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Wiener index of tori $T_{p,q}[C_4, C_8]$ covered by C_4 and C_8[1]

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Abstract

A $T_{p,q}[C_4, C_8]$ is a torus covered by alternating C_4 and C_8. This paper presents a method for deriving formulas for calculating the Wiener index of the torus $T_{p,q}[C_4, C_8]$.

1 Introduction

The Wiener index is a graph invariant based on distances in a graph. It is denoted by $W(G)$ and defined as the sum of distances between all pairs of vertices in a connected graph G:

$$W(G) = \sum_{\{u,v\} \subseteq V(G)} d_G(u,v) = \frac{1}{2} \sum_{v \in V(G)} d_G(v) \qquad (1)$$

where $V(G)$ is the vertex set of G and $d_G(u,v)$ denotes the distance between the vertices $u, v \in V(G)$ and $d_G(v)$ is the sum of distances between v and all other vertices of G.

The Wiener index is much studied in the chemical literature, since Harold Wiener [1], in 1947, was the first to consider it. Wiener's original definition

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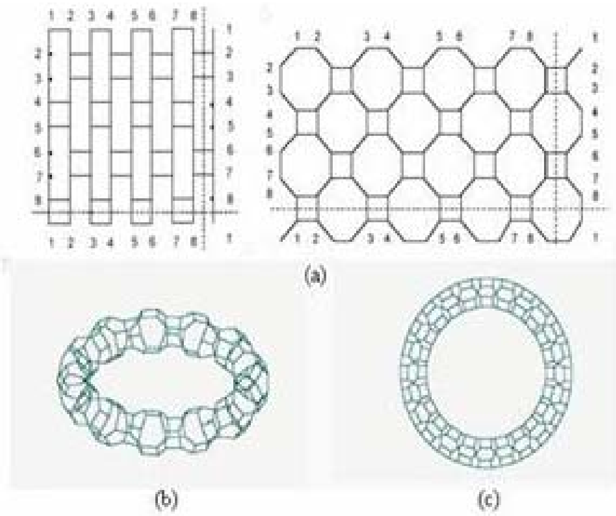

Figure 1. (a) a cutting of $G = T_{4,2}[C_4, C_8]$;
(b) side view; (c) top view.

Starting from the middle of the 1970s, the Wiener index gained much popularity and, since then, new results related to it are constantly being reported. For a review, historical details and further bibliography on the chemical applications of the Wiener index see [5,6,7,11,12,13,18]. Results on the Wiener index of trees and hexagonal systems were summarized in [3,4,8,9,10].

The primary aim of this article presents a method for deriving formulas for calculating the Wiener index of the torus $T_{p,q}[C_4, C_8]$ covered by C_4 and C_8. Some literatures related this topic in chemical graph theory, see [14-17,19-21].

Our notations are mainly taken from [7,8]. Throughout this paper $G = T_{p,q}[C_4, C_8]$ denotes a torus covered by C_4 and C_8 with $4q$ rows and $2p$ columns in its cutting, see Figure 1.

$G = T_{p,q}[C_4, C_8]$ has $8pq$ vertices. To compute the Wiener index of this graph, we assume that x_{ij} denotes the vertex in row i and column j of the 2-dimensional lattice of G, see Figure 2. From the symmetry of the tori G, x_{ij} can be changed into x_{rt} by using the rotations and reflections. Then, $d_G(x_{ij}) = d_G(x_{rt})$. So, we have

Lemma 1. Let $G = T_{p,q}[C_4, C_8]$, then $W(G) = 4pqd_G(x)$, where x be any vertex of G.

To calculating $W(G)$, we need only calculate $d_G(x)$ for a vertex x of $G = T_{p,q}[C_4, C_8]$.

2 The distances in $G = T_{p,q}[C_4, C_8]$

In this section, we will give a formula for calculating the distances between x_{11} and any other vertices of $G = T_{p,q}[C_4, C_8]$. The elementary results come from [16].

We first consider four graphs G_1, G_2, G_3 and G_4, where G_1 is obtaining from $G = T_{p,q}[C_4, C_8]$ by deleting the horizontal edges between columns 1 and 2p and the vertical edges between rows 1 and 4q (see Figure 2), G_2 is obtaining from $G = T_{p,q}[C_4, C_8]$ by deleting the horizontal edges between columns 1 and 2 and the vertical edges between rows 1 and 4q, G_3 is obtaining from $G = T_{p,q}[C_4, C_8]$ by deleting the horizontal edges between columns 1 and 2p and the vertical edges between rows 3 and 4, G_4 is obtaining from $G = T_{p,q}[C_4, C_8]$ by deleting the horizontal edges between columns 1 and 2 and the vertical edges between rows 3 and 4. And the distances from x_{11} in G is the minimum of the ones in G_1, G_2, G_3 and G_4.

