 \le r \le 4q$.

Next, we consider the distances from x_{21}. Using the same methods as above, we can calculate the distances from x_{21} in G_1, G_2, G_3 and G_4 (see Figure 3).

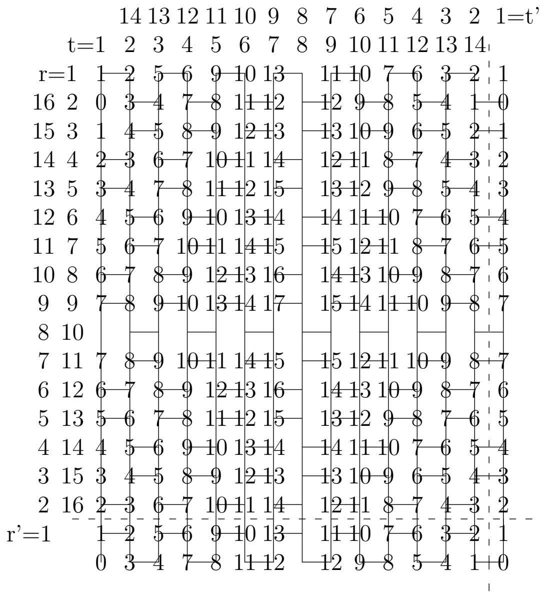

Figure 3. Some distances from the vertex x_{21} in G_1, G_2, G_3 and G_4.

Table 5. The values of $d_1(x_{21}, x_{rt}) - t$.

	1	2	3	4	5	6	7	8	9	10	11	12
1	0	0	2	2	4	4	6	6	8	8	10	10
2	-1	1	1	3	3	5	5	7	7	9	9	11
3	0	2	2	4	4	6	6	8	8	10	10	12
4	1	1	3	3	5	5	7	7	9	9	11	11
5	2	2	4	4	6	6	8	8	10	10	12	12
6	3	3	3	5	5	7	7	9	9	11	11	13
7	4	4	4	6	6	8	8	10	10	12	12	14
8	5	5	5	5	7	7	9	9	11	11	13	13
9	6	6	6	6	8	8	10	10	12	12	14	14

We can see from Table 5 that

$$d_1(x_{21}, x_{rt}) - t = \begin{cases} r-3, & 1 \le t \le [\frac{r}{2}]; \\ 2[\frac{2t+r-2}{4}] - 1, & t \ge [\frac{r}{2}] + 1 \text{ and r is even;} \\ 2[\frac{2t+r-3}{4}], & t \ge [\frac{r}{2}] + 1 \text{ and r is odd.} \end{cases}$$

So, we have

Lemma 8. $d_1(x_{21}, x_{rt}) = t + \begin{cases} r-3, & 1 \le t \le [\frac{r}{2}]; \\ 2[\frac{2t+r-2}{4}] - 1, & t \ge [\frac{r}{2}] + 1 \text{ and r is even;} \\ 2[\frac{2t+r-3}{4}], & t \ge [\frac{r}{2}] + 1 \text{ and r is odd.} \end{cases}$

Table 6. The values of $d_2(x_{21}, x_{rt'}) - t'$.

	1	2	3	4	5	6	7	8	9	10	11	12
1	0	0	0	2	2	4	4	6	6	8	8	10
2	-1	-1	1	1	3	3	5	5	7	7	9	9
3	0	0	2	2	4	4	6	6	8	8	10	10
4	1	1	1	3	3	5	5	7	7	9	9	11
5	2	2	2	4	4	6	6	8	8	10	10	12
6	3	3	3	3	5	5	7	7	9	9	11	11
7	4	4	4	4	6	6	8	8	10	10	12	12
8	5	5	5	5	5	7	7	9	9	11	11	13
9	6	6	6	6	6	8	8	10	10	12	12	14

We can see from Table 6 that $(r \ge 2)$

$$d_2(x_{21}, x_{rt'}) - t' = \begin{cases} r-3, & 1 \le t' \le [\frac{r}{2}] + 1; \\ 2[\frac{2t'+r}{4}] - 3, & t' \ge [\frac{r}{2}] + 2 \text{ and r is even;} \\ 2[\frac{2t'+r-1}{4}] - 2, & t' \ge [\frac{r}{2}] + 2 \text{ and r is odd.} \end{cases}$$

So, we have

Lemma 9. If $r = 1$, then $d_2(x_{21}, x_{rt'}) = \begin{cases} 0, & t' = 1; \\ t' + 2[\frac{t'}{2}] - 2, & t' \ge 2. \end{cases}$

If $r \geq 2$, then $d_2(x_{21}, x_{rt'}) = t' + \begin{cases} r-3, & 1 \leq t' \leq [\frac{r}{2}]+1; \\ 2[\frac{2t'+r}{4}]-3, & t' \geq [\frac{r}{2}]+2 \text{ and r is even;} \\ 2[\frac{2t'+r-1}{4}]-2, & t' \geq [\frac{r}{2}]+2 \text{ and r is odd.} \end{cases}$

Table 7. The values of $d_3(x_{21}, x_{r't}) - t$.

