MATCH
Communications in Mathematical and in Computer Chemistry

MATCH Commun. Math. Comput. Chem. **57** (2007) 363-370

ISSN 0340 - 6253

EQUIENERGETIC BIPARTITE GRAPHS

Yaoping Hou[a] and **Lixin Xu**[b]

[a]*Department of Mathematics, Hunan Normal University, Changsha, Hunan 410081, China*
email: yphou@hunnu.edu.cn

[b]*Department of Mathematics, Shaoyang University, Shaoyang, Hunan, China*

(Received September 19, 2006)

Abstract

The energy of a graph is the sum of the absolute values of its eigenvalues. Two graphs are said to be equienergetic if their energies are equal. We show how infinitely many pairs of equienergetic bipartite graphs can be constructed, such that these bipartite graphs are connected, possess equal number of vertices, equal number of edges, and are not cospectral.

INTRODUCTION

The concept of graph energy was introduced by Gutman long times ago. Recently this concept started to attract considerable attention of mathematicians involved in the study of graph spectral theory; for recent mathematical works on the energy of graphs see [2,3,5,7–10,13–15] and the references of quoted therein.

Using the notation and terminology of the paper [10], we denote by $\lambda_1, \lambda_2, \ldots, \lambda_n$ the eigenvalues of a graph G and by n the number of its vertices. The energy of the graph G is then defined as $E(G) = |\lambda_1| + |\lambda_2| + \cdots + |\lambda_n|$. Two graphs G_1 and G_2 are said to be equienergetic if $E(G_1) = E(G_2)$.

Let $L^2(G)$ denote the second iterated line graph of the graph G . In the papers [10, 11, 12] the following theorem has been proved.

Theorem 1. *Let G_1 and G_2 be two regular graphs, both on n vertices, both of degree $r \geq 3$. Then*

(1). $L^2(G_1)$ and $L^2(G_2)$ are equienergetic, and $E(L^2(G_1)) = E(L^2(G_2)) = 2nr(r-2)$.

(2). $\overline{L^2(G_1)}$ and $\overline{L^2(G_2)}$ are equienergetic, and $E(\overline{L^2(G_1)}) = E(\overline{L^2(G_2)}) = (nr-4)(2r-3)-2$, where $\overline{G}$ denotes the complement of the graph G .

Theorem 1 gives a systematic method for constructing pairs of equienergetic graphs. On the other hand, until now no systematic method for constructing pairs of equienergetic bipartite graphs was reported. Here we are now able to offer one, that is similar to Theorem 1.

For studying networks, N. Alon [1] introduced the extended double cover of a graph to obtain expanders from magnifier. This motivated our interest in studying the energies of the extended double graphs.

Let G be a simple graph with vertex set $V = \{v_1, v_2, \ldots, v_n\}$. The extended double cover of G , denote by G^* is the bipartite graph with bipartition (X, Y) where $X = \{x_1, x_2, \ldots, x_n\}$ and $Y = \{y_1, y_2, \ldots, y_n\}$, in which x_i and y_j are adjacent if and only if $i = j$ or v_i and v_j are adjacent in G . It is easy to see that G^* is connected if and only if G is connected and G^* is regular of degree $r+1$ if and only if G is regular of degree r .

The following is the main result of this note.

Theorem 2. *Let G_1 and G_2 be two regular graphs, both on n vertices, both of degree $r \geq 3$. Then*

(1). $(L^2(G_1))^*$ and $(L^2(G_2))^*$ are equienergetic bipartite graphs, and $E((L^2(G_1))^*) = E((L^2(G_2))^*) = nr(3r-5)$.

(2). $(\overline{L^2(G_1)})^*$ and $(\overline{L^2(G_2)})^*$ are equienergetic bipartite graphs, and $E((\overline{L^2(G_1)})^*) = E((\overline{L^2(G_2)})^*) = (5nr-16)(r-2)+nr-8$.

(3). $\overline{(L^2(G_1))^*}$ and $\overline{(L^2(G_2))^*}$ are equienergetic graphs, and $E(\overline{(L^2(G_1))^*}) = E(\overline{(L^2(G_2))^*}) = (2nr-4)(2r-3)-2$.