In the following, $d(x_{11}, x_{rt})$, $d_1(x_{11}, x_{rt})$, $d_2(x_{11}, x_{rt})$, $d_3(x_{11}, x_{rt})$ and $d_4(x_{11}, x_{rt})$ denote the distance between x_{11} and x_{rt} in G, G_1, G_2, G_3 and G_4, respectively.

The distances from x_{11} to other vertices in G_1 as showing in Figure 2. And we have

Lemma 2([16]).

$$d_1(x_{11}, x_{rt}) = t + \begin{cases} r-2, & 1 \le t \le [\frac{r}{2}] + 2; \\ 2[\frac{2t+r-3}{4}] - 1, & t \ge [\frac{r}{2}] + 3 \text{ and r is odd;} \\ 2[\frac{2t+r-2}{4}] - 2, & t \ge [\frac{r}{2}] + 3 \text{ and r is even} \end{cases}$$

where $[x]$ denotes the maximum integer not larger than x over all the paper.

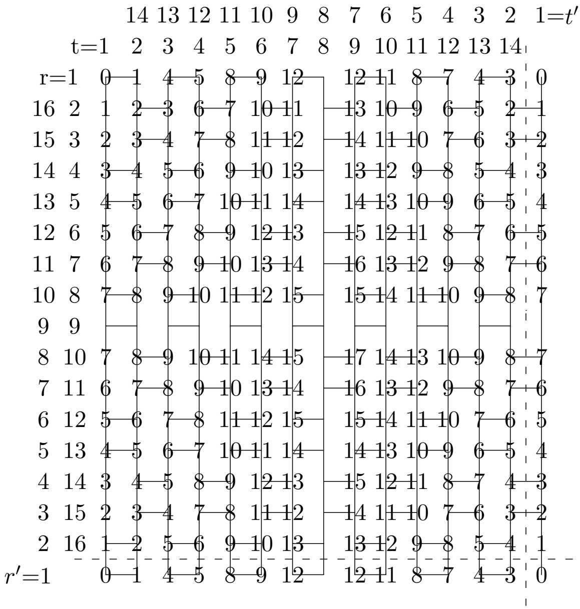

Figure 2. Some distances from the vertex x_{11} in G_1, G_2, G_3 and G_4, where p=7 and q=4.

The distances from x_{11} to other vertices in G_2 as showing in Figure 2. And we have

Lemma 3([16]).

$$d_2(x_{11}, x_{rt'}) = t' + \begin{cases} r-2, & 1 \le t' \le [\frac{r}{2}]+1; \\ 2[\frac{2t'+r-1}{4}]-1, & t' \ge [\frac{r}{2}]+2 \text{ and r is odd;} \\ 2[\frac{2t'+r}{4}]-2, & t' \ge [\frac{r}{2}]+2 \text{ and r is even} \end{cases}$$

where $d_2(x_{11}, x_{rt'})$ is the distance between x_{11} and $x_{rt'}$ in G_2 and

$$t' = \begin{cases} 1, & t=1 \\ 2p+2-t, & t \ge 2 \end{cases}$$

Also, we have

Lemma 4([16]).

$$d_3(x_{11}, x_{r't}) = t + \begin{cases} r'-2, & 1 \le t \le [\frac{r'-1}{2}]+2; \\ 2[\frac{2t+r'+1}{4}]-3, & t \ge [\frac{r'-1}{2}]+3 \text{ and } r' \text{ is odd;} \\ 2[\frac{2t+r'}{4}]-2, & t \ge [\frac{r'-1}{2}]+3 \text{ and } r' \text{ is even.} \end{cases}$$

where $d_3(x_{11}, x_{r't})$ is the distance between x_{11} and $x_{r't}$ in G_3, and

$$r' = \begin{cases} 1, & r=1 \\ 4q+2-r, & r \ge 2 \end{cases}$$

.

Lemma 5([16]).

$$d_4(x_{11}, x_{r't'}) = t' + \begin{cases} r' - 2, & 1 \le t' \le [\frac{r'-1}{2}] + 1; \\ 2[\frac{2t'+r'-1}{4}] - 1, & t' \ge [\frac{r'-1}{2}] + 2 \text{ and } r' \text{ is odd;} \\ 2[\frac{2t'+r'-2}{4}], & t' \ge [\frac{r'-1}{2}] + 2 \text{ and } r' \text{ is even.} \end{cases}$$

where $d_4(x_{11}, x_{r't'})$ is the distance between x_{11} and $x_{r't'}$ in G_4.