	1	2	3	4	5	6	7	8	9	10	11	12
1	0	0	2	2	4	4	6	6	8	8	10	10
2	1	1	3	3	5	5	7	7	9	9	11	11
3	2	2	2	4	4	6	6	8	8	10	10	12
4	3	3	3	5	5	7	7	9	9	11	11	13
5	4	4	4	4	6	6	8	8	10	10	12	12
6	5	5	5	5	7	7	9	9	11	11	13	13
7	6	6	6	6	6	8	8	10	10	12	12	14
8	7	7	7	7	7	9	9	11	11	13	13	15
9	8	8	8	8	8	8	10	10	12	12	14	14

We can see that from Table 7

$$d_3(x_{21}, x_{r't}) - t = \begin{cases} r'-1, & 1 \leq t \leq [\frac{r'-1}{2}]+2; \\ 2[\frac{2t+r'+1}{4}]-2, & t \geq [\frac{r'-1}{2}]+3 \text{ and } r' \text{ is odd;} \\ 2[\frac{2t+r'}{4}]-1, & t \geq [\frac{r'-1}{2}]+3 \text{ and } r' \text{ is even.} \end{cases}$$

So, we have

Lemma 10. $d_3(x_{21}, x_{r't}) = t + \begin{cases} r'-1, & 1 \leq t \leq [\frac{r'-1}{2}]+2; \\ 2[\frac{2t+r'+1}{4}]-2, & t \geq [\frac{r'-1}{2}]+3 \text{ and } r' \text{ is odd;} \\ 2[\frac{2t+r'}{4}]-1, & t \geq [\frac{r'-1}{2}]+3 \text{ and } r' \text{ is even.} \end{cases}$

Table 8. The values of $d_4(x_{21}, x_{r't'}) - t'$.

	1	2	3	4	5	6	7	8	9	10	11	12
1	0	0	0	2	2	4	4	6	6	8	8	10
2	1	1	1	3	3	5	5	7	7	9	9	11
3	2	2	2	2	4	4	6	6	8	8	10	10
4	3	3	3	3	5	5	7	7	9	9	11	11
5	4	4	4	4	4	6	6	8	8	10	10	12
6	5	5	5	5	5	7	7	9	9	11	11	13
7	6	6	6	6	6	6	8	8	10	10	12	12
8	7	7	7	7	7	7	9	9	11	11	13	13
9	8	8	8	8	8	8	8	10	10	12	12	14

We can see that from Table 8 ($r' \geq 2$)

$$d_4(x_{21}, x_{r't'}) - t' = \begin{cases} r'-1, & 1 \leq t' \leq [\frac{r'-1}{2}]+3; \\ 2[\frac{2t'+r'-1}{4}]-2, & t' \geq [\frac{r'-1}{2}]+4 \text{ and } r' \text{ is odd;} \\ 2[\frac{2t'+r'-2}{4}]-1, & t' \geq [\frac{r'-1}{2}]+4 \text{ and } r' \text{ is even.} \end{cases}$$

So, we have

Lemma 11. If $r'=1$, then $d_4(x_{21},x_{rt'})=\begin{cases}0, & t'=1;\\ t'+2[\frac{t'}{2}]-2, & t'\geq 2.\end{cases}$

If $r'\geq 2$, then $d_4(x_{21},x_{r't'})=t'+\begin{cases}r'-1, & 1\leq t'\leq [\frac{r'-1}{2}]+3;\\ 2[\frac{2t'+r'-1}{4}]-2, & t'\geq [\frac{r'-1}{2}]+4 \text{ and } r' \text{ is odd};\\ 2[\frac{2t'+r'-2}{4}]-1, & t'\geq [\frac{r'-1}{2}]+4 \text{ and } r' \text{ is even}.\end{cases}$

