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CONSTRUCTION OF EQUIENERGETIC GRAPHS

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Abstract

The energy of a graph G is the sum of the absolute values of its eigenvalues. Two non-isomorphic graphs of same order are said to be equienergetic if their energies are equal. In this paper we construct pairs of connected, noncospectral, equienergetic graphs of order n for all $n \geq 9$.

Introduction:

Let G be a simple undirected graph on n vertices and m edges. The characteristic polynomial of the adjacency matrix of G is the characteristic polynomial of G, denoted by $\Phi(G : \lambda)$. The roots of the equation $\Phi(G : \lambda) = 0$, denoted by $\lambda_1, \lambda_2, \ldots, \lambda_n$ are said to be eigenvalues of G and their collection is the spectrum of G [6]. Two non-isomorphic graphs are said to be cospectral if they have same spectra.

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The energy of a graph G is defined as

$$E(G) = \sum_{i=1}^{n} |\lambda_i| .$$

It was introduced by I. Gutman long time ago [9]. In chemistry the energy of a graph is intensively studied since it can be used to approximate the total π-electron energy of a molecule [5, 9, 10, 14]. For recent mathematical and chemical work on the energy of a graph, see [1 – 4, 7, 8, 10 – 13, 15 – 28, 30 – 38].

Two non-isomorphic graphs G_1 and G_2 of same order are said to be equienergetic if $E(G_1) = E(G_2)$. Certainly, cospectral graphs are equienergetic. Such case is of no interest. Recently classes of non-cospectral equienergetic graphs were designed. R. Balakrishnan [1] proved that for any positive integer $n \geq 3$, there exists non-cospectral, equienergetic graphs of order $4n$. H. S. Ramane et al. [26, 27] proved that if G is regular graph of order n and of degree $r \geq 3$ then $E(L^2(G)) = 2nr(r - 2)$ and $E(\overline{L^2(G)}) = (nr - 4)(2r - 3) - 2$, where $L^2(G)$ is the second line graph of G and $\overline{G}$ is the complement of G. Thus they constructed large families of non-cospectral, equienergetic graphs of order $nr(r - 1)/2$. Pairs of equienergetic chemical trees were first time designed by V. Brankov, D. Stevanovic, I. Gutman [3]. For other results on equienergetic graphs see [21, 28, 30]. In the following we construct pairs of connected, non-cospectral, equienergetic graphs for all $n \geq 9$.

Energy of complete product of regular graphs:

Definition [6]: The complete product $G_1 \nabla G_2$ of two graphs G_1 and G_2 is the graph obtained by joining every vertex of G_1 with every vertex of G_2.

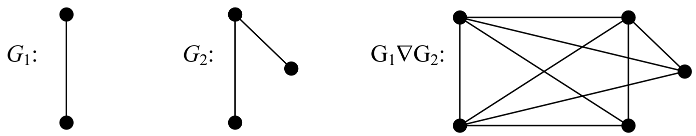

Fig. 1

Lemma 1: *If G_i is a regular graph of degree r_i with n_i vertices, $i = 1, 2$ then*

$$E(G_1\nabla G_2) = E(G_1) + E(G_2) + \sqrt{(r_1 + r_2)^2 + 4(n_1n_2 - r_1r_2)} - (r_1 + r_2).$$

Proof: If G_i is a regular graph of degree r_i with n_i vertices, $i = 1, 2$ then [6]

$$\Phi(G_1\nabla G_2 : \lambda) = \frac{\Phi(G_1 : \lambda)\Phi(G_2 : \lambda)}{(\lambda - r_1)(\lambda - r_2)}[(\lambda - r_1)(\lambda - r_2) - n_1n_2],$$

which gives

$$(\lambda - r_1)(\lambda - r_2)\Phi(G_1\nabla G_2 : \lambda) = \Phi(G_1 : \lambda)\Phi(G_2 : \lambda)[(\lambda - r_1)(\lambda - r_2) - n_1n_2].$$

Let $P_1 = (\lambda - r_1)(\lambda - r_2)\Phi(G_1\nabla G_2 : \lambda)$

and $P_2 = \Phi(G_1 : \lambda)\Phi(G_2 : \lambda)[(\lambda - r_1)(\lambda - r_2) - n_1n_2]$.

The roots of $P_1 = 0$ are r_1, r_2 and the eigenvalues of $G_1\nabla G_2$. Therefore the sum of the absolute values of the roots of $P_1 = 0$ is

$$E(G_1\nabla G_2) + r_1 + r_2. \tag{1}$$

The roots of $P_2 = 0$ are the eigenvalues of G_1 and G_2 and

$$\frac{r_1 + r_2 \pm \sqrt{(r_1 + r_2)^2 + 4(n_1n_2 - r_1r_2)}}{2}.$$

Therefore the sum of the absolute values of the roots of $P_2 = 0$ is

$$E(G_1) + E(G_2) + \left|\frac{r_1 + r_2 + \sqrt{(r_1 + r_2)^2 + 4(n_1n_2 - r_1r_2)}}{2}\right|$$

$$+ \left|\frac{r_1 + r_2 - \sqrt{(r_1 + r_2)^2 + 4(n_1n_2 - r_1r_2)}}{2}\right|$$

$$= E(G_1) + E(G_2) + \sqrt{(r_1 + r_2)^2 + 4(n_1n_2 - r_1r_2)}. \tag{2}$$

Since $P_1 = P_2$, equating (1) and (2) we get

$$E(G_1\nabla G_2) = E(G_1) + E(G_2) + \sqrt{(r_1 + r_2)^2 + 4(n_1n_2 - r_1r_2)} - (r_1 + r_2). \qquad \square$$