Now we can give a formula of calculating the distances from x_{11} to other vertices in $G = T_{p,q}[C_4, C_8]$.

Theorem 1([16]). (i) $d(x_{11}, x_{rt}) = d_1(x_{11}, x_{rt})$ if $1 \le t \le p+1$ and $1 \le r \le 2q+1$;

(ii) $d(x_{11}, x_{rt}) = d_2(x_{11}, x_{rt})$ if $p+2 \le t \le 2p$ and $1 \le r \le 2q+1$;

(iii) $d(x_{11}, x_{rt}) = d_3(x_{11}, x_{rt})$ if $1 \le t \le p+1$ and $2q+2 \le r \le 4q$;

(iv) $d(x_{11}, x_{rt}) = d_4(x_{11}, x_{rt})$ if $p+2 \le t \le 2p$ and $2q+2 \le r \le 4q$.

3 A formula for calculating Wiener index of $G = T_{p,q}[C_4, C_8]$

In this section, we will give a formula for calculating Wiener index of $G = T_{p,q}[C_4, C_8]$.

Let $A(p,q) = (a_{rt})_{4q\times 2p}$ be a matrix with $4q$ rows and $2p$ columns, where $a_{rt} = d(x_{11}, x_{rt})$ is the distance between x_{11} and x_{rt} in G, and

$$D_1(r) = \sum_{t=1}^{p+1} a_{rt};\ D_2(r) = \sum_{t=p+2}^{2p} a_{rt};\ D(r) = D_1(r) + D_2(r) = \sum_{t=1}^{2p} a_{rt}$$

then $D(r)$ is the sum of the elements on row r in $A(p,q)$, i.e., the sum of the distances from x_{11} to the vertices in row r of the 2-dimensional lattice of G. And $d_G(x_{11}) = \sum_{r=1}^{4q}\sum_{t=1}^{2p} a_{rt} = \sum_{r=1}^{4q} D(r)$.

Lemma 6. If $p \ge q+2$, then $d_G(x_{11}) = \frac{16}{3}q^3 + 3p^2q + 8q^2 + \frac{11}{3}q + 8\sum_{k=2}^{p}[\frac{k}{2}] + 8\sum_{k=q+2}^{p}[\frac{k+q}{2}] + 16\sum_{l=1}^{q-1}\sum_{k=l+1}^{p}[\frac{k+l}{2}]$.

Proof. Using Theorem 1, we first compute $D(r)$.

Case I. $1 \le r \le 2q+1$. Then $\frac{r+5}{2} \le p+1$ since $p \ge q+2$.

(i) If r is even, then

$$\begin{aligned} D_1(r) &= \sum_{t=1}^{p+1} d_1(x_{11}, x_{rt}) = \sum_{t=1}^{\frac{r}{2}+2}(t+r-2) + \sum_{t=\frac{r}{2}+3}^{p+1}(t + 2[\tfrac{2t+r-2}{4}] - 2) \\ &= \tfrac{1}{2}p^2 - \tfrac{1}{2}p + \tfrac{1}{2}r^2 + 2r - 1 + 2\sum_{t=\frac{r}{2}+3}^{p+1}[\tfrac{2t+r-2}{4}] \\ &= \tfrac{1}{2}p^2 - \tfrac{1}{2}p + \tfrac{1}{2}r^2 + 2r - 1 + 2\sum_{t=\frac{r}{2}+2}^{p}[\tfrac{2t+r}{4}] \end{aligned}$$

$$\begin{aligned}D_2(r) &= \sum_{t=p+2}^{2p} d(x_{11}, x_{rt}) = \sum_{t'=2}^{p} d_2(x_{11}, x_{rt'}) \\ &= \sum_{t'=2}^{\frac{r}{2}+1} (t'+r-2) + \sum_{t'=\frac{r}{2}+2}^{p} (t'+2[\tfrac{2t'+r}{4}]-2) \\ &= \tfrac{1}{2}p^2 - \tfrac{3}{2}p + \tfrac{1}{2}r^2 + 1 + 2\sum_{t'=\frac{r}{2}+2}^{p} [\tfrac{2t'+r}{4}]\end{aligned}$$

And, $D(r) = D_1(r) + D_2(r) = p^2 - 2p + r^2 + 2r + 4\sum_{k=\frac{r}{2}+2}^{p} [\frac{2k+r}{4}]$.