PROOF OF THEOREM 2

In order to prove Theorem 2, we first give the relation between the eigenvalues of a graph G and its extended double cover G^*.

Lemma 3. *Let $\lambda_1, \lambda_2, \ldots, \lambda_n$ be the eigenvalues of the graph G. Then the eigenvalues of G^* are $\pm(\lambda_1+1), \pm(\lambda_2+1), \ldots, \pm(\lambda_n+1)$.*

Proof. Let the adjacency matrix of G be A. Then G^* has adjacency matrix $\begin{pmatrix} 0 & A+I \\ A+I & 0 \end{pmatrix}$, where I is the unit matrix. Suppose that λ is an eigenvalue of G and x is an eigenvector corresponding to λ, that is, $Ax = \lambda x$. Then we have

$$\begin{pmatrix} 0 & A+I \\ A+I & 0 \end{pmatrix} \begin{pmatrix} x \\ x \end{pmatrix} = (\lambda+1) \begin{pmatrix} x \\ x \end{pmatrix}$$

$$\begin{pmatrix} 0 & A+I \\ A+I & 0 \end{pmatrix} \begin{pmatrix} x \\ -x \end{pmatrix} = -(\lambda+1) \begin{pmatrix} x \\ -x \end{pmatrix} .$$

Thus G^* has two eigenvalues $\pm(\lambda+1)$ corresponding to the eigenvalue λ of G. Hence the lemma follows. $\square$

Proof of Theorem 2. From [11] we know that if G is a regular graph of order n and degree r, then $L^2(G)$ is a regular graph of order $nr(r-1)/2$ and degree $4r-6$. If $\lambda_1, \lambda_2, \ldots, \lambda_n$ are the eigenvalues of G, then the spectrum of $L^2(G)$ consists of the numbers

$$\left.\begin{array}{cc} \lambda_i + 3r - 6 & i = 1, 2, \ldots, n \\ 2r - 6 & n(r-2)/2 \; times \\ -2 & nr(r-2)/2 \; times \end{array}\right\} \tag{1}$$

Combining Lemma 3 and (1), we obtain that the eigenvalues of $(L^2(G))^*$ are:

$$\left.\begin{array}{cc} \pm(\lambda_i + 3r - 5) & i = 1, 2, \ldots, n \\ \pm(2r - 5) & n(r-2)/2 \;\; times \\ \pm 1 & nr(r-2)/2 \;\; times \end{array}\right\} \tag{2}$$

Recall that all eigenvalues of a regular graph of degree r lie in the interval $[-r, +r]$. Therefore, we obtain $\lambda_i + 3r - 5 \geq 0$. From (2) we can express the energy of $(L^2(G))^*$ as:

$$\begin{aligned} E((L^2(G))^*) &= 2\left[\sum_{i=1}^{n}(\lambda_i + 3r - 5) + (2r-5)n(r-2)/2 + nr(r-2)/2\right] \\ &= nr(3r-5) \qquad \left(\text{recall that } \sum_{i=1}^{n} \lambda_i = 0\right) . \end{aligned}$$

If G is regular of order n and degree r, then its complement $\overline{G}$ is a regular graph of order n and degree $n - r - 1$. The spectrum of $\overline{G}$ consists of the numbers (see [4]):

$$\left.\begin{array}{cc} n - r - 1 & \\ -\lambda_i - 1 & i = 2, 3, \ldots, n \end{array}\right\} \tag{3}$$

Combining (1) and (3), we obtain that the eigenvalues of $\overline{L^2(G)}$ are:

$$\left.\begin{array}{cc} nr(r-1)/2 - 4r + 5 & \\ -\lambda_i - 3r + 5 & i = 2, 3, \ldots, n \\ -2r + 5 & n(r-2)/2 \;\; times \\ 1 & nr(r-2)/2 \;\; times \end{array}\right\} \tag{4}$$

Combining Lemma 3 and (4), we obtain that the eigenvalues $(\overline{L^2(G)})^*$ are:

$$\left.\begin{array}{cc} \pm(nr(r-1)/2 - 4r + 6) & \\ \pm(-\lambda_i - 3r + 6) & i = 2, 3, \ldots, n \\ \pm(-2r + 6) & n(r-2)/2 \;\; times \\ \pm 2 & nr(r-2)/2 \;\; times \end{array}\right\} \tag{5}$$