Corollary 2: *If $G_1, G_2, \ldots, G_k$, $k \geq 3$ be the equienergetic regular graphs of same order and of same degree then $E(G_a\nabla G_b) = E(G_c\nabla G_d)$ for all $1 \leq a, b, c, d \leq k$.* $\square$

Constructing equienergetic graphs:

Consider the graphs H_1 and H_2 as shown in Fig. 2.

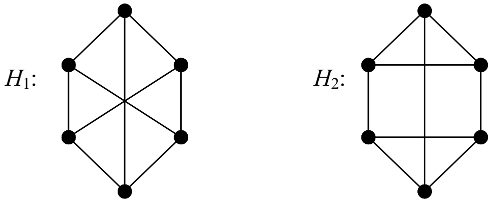

Fig. 2

The characteristic polynomials of H_1 and H_2 are
$\Phi(H_1 : \lambda) = (\lambda - 3)\lambda^4(\lambda + 3)$ and $\Phi(H_2 : \lambda) = (\lambda - 3)(\lambda - 1)\lambda^2(\lambda + 2)^2$.
Let $G_1 = L(H_1)$ and $G_2 = L(H_2)$ (See Fig. 3).

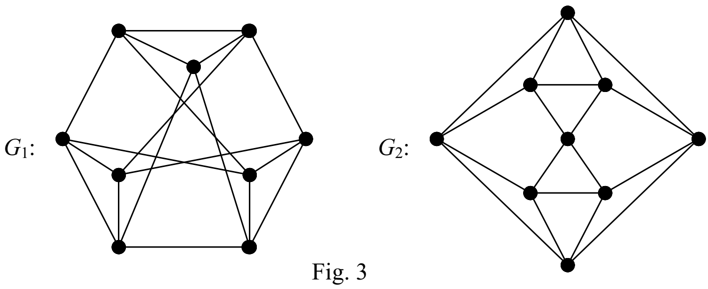

Fig. 3

According to the theorem by H. Sachs [6, 29], the characteristic polynomial of regular graph G and its line graph $L(G)$ are related as
$\Phi(L(G) : \lambda) = (\lambda + 2)^{n(r-2)/2}\, \Phi(G : \lambda - r + 2)$
where n is the order and r is the degree of G. Using this result we get characteristic polynomials of G_1 and G_2 as

$$\Phi(G_1 : \lambda) = \lambda^9 - 18\lambda^7 - 12\lambda^6 + 81\lambda^5 - 156\lambda^4 + 600\lambda^3 + 144\lambda - 64$$
$$= (\lambda - 4)(\lambda - 1)^4(\lambda + 2)^4. \qquad (3)$$

and $\Phi(G_2 : \lambda) = \lambda^9 - 18\lambda^7 - 16\lambda^6 + 81\lambda^5 + 96\lambda^4 - 112\lambda^3 - 144\lambda^2 + 48\lambda + 64$

$$= (\lambda - 4)(\lambda - 2)(\lambda - 1)^2(\lambda + 1)^2(\lambda + 2)^3. \quad (4)$$

Theorem 3: *There exists a pair of connected non-cospectral, equienergetic graphs with n vertices for all* $n \geq 9$.

Proof: Consider the graphs G_1 and G_2 as shown in Fig. 3. Both G_1 and G_2 are connected regular graphs on nine vertices and of degree four. From equations (3) and (4), $E(G_1) = E(G_2) = 16$. A complete graph K_p is regular graph on p vertices and of degree $p - 1$.

Knowing $\Phi(K_p : \lambda) = (\lambda - p + 1)(\lambda + 1)^{p-1}$, $E(K_p) = 2(p - 1)$.

From Lemma 1, we have

$$E(G_1 \nabla K_p) = E(G_2 \nabla K_p) = 16 + 2(p - 1) + \sqrt{(4 + p - 1)^2 + 4(9p - 4(p - 1))} - (4 + p - 1)$$

$$= 11 + p + \sqrt{(p + 3)^2 + 4(5p + 4)}\ .$$

Thus $G_1 \nabla K_p$ and $G_2 \nabla K_p$ are equienergetic. By equations (3) and (4) G_1 and G_2 are non-cospectral, so $G_1 \nabla K_p$ and $G_2 \nabla K_p$. Further $G_1 \nabla K_p$ and $G_2 \nabla K_p$ are connected and possess equal number of vertices $n = 9 + p$, $p = 0, 1, 2, \ldots$ □

Conclusion: Corollary 2 and Theorem 3 shows that there exist pairs of connected, non-cospectral, equienergetic graphs with n vertices for all $n \geq 9$. Further this method leads to construction of pairs of connected, nonregular, non-cospectral, equienergetic graphs of order n for $n \geq 10$.

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