(ii) If r is odd, then

$$\begin{aligned}D_1(r) &= \sum_{t=1}^{p+1} d_1(x_{11}, x_{rt}) = \sum_{t=1}^{\frac{r+3}{2}} (t+r-2) + \sum_{t=\frac{r+5}{2}}^{p+1} (t+2[\tfrac{2t+r-3}{4}]-1) \\ &= \tfrac{1}{2}p^2 + \tfrac{1}{2}p + \tfrac{1}{2}r^2 + r - \tfrac{3}{2} + 2\sum_{t=\frac{r+3}{2}}^{p} [\tfrac{2t+r-1}{4}]\end{aligned}$$

$$\begin{aligned}D_2(r) &= \sum_{t=p+2}^{2p} d(x_{11}, x_{rt}) = \sum_{t'=2}^{p} d_2(x_{11}, x_{rt'}) \\ &= \sum_{t'=2}^{\frac{r+1}{2}} (t'+r-2) + \sum_{t'=\frac{r+3}{2}}^{p} (t'+2[\tfrac{2t'+r-1}{4}]-1) \\ &= \tfrac{1}{2}p^2 - \tfrac{1}{2}p + \tfrac{1}{2}r^2 - r + \tfrac{1}{2} + 2\sum_{t'=\frac{r+3}{2}}^{p} [\tfrac{2t'+r-1}{4}]\end{aligned}$$

And, $D(r) = D_1(r) + D_2(r) = p^2 + r^2 - 1 + 4\sum_{k=\frac{r+3}{2}}^{p} [\frac{2k+r-1}{4}]$.

$$\begin{aligned}\sum_{r=1}^{2q+1} D(r) = & \sum_{\text{r is even}} (p^2 - 2p + r^2 + 2r + 4\sum_{k=\frac{r}{2}+2}^{p} [\tfrac{2k+r}{4}]) \\ & + \sum_{\text{r is odd}} (p^2 + r^2 - 1 + 4\sum_{k=\frac{r+3}{2}}^{p} [\tfrac{2k+r-1}{4}]) \\ = & \ p^2(2q+1) - 2pq + \sum_{r=1}^{2q+1} r^2 + 2(2+4+6+\cdots+2q) \\ & -(q+1) + 4(\sum_{k=3}^{p} [\tfrac{2k+2}{4}] + \sum_{k=4}^{p} [\tfrac{2k+4}{4}] + \cdots + \sum_{k=q+2}^{p} [\tfrac{2k+2q}{4}]) \\ & +4(\sum_{k=2}^{p} [\tfrac{2k}{4}] + \sum_{k=3}^{p} [\tfrac{2k+2}{4}] + \cdots + \sum_{k=q+2}^{p} [\tfrac{2k+2q}{4}]) \\ = & \ p^2(2q+1) - 2pq + \tfrac{1}{6}(2q+1)(2q+2)(4q+3) \\ & +(2q-1)(q+1) + 4\sum_{k=2}^{p} [\tfrac{k}{2}] + 8(\sum_{k=3}^{p} [\tfrac{k+1}{2}] \\ & + \sum_{k=4}^{p} [\tfrac{k+2}{2}] + \cdots + \sum_{k=q+2}^{p} [\tfrac{k+q}{2}]) \\ = & \ \tfrac{8}{3}q^3 + 2p^2q + 8q^2 + p^2 - 2pq + \tfrac{16}{3}q \\ & +4\sum_{k=2}^{p} [\tfrac{k}{2}] + 8\sum_{l=1}^{q} \sum_{k=l+2}^{p} [\tfrac{k+l}{2}]\end{aligned}$$

Case II. $2q+2 \le r \le 4q$, i.e., $2 \le r' \le 2q$. Then $\frac{r'+5}{2} \le p+1$ since $p \ge q+2$.

(i) If r is even, then $r' = 4q+2-r$ is also even. And

$$\begin{aligned}D_1(r) &= \sum_{t=1}^{p+1} d_3(x_{11}, x_{r't}) = \sum_{t=1}^{\frac{r'+2}{2}} (t+r'-2) + \sum_{t=\frac{r'+4}{2}}^{p+1} (t+2[\tfrac{2t+r'}{4}]-2)\\ &= \sum_{t=1}^{\frac{r'+2}{2}} (t+r'-2) + \sum_{t=\frac{r'+4}{2}}^{p+1} (t+2[\tfrac{2(t-1)+r'-2}{4}])\\ &= \tfrac{1}{2}p^2 + \tfrac{3}{2}p + \tfrac{1}{2}r'^2 - 1 + 2\sum_{t=\frac{r'+2}{2}}^{p} [\tfrac{2t+r'-2}{4}]\end{aligned}$$

$$\begin{aligned}D_2(r) &= \sum_{t=p+2}^{2p} d(x_{11}, x_{rt}) = \sum_{t'=2}^{p} d_4(x_{11}, x_{r't'})\\ &= \sum_{t'=2}^{\frac{r'}{2}} (t'+r-2) + \sum_{t'=\frac{r'+2}{2}}^{p} (t'+2[\tfrac{2t'+r'-2}{4}])\\ &= \tfrac{1}{2}p^2 + \tfrac{1}{2}p + \tfrac{1}{2}r'^2 - 2r' + 1 + 2\sum_{t'=\frac{r'+2}{2}}^{p} [\tfrac{2t'+r-2}{4}]\end{aligned}$$