From (5) we can express the energy of $\overline{L^2(G)})^*$ as:

$$\begin{aligned} E(\overline{L^2(G)})^*) &= 2[nr(r-1)/2 - 4r + 6 + \sum_{i=2}^{n}(\lambda_i + 3r - 6) \\ &\quad +(2r-6)n(r-2)/2 + 2 \times nr(r-2)/2] \\ &= (5nr-16)(r-2) + nr - 8 \end{aligned}$$

Combining Lemma 3 and (5), we obtain that the eigenvalues of the of $\overline{(L^2(G))^*}$ are:

$$\left.\begin{array}{cc} nr(r-1) - 4r - 5 - 1 & \\ -(4r-5) - 1 & \\ \pm(-\lambda_i - 3r + 5) - 1 & i = 2, 3, \ldots, n \\ \pm(2r-5) - 1 & n(r-2)/2 \ \ times \\ \pm 1 - 1 & nr(r-2)/2 \ \ times \end{array}\right\} \tag{6}$$

From (6) we can express the energy of $\overline{(L^2(G))^*}$ as:

$$\begin{aligned} E(\overline{(L^2(G))^*}) &= 2[nr(r-1)/2 - 4r + 4 + 4r - 4 + \sum_{i=2}^{n}(\lambda_i + 3r - 6) \\ &+ (2r-6)n(r-2)/2 + nr(r-2)/2 \times 2] \\ &= (2nr-4)(2r-3) - 2\ . \end{aligned}$$

Hence Theorem 2 follows. □

DISCUSSION

In full analogy with the corollaries of Theorem 1 (stated in [10] and [12]), we now have:

Corollary 2.1. *Let G_1 and G_2 be two regular graphs, both on n vertices, both of degree $r \geq 3$. Then for any $k \geq 2$, the following pairs of graphs are equienergetic:*

(1) $(L^k(G_1))^*$ *and* $(L^k(G_2))^*$;

(2) $(\overline{L^k(G_1)})^*$ *and* $(\overline{L^k(G_2)})^*$;

(3) $\overline{(L^k(G_1))^*}$ *and* $\overline{(L^k(G_2))^*}$.

Corollary 2.2. *Let G_1 and G_2 be two connected and non-cospectral regular graphs, both on n vertices, both of degree $r \geq 3$. Then for any $k \geq 2$, both $(L^k(G_1))^*$ and $(L^k(G_2))^*$ are regular, bipartite, connected, non-cospectral, and equienergetic. Furthermore, $(L^k(G_1))^*$ and $(L^k(G_2))^*$ possess the same number of vertices, and the same number of edges.*

Within Theorem 2 we obtained the expression (in terms of n and r) for the energy of the extended double cover of the second iterated line graph of a regular graph. Analogous (yet much less simple) expressions could be calculated also for $E((L^k(G))^*)$, $k \geq 2$, i. e., the energy of the extended double cover of the k-th iterated line graph, $k \geq 2$, of a regular graph on n vertices and of degree $r \geq 3$ is also fully determined by the parameters n and r.

A graph G on n vertex is called hyperenergetic if $E(G) > 2n - 2$. In [6] it was shown that if G has more than $2n-1$ edges then all its iterated line graphs $L^k(G)$ are hyperenergetic. From the expressions for the energies of the graphs $L^2(G)$, $\overline{L^2(G)}$, $(L^2(G))^*$, $\overline{(L^2(G))^*}$, $(\overline{L^2(G)})^*$, it is not difficult to see that these all are hyperenergetic for $r \geq 3$ and $n \geq 5$.