As in Lemma 7, we can prove the following result by using Lemmas $8\sim 11$

Lemma 12. (i)If $t=1$, then $d_1(x_{21},x_{rt})=d_2(x_{21},x_{rt'})$ and $d_3(x_{21},x_{r't})=d_4(x_{21},x_{r't'})$;

(ii) If $2\leq t\leq p$, then $d_1(x_{21},x_{rt})\leq d_2(x_{21},x_{rt'})$ and $d_3(x_{21},x_{r't})\leq d_4(x_{21},x_{r't'})$;

(iii) If $p+1\leq t\leq 2p$, then $d_1(x_{21},x_{rt})\geq d_2(x_{21},x_{rt'})$ and $d_3(x_{21},x_{r't})\geq d_4(x_{21},x_{r't'})$;

(iv)If $r=1$, then $d_1(x_{21},x_{rt})=d_3(x_{21},x_{r't})$ and $d_2(x_{21},x_{rt'})=d_4(x_{21},x_{r't'})$;

(v) If $3\leq r\leq 2q+1$, then $d_1(x_{21},x_{rt})\leq d_3(x_{21},x_{r't})$ and $d_2(x_{21},x_{rt'})\leq d_4(x_{21},x_{r't'})$;

(vi) If $2q+2\leq r\leq 4q$, then $d_1(x_{21},x_{rt})\geq d_3(x_{21},x_{r't})$ and $d_2(x_{21},x_{rt'})\geq d_4(x_{21},x_{r't'})$.

And now, we can give a formula of calculating the distances from x_{21} in $G=T_{p,q}[C_4,C_8]$ by Lemma 12.

Theorem 2. (i) $d(x_{21},x_{rt})=d_1(x_{21},x_{rt})$ if $1\leq t\leq p$ and $1\leq r\leq 2q+1$;
(ii) $d(x_{21},x_{rt})=d_2(x_{21},x_{rt})$ if $p+1\leq t\leq 2p$ and $1\leq r\leq 2q+1$;
(iii) $d(x_{21},x_{rt})=d_3(x_{21},x_{rt})$ if $1\leq t\leq p$ and $2q+2\leq r\leq 4q$;
(iv) $d(x_{21},x_{rt})=d_4(x_{21},x_{rt})$ if $p+1\leq t\leq 2p$ and $2q+2\leq r\leq 4q$.

The methods of calculating $d(x_{11},x_{rt})$ and $d(x_{21},x_{rt})$ in Theorems 1 and 2 can be showed in Figure 4.

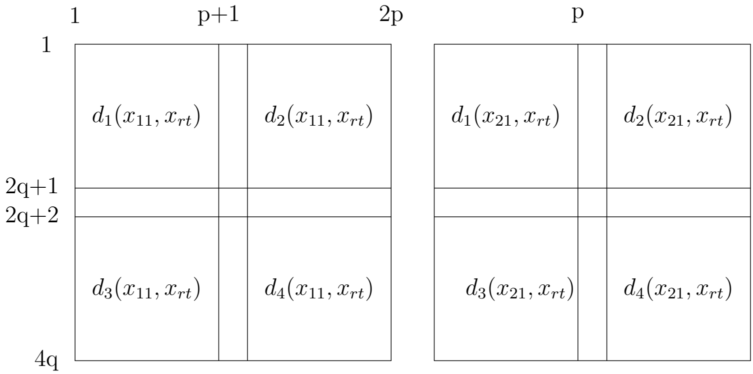

Figure 4.

4 A formula for calculating PI index of $G = T_{p,q}[C_4, C_8]$

In this section, we first find the subset X of vertices of $V(G)$ which are closer to x_{11} than x_{21} and the subset Y of vertices which are closer to x_{21} than x_{11} in G, and give the formula of calculating $n(e)$ for all oblique edges e. And then we calculate the PI index of $G = T_{p,q}[C_4, C_8]$.

Let $D = d(x_{11}, x_{rt}) - d(x_{21}, x_{rt})$. Then $x_{rt} \in X$ if and only if $D < 0$.

Case I. $1 \leq t \leq p$ and $1 \leq r \leq 2q + 1$.

By Theorems 1 and 2, we have $D = d_1(x_{11}, x_{rt}) - d_1(x_{21}, x_{rt})$, and

(i) $D < 0$ for $1 \leq r \leq 2t - 1$;

(ii) $D > 0$ for $2t \leq r \leq 2q + 1$.