And, $D(r) = D_1(r) + D_2(r) = p^2 + 2p + r'^2 - 2r' + 4\sum_{k=\frac{r'+2}{2}}^{p} [\frac{2k+r'-2}{4}]$.

(ii) If r is odd, then

$$\begin{aligned}D_1(r) &= \sum_{t=1}^{p+1} d_3(x_{11}, x_{r't}) = \sum_{t=1}^{\frac{r'+3}{2}} (t+r'-2) + \sum_{t=\frac{r'+5}{2}}^{p+1} (t+2[\tfrac{2t+r'+1}{4}]-3)\\ &= \sum_{t=1}^{\frac{r'+3}{2}} (t+r'-2) + \sum_{t=\frac{r'+5}{2}}^{p+1} (t+2[\tfrac{2(t-1)+r'-1}{4}]-1)\\ &= \tfrac{1}{2}p^2 + \tfrac{1}{2}p + \tfrac{1}{2}r'^2 + r' - \tfrac{3}{2} + 2\sum_{t=\frac{r'+3}{2}}^{p} [\tfrac{2t+r'-1}{4}]\end{aligned}$$

$$\begin{aligned}D_2(r) &= \sum_{t=p+2}^{2p} d(x_{11}, x_{rt}) = \sum_{t'=2}^{p} d_4(x_{11}, x_{r't'})\\ &= \sum_{t'=2}^{\frac{r'+1}{2}} (t'+r'-2) + \sum_{t'=\frac{r'+3}{2}}^{p} (t'+2[\tfrac{2t'+r'-1}{4}]-1)\\ &= \tfrac{1}{2}p^2 - \tfrac{1}{2}p + \tfrac{1}{2}r'^2 - r' + \tfrac{1}{2} + 2\sum_{t'=\frac{r'+3}{2}}^{p} [\tfrac{2t'+r'-1}{4}]\end{aligned}$$

And, $D(r) = D_1(r) + D_2(r) = p^2 + r'^2 - 1 + 4\sum_{k=\frac{r'+3}{2}}^{p} [\frac{2k+r'-1}{4}]$.

$$\begin{aligned}\sum_{r=2q+2}^{4q} D(r) &= \sum_{r'=2}^{2q} D(r)\\ &= \sum_{r' \text{ is even}} (p^2 + 2p + r'^2 - 2r' + 4\sum_{k=\frac{r'+2}{2}}^{p} [\tfrac{2k+r'-2}{4}])\\ &\quad + \sum_{r' \text{ is odd}} (p^2 + r'^2 - 1 + 4\sum_{k=\frac{r'+3}{2}}^{p} [\tfrac{2k+r'-1}{4}])\\ &= \tfrac{8}{3}q^3 + 2p^2q - p^2 + 2pq - \tfrac{8}{3}q + 4\sum_{k=2}^{p} [\tfrac{k}{2}] + 8\sum_{l=1}^{q-1}\sum_{k=l+2}^{p} [\tfrac{k+l}{2}]\end{aligned}$$

Therefore, we have

$$d_G(x_{11}) = \sum_{r=1}^{4q}\sum_{t=1}^{2p} a_{rt} = \sum_{r=1}^{4q} D(r) = \sum_{r=1}^{2q+1} D(r) + \sum_{r=2q+2}^{4q} D(r)$$

$$= \tfrac{8}{3}q^3 + 2p^2q + 8q^2 + p^2 - 2pq + \tfrac{16}{3}q + 4\sum_{k=2}^{p}[\tfrac{k}{2}] + 8\sum_{l=1}^{q}\sum_{k=l+2}^{p}[\tfrac{k+l}{2}]$$

$$+\tfrac{8}{3}q^3 + 2p^2q - p^2 + 2pq - \tfrac{8}{3}q + 4\sum_{k=2}^{p}[\tfrac{k}{2}] + 8\sum_{l=1}^{q-1}\sum_{k=l+2}^{p}[\tfrac{k+l}{2}]$$

$$= \tfrac{16}{3}q^3 + 4p^2q + 8q^2 + \tfrac{8}{3}q + 8\sum_{k=2}^{p}[\tfrac{k}{2}] + 8\sum_{k=q+2}^{p}[\tfrac{k+q}{2}] + 16\sum_{l=1}^{q-1}\sum_{k=l+1}^{p}[\tfrac{k+l}{2}]$$

For an exact expression, we need the following lemma.