Similar to the extended double cover of a graph, we may define the extended m-cover of a graph as follows:

Let G be a simple graph with vertex set $V = \{v_1, v_2, \ldots, v_n\}$. The extended m-cover ($m \geq 2$) of G, denote by $G^{(m)}$, is the m-partite graph with m-partition $(X_1, X_2, \ldots, X_m)$ where $X_1 = \{x_1^{(1)}, x_2^{(1)}, \ldots, x_n^{(1)}\}$, $X_2 = \{x_1^{(2)}, x_2^{(2)}, \ldots, x_n^{(2)}\}$, $\ldots$, $X_m = \{x_1^{(m)}, x_2^{(m)}, \ldots, x_n^{(m)}\}$, in which x_i^k and x_j^ℓ are adjacent if and only if $i = j$ or v_i and v_j are adjacent in G (for any $k \neq \ell$). It is not difficult to prove that if the characteristic polynomial (of the adjacency matrix) of G is $\phi(G, x)$, then the characteristic polynomial (of the adjacency matrix) of $G^{(m)}$ is

$$\phi(G^{(m)}, x) = \phi^{m-1}(G, -x-1)\phi\left(G, \frac{x}{m-1} - 1\right) .$$

In other words, if the eigenvalues of G are $\lambda_1, \lambda_2, \ldots, \lambda_n$, then the eigenvalues of $G^{(m)}$ are $(m-1)(\lambda_i+1)$ and $-(\lambda_i+1)$ ($m-1$ times), $i = 1, 2, \ldots, n$. From this fact, we can show that if G is an r-regular graph with n vertices, then the energy of $(L^2(G))^{(m)}$ is $[mnr(3r-5)]/2$, which is independent of the structure of the graph G.

Acknowledgement: This work was supported by NSF of China (10471037) and by SRF of Hunan Provincial Education Department.

References

[1] N. Alon, Eigenvalues and expanders, *Combinatorica* **6** (1986) 83–96.

[2] R. Balakrishnan, The energy of a graph, *Lin. Algebra Appl.* **387** (2004) 287–295.

[3] A. Chen, A. Chang, W. C. Shiu, Energy ordering of unicyclic graphs, *MATCH Commun. Math. Comput. Chem.* **55** (2006) 95–102.

[4] D. Cvetković, M. Doob, H. Sachs, *Spectra of Graphs – Theory and Application*, Academic Press, New York, 1980; 2nd revised ed. Barth, Heidelberg, 1995.

[5] I. Gutman, The energy of a graph: old and new results, in: A. Betten, A. Kohnert, R. Laue, A. Wassermann (Eds.), *Algebraic Combinatorics and Applications*, Springer–Verlag, Berlin, 2001, pp. 196–211.

[6] Y. Hou, I. Gutman, Hyperenergetic line graphs, *MATCH Commun. Math. Comput. Chem.* **43** (2001) 29–39.

[7] G. Indulal, A. Vijayakumar, On a pair of equienergetic graphs, *MATCH Commun. Math. Comput. Chem.* **55** (2006) 83–90.

[8] F. Li, B. Zhou, Minimal energy of bipartite unicyclic graphs of a given bipartition, *MATCH Commun. Math. Comput. Chem.* **54** (2005) 379–388.

[9] W. Lin, X. Guo, H. Li, On the extremal energies of trees with a given maximum degree, *MATCH Commun. Math. Comput. Chem.* **54** (2005) 363–378.

[10] H. S. Ramane, H. B. Walikar, S. B. Rao, B. D. Acharya, P. R. Hampiholi, S. R. Jog, I. Gutman, Equienergetic graphs, *Kragujevac J. Math.* **26** (2004) 5–13.

[11] H. S. Ramane, H. B. Walikar, S. B. Rao, B. D. Acharya, P. R. Hampiholi, S. R. Jog, I. Gutman, Spectra and energies of iterated line graphs of regular graphs, *Appl. Math. Lett.* **18** (2005) 679–682.

[12] H. S. Ramane, I. Gutman, H. B. Walikar, S. B. Halkarni, Another class of equienergetic graphs, *Kragujevac J. Math.* **26** (2004) 15–18.

[13] W. Yan, L. Ye, On the maximal energy and the Hosoya index of a type of trees with many pendant vertices, *MATCH Commun. Math. Comput. Chem.* **53** (2005) 449–459.

[14] A. Yu, M. Lu, F. Tian, New upper bounds for the energy of graphs, *MATCH Commun. Math. Comput. Chem.* **53** (2005) 441–448.

[15] B. Zhou, Lower bounds for energy of quadrangle–free graphs, *MATCH Commun. Math. Comput. Chem.* **55** (2006) 91–94.