Case II. $1 \leq t \leq p$ and $2q + 2 \leq r \leq 4q$.

By Theorems 1 and 2, we have $D = d_3(x_{11}, x_{r't}) - d_3(x_{21}, x_{r't}) < 0$.

Case III. $p + 2 \leq t \leq 2p$ and $1 \leq r \leq 2q + 1$.

By Theorems 1 and 2, we have $D = d_2(x_{11}, x_{rt'}) - d_2(x_{21}, x_{rt'}) > 0$.

Case IV. $p + 2 \leq t \leq 2p$ and $2q + 2 \leq r \leq 4q$.

By Theorems 1 and 2, we have $D = d_4(x_{11}, x_{r't'}) - d_4(x_{21}, x_{r't'})$, and

(i) $D < 0$ for $2t' - 1 \leq r' \leq 2q$;

(ii) $D > 0$ for $2 \leq r' \leq 2t' - 2$.

Case V. $t = p + 1$ ($t' = p + 1$).

(i) $1 \leq r \leq 2q + 1$.

By Theorems 1 and 2, we have $D = d_1(x_{11}, x_{rt}) - d_2(x_{21}, x_{rt'})$.

$$\text{When } 1 \le r \le 2p-3, \begin{cases} D > 0, & 2p + r \equiv 1(mod4) \text{ and r is odd;} \\ D < 0, & 2p + r \equiv 3(mod4) \text{ and r is odd;} \\ D > 0, & 2p + r \equiv 0(mod4) \text{ and r is even;} \\ D < 0, & 2p + r \equiv 2(mod4) \text{ and r is even.} \end{cases}$$

When $r = 2p-2, 2p-1$, $D < 0$.
When $r \ge 2p$, $D > 0$.
(ii) $2q + 2 \le r \le 4q$.
By Theorems 1 and 2, we have $D = d_3(x_{11}, x_{r't}) - d_4(x_{21}, x_{r't'})$.
When $2 \le r' \le 2p-4$ (i.e., $4q - 2p + 6 \le r \le 4q$),

$$\begin{cases} D > 0, & 2p + r' \equiv 1(mod4) \text{ and } r' \text{ is odd;} \\ D < 0, & 2p + r' \equiv 3(mod4) \text{ and } r' \text{ is odd;} \\ D > 0, & 2p + r' \equiv 2(mod4) \text{ and } r' \text{ is even;} \\ D < 0, & 2p + r' \equiv 0(mod4) \text{ and } r' \text{ is even.} \end{cases}$$

When $r' = 2p-3, 2p-2$ (i.e., $r = 4q - 2p + 5, 4q - 2p + 4$), $D > 0$.
When $2p - 1 \le r' \le 2q$ (i.e., $2q + 2 \le r \le 4q - 2p + 3$), $D < 0$.

A example for $p = 7$ and $q = 4$ is showed in Figure 5, where X is the set of large dots.

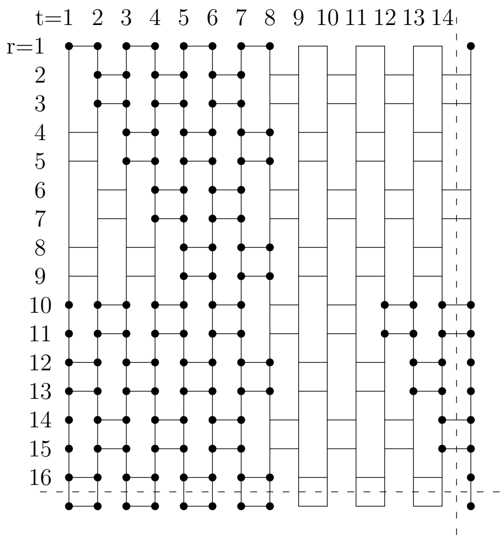

Figure 5.
Now, we calculate $n(e) = |[X, Y]|$ for the oblique edge $e = x_{11}x_{21}$.

Lemma 13. Let e be an oblique edge, then $n(e) = \begin{cases} 6q - 2, & \text{if } q \le p; \\ 6p - 2, & \text{if } q > p. \end{cases}$

Proof. We need only calculate $n(e)$ for $e = x_{11}x_{21}$ by the symmetry of G.

(1) If $q \leq p$, then there is no edge in $[X, Y]$ for $t \in \{q+1, q+2, \cdots, 2p-q+1\}$ and $t \neq p+1$, and there are two edges in $[X, Y]$ for $t \in \{1, 2, \cdots, q\} \cup \{2p-q+2, 2p-q+3, \cdots, 2p\}$ from Cases I-IV above.