Lemma 7. If $x \le y$, then

$$[\tfrac{x}{2}] + [\tfrac{x+1}{2}] + [\tfrac{x+2}{2}] + \cdots + [\tfrac{y}{2}]$$

$$= \begin{cases} \tfrac{1}{4}(y^2 - x^2) + \tfrac{1}{2}x, & \text{if } x \text{ and } y \text{ are even;} \\ \tfrac{1}{4}(y^2 - x^2) + \tfrac{1}{2}x - \tfrac{1}{4}, & \text{if the parity of } x \text{ and } y \text{ are different;} \\ \tfrac{1}{4}(y^2 - x^2) + \tfrac{1}{2}x - \tfrac{1}{2}, & \text{if } x \text{ and } y \text{ are odd.} \end{cases}$$

The proof of Lemma 7 is easy and omitted here.

Using Lemmas 1, 6 and 7, we can give a formula for calculating Wiener index of $G = T_{p,q}[C_4, C_8]$.

Theorem 2. Let $p \ge q + 2$. $W(G)$ is the Wiener index of $G = T_{p,q}[C_4, C_8]$.

(i)If p and q are even, then

$$W(G) = -48pq - \frac{40}{3}pq^2 + 32p^3q^2 + 16pq^3 + 16p^2q^3 + \frac{16}{3}pq^4;$$

(ii)If p is even and q is odd, then

$$W(G) = -64pq - \frac{136}{3}pq^2 + 32p^3q^2 + 16pq^3 + 16p^2q^3 + \frac{16}{3}pq^4;$$

(iii)If p is odd and q is even, then

$$W(G) = -96pq - \frac{136}{3}pq^2 + 32p^3q^2 + 16pq^3 + 16p^2q^3 + \frac{16}{3}pq^4;$$

(iv)If p and q are odd, then

$$W(G) = -120pq - \frac{40}{3}pq^2 + 32p^3q^2 + 16pq^3 + 16p^2q^3 + \frac{16}{3}pq^4.$$

Proof. (1) If p and q are even, then by Lemma 7, we have

$\sum_{k=2}^{p}[\tfrac{k}{2}] = \tfrac{1}{4}(p^2 - 2^2) + 1 = \tfrac{1}{4}p^2,$

$\sum\limits_{k=3}^{p}[\frac{k+1}{2}] = \frac{1}{4}((p+1)^2 - 4^2) + \frac{7}{4} = \frac{1}{4}p^2 + \frac{1}{2}p - 2,$
$\sum\limits_{k=4}^{p}[\frac{k+2}{2}] = \frac{1}{4}((p+2)^2 - 6^2) + 3 = \frac{1}{4}p^2 + p - 5,$
$\sum\limits_{k=5}^{p}[\frac{k+3}{2}] = \frac{1}{4}((p+3)^2 - 8^2) + \frac{15}{4} = \frac{1}{4}p^2 + \frac{3}{2}p - 10,$
$\sum\limits_{k=6}^{p}[\frac{k+4}{2}] = \frac{1}{4}((p+4)^2 - 10^2) + 5 = \frac{1}{4}p^2 + 2p - 16,$
$\cdots\cdots$
$\sum\limits_{k=q}^{p}[\frac{k+q-2}{2}] = \frac{1}{4}((p+q-2)^2 - (2q-2)^2) + (q-1),$
$\sum\limits_{k=q+1}^{p}[\frac{k+q-1}{2}] = \frac{1}{4}((p+q-1)^2 - (2q)^2) + \frac{1}{4}(4q-1),$
$\sum\limits_{k=q+2}^{p}[\frac{k+q}{2}] = \frac{1}{4}((p+q)^2 - (2q+2)^2) + (q+1) = \frac{1}{4}p^2 + \frac{1}{2}pq - \frac{3}{4}q^2 - q.$

$$\sum_{l=1}^{q-1}\sum_{k=l+2}^{p}[\frac{k+l}{2}] = \frac{1}{4}(\sum_{l=1}^{q-1}((p+l)^2 - (2(l+1))^2) + \sum_{t=1}^{\frac{q-2}{2}}(2t+1) + \frac{1}{4}\sum_{r=2}^{q}(4r-1)$$
$$= -\frac{3}{4} - \frac{p^2}{4} + \frac{q}{8} - \frac{pq}{4} + \frac{p^2q}{4} + \frac{q^2}{8} + \frac{pq^2}{4} - \frac{q^3}{4}$$

By Lemma 6,

$$\begin{aligned} d_G(x_{11}) = \ & \frac{16}{3}q^3 + 4p^2q + 8q^2 + \frac{8}{3}q + 8\sum_{k=2}^{p}[\frac{k}{2}] + 8\sum_{k=q+2}^{p}[\frac{k+q}{2}] \\ & +16\sum_{l=1}^{q-1}\sum_{k=l+1}^{p}[\frac{k+l}{2}] \\ = \ & \frac{16}{3}q^3 + 4p^2q + 8q^2 + \frac{8}{3}q + 2p^2 + (2p^2 + 4pq - 6q^2 - 8q) \\ & +(-12 - 4p^2 + 2q - 4pq + 4p^2q + 2q^2 + 4pq^2 - 4q^3) \\ = \ & -12 - \frac{10}{3}q + 8p^2q + 4q^2 + 4pq^2 + \frac{4}{3}q^3 \end{aligned}$$

By Lemma 1, we have
$W(G) = 4pqd_G(x_{11}) = -48pq - \frac{40}{3}pq^2 + 32p^3q^2 + 16pq^3 + 16p^2q^3 + \frac{16}{3}pq^4.$

(2) If p is even and q is odd, then by Lemma 7, we have

$\sum\limits_{k=2}^{p}[\frac{k}{2}] = \frac{1}{4}(p^2 - 2^2) + 1 = \frac{1}{4}p^2,$

$\sum\limits_{k=3}^{p}[\frac{k+1}{2}] = \frac{1}{4}((p+1)^2 - 4^2) + \frac{7}{4} = \frac{1}{4}p^2 + \frac{1}{2}p - 2,$

$\sum\limits_{k=4}^{p}[\frac{k+2}{2}] = \frac{1}{4}((p+2)^2 - 6^2) + 3 = \frac{1}{4}p^2 + p - 5,$

$\sum\limits_{k=5}^{p}[\frac{k+3}{2}] = \frac{1}{4}((p+3)^2 - 8^2) + \frac{15}{4} = \frac{1}{4}p^2 + \frac{3}{2}p - 10,$

$\sum\limits_{k=6}^{p}[\frac{k+4}{2}] = \frac{1}{4}((p+4)^2 - 10^2) + 5 = \frac{1}{4}p^2 + 2p - 16,$

$\cdots\cdots$

$\sum\limits_{k=q}^{p}[\frac{k+q-2}{2}] = \frac{1}{4}((p+q-2)^2 - (2q-2)^2) + \frac{1}{4}(4(q-1)-1),$

$$\sum_{k=q+1}^{p}[\frac{k+q-1}{2}] = \frac{1}{4}((p+q-1)^2-(2q)^2)+q,$$

$$\begin{aligned}\sum_{k=q+2}^{p}[\frac{k+q}{2}] = & \frac{1}{4}((p+q)^2-(2q+2)^2)+\frac{1}{4}(4(q+1)-1) \\ = & -\frac{1}{4}+\frac{1}{4}p^2-q+\frac{1}{2}pq-\frac{3}{4}q^2.\end{aligned}$$

$$\begin{aligned}\sum_{l=1}^{q-1}\sum_{k=l+2}^{p}[\frac{k+l}{2}] = & \frac{1}{4}(\sum_{l=1}^{q-1}((p+l)^2-(2(l+1))^2)+\sum_{t=1}^{\frac{q-1}{2}}(2t+1) \\ & +\frac{1}{4}\sum_{r=2}^{q-1}(4r-1) \\ = & -\frac{1}{4}-\frac{p^2}{4}-\frac{3q}{8}-\frac{pq}{4}+\frac{p^2q}{4}+\frac{q^2}{8}+\frac{pq^2}{4}-\frac{q^3}{4}\end{aligned}$$

By Lemma 6,

$$\begin{aligned}d_G(x_{11}) = & \frac{16}{3}q^3+4p^2q+8q^2+\frac{8}{3}q+8\sum_{k=2}^{p}[\frac{k}{2}]+8\sum_{k=q+2}^{p}[\frac{k+q}{2}] \\ & +16\sum_{l=1}^{q-1}\sum_{k=l+1}^{p}[\frac{k+l}{2}] \\ = & \frac{16}{3}q^3+4p^2q+8q^2+\frac{8}{3}q+2p^2+(-2+2p^2-8q+4pq-6q^2) \\ & +(-4-4p^2-6q-4pq+4p^2q+2q^2+4pq^2-4q^3) \\ = & -16-\frac{34}{3}q+8p^2q+4q^2+4pq^2+\frac{4}{3}q^3\end{aligned}$$