For $t = p+1$, if p is even, then $D < 0$ if and only if $r \equiv 2, 3(mod4)$; and if p is odd, then $D < 0$ if and only if $r \equiv 1, 0(mod4)$, from Case V above.

So, $n(e) = 2(q + (q-1)) + 2q = 6q - 2$.

(2) If $q > p$, then there are two edges in $[X, Y]$ for $t \in \{1, 2, \cdots, p\} \cup \{p+2, p+3, \cdots, 2p\}$ from Cases I-IV above.

For $t = p+1$, by Case V above, we have

(i) p is even.

$D < 0$ for $r \equiv 2, 3(mod4)$ and $r \in \{1, 2, \cdots, 2p-3\} \cup \{4q-2p+6, \cdots, 4q\}$;

$D < 0$ for $r = 2p-2, 2p-1$;

$D > 0$ for $r = 2p, 2p+1, \cdots, 2q+1$;

$D < 0$ for $r = 2q+2, 2q+3, \cdots, 4q-2p+3$;

$D > 0$ for $r = 4q-2p+4, 4q-2p+5$.

Then, there are p edges in $[X, Y]$ when r goes from 1 to $2p$, no edge in $[X, Y]$ when r goes from $2p$ to $2q+1$, one edge in $[X, Y]$ when r goes from $2q+1$ to $2q+2$, no edge in $[X, Y]$ when r goes from $2q+2$ to $4q-2p+3$, one edge in $[X, Y]$ when r goes from $4q-2p+3$ to $4q-2p+4$, $p-2$ edges in $[X, Y]$ when r goes from $4q-2p+4$ to $4p$; i.e., there are $2p$ edges in $[X, Y]$ when $t = p+1$.

So, $n(e) = 2(p + (p-1)) + 2p = 6p - 2$.

(ii) p is odd.

$D < 0$ for $r \equiv 1, 0(mod4)$ and $r \in \{1, 2, \cdots, 2p-3\} \cup \{4q-2p+6, \cdots, 4q\}$;

$D < 0$ for $r = 2p-2, 2p-1$;

$D > 0$ for $r = 2p, 2p+1, \cdots, 2q+1$;

$D < 0$ for $r = 2q+2, 2q+3, \cdots, 4q-2p+3$;

$D > 0$ for $r = 4q-2p+4, 4q-2p+5$.

Then, there are p edges in $[X, Y]$ when r goes from 1 to $2p$, no edge in $[X, Y]$ when r goes from $2p$ to $2q+1$, one edge in $[X, Y]$ when r goes from $2q+1$ to $2q+2$, no edge in $[X, Y]$ when r goes from $2q+2$ to $4q-2p+3$, $p-1$ edges in $[X, Y]$ when r goes from $4q-2p+3$ to $4p$; i.e., there are $2p$ edges in $[X, Y]$ when $t = p+1$.

So, $n(e) = 2(p + (p-1)) + 2p = 6p - 2$.

Using Lemmas 1,2 and 13, we can give a formula for calculating PI index of $G = T_{p,q}[C_4, C_8]$.

Theorem 3. The PI index of $G = T_{p,q}[C_4, C_8]$ is

$$PI(G) = \begin{cases} 144p^2q^2 - 16p^2q - 40pq^2 + 8pq, & \text{if } q \leq p; \\ 144p^2q^2 - 40p^2q - 16pq^2 + 8pq, & \text{if } q \geq p+1. \end{cases}$$

Proof. Let E_1, E_2 and E_3 be the sets of horizontal edges, vertical edges and oblique edges, respectively. Then we have $|E(G)| = 12pq$, and $|E_1(G)| = |E_2(G)| = E_3(G)| = 4pq$. By Lemmas 1,2 and 13, we have that

$$\begin{aligned} PI(G) &= |E(G)|^2 - \sum_{e\in E(G)} n(e) \\ &= 144p^2q^2 - \sum_{e\in E_1} n(e) - \sum_{e\in E_2} n(e) - \sum_{e\in E_3} n(e) \\ &= \begin{cases} 144p^2q^2 - 16p^2q - 40pq^2 + 8pq, & \text{if } q \le p; \\ 144p^2q^2 - 40p^2q - 16pq^2 + 8pq, & \text{if } q \ge p+1. \end{cases} \end{aligned}$$

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