By Lemma 1, we have

$$W(G)=4pqd_G(x_{11})=-64pq-\frac{136}{3}pq^2+32p^3q^2+16pq^3+16p^2q^3+\frac{16}{3}pq^4.$$

(3) If p is odd and q is even, then by Lemma 7, we have

$$\sum_{k=2}^{p}[\frac{k}{2}] = \frac{1}{4}(p^2-2^2)+\frac{3}{4} = \frac{1}{4}p^2-\frac{1}{4},$$

$$\sum_{k=3}^{p}[\frac{k+1}{2}] = \frac{1}{4}((p+1)^2-4^2)+2,$$

$$\sum_{k=4}^{p}[\frac{k+2}{2}] = \frac{1}{4}((p+2)^2-6^2)+\frac{11}{4},$$

$$\sum_{k=5}^{p}[\frac{k+3}{2}] = \frac{1}{4}((p+3)^2-8^2)+4,$$

$$\sum_{k=6}^{p}[\frac{k+4}{2}] = \frac{1}{4}((p+4)^2-10^2)+\frac{19}{4},$$

……

$$\sum_{k=q}^{p}[\frac{k+q-2}{2}] = \frac{1}{4}((p+q-2)^2-(2q-2)^2)+\frac{1}{4}(4(q-1)-1),$$

$$\sum_{k=q+1}^{p}[\frac{k+q-1}{2}] = \frac{1}{4}((p+q-1)^2-(2q)^2)+q,$$

$$\begin{aligned}\sum_{k=q+2}^{p}[\frac{k+q}{2}] & = \frac{1}{4}((p+q)^2-(2q+2)^2)+\frac{1}{4}(4(q+1)-1) \\ & = -\frac{1}{4}+\frac{1}{4}p^2-q+\frac{1}{2}pq-\frac{3}{4}q^2.\end{aligned}$$

$$\sum_{l=1}^{q-1}\sum_{k=l+2}^{p}[\frac{k+l}{2}] = \frac{1}{4}(\sum_{l=1}^{q-1}((p+l)^2-(2(l+1))^2)+\sum_{t=1}^{\frac{q}{2}}2t+\frac{1}{4}\sum_{r=3}^{q-1}(4r-1)$$
$$= -\frac{5}{4}-\frac{p^2}{4}-\frac{3q}{8}-\frac{pq}{4}+\frac{p^2q}{4}+\frac{q^2}{8}+\frac{pq^2}{4}-\frac{q^3}{4}$$

By Lemma 6,

$$d_G(x_{11}) = \frac{16}{3}q^3+4p^2q+8q^2+\frac{8}{3}q+8\sum_{k=2}^{p}[\frac{k}{2}]+8\sum_{k=q+2}^{p}[\frac{k+q}{2}]$$
$$+16\sum_{l=1}^{q-1}\sum_{k=l+1}^{p}[\frac{k+l}{2}]$$
$$= \frac{16}{3}q^3+4p^2q+8q^2+\frac{8}{3}q+(2p^2-2)+(-2+2p^2-8q+4pq$$
$$-6q^2)+(-20-4p^2-6q-4pq+4p^2q+2q^2+4pq^2-4q^3)$$
$$= -24-\frac{34}{3}q+8p^2q+4q^2+4pq^2+\frac{4}{3}q^3$$

By Lemma 1, we have
$W(G) = 4pqd_G(x_{11}) = -96pq-\frac{136}{3}pq^2+32p^3q^2+16pq^3+16p^2q^3+\frac{16}{3}pq^4.$

(4) If p and q are odd, then by Lemma 7, we have

$$\sum_{k=2}^{p}[\frac{k}{2}] = \frac{1}{4}(p^2-2^2)+\frac{3}{4} = \frac{1}{4}p^2-\frac{1}{4},$$
$$\sum_{k=3}^{p}[\frac{k+1}{2}] = \frac{1}{4}((p+1)^2-4^2)+2,$$
$$\sum_{k=4}^{p}[\frac{k+2}{2}] = \frac{1}{4}((p+2)^2-6^2)+\frac{11}{4},$$
$$\sum_{k=5}^{p}[\frac{k+3}{2}] = \frac{1}{4}((p+3)^2-8^2)+4,$$
$$\sum_{k=6}^{p}[\frac{k+4}{2}] = \frac{1}{4}((p+4)^2-10^2)+\frac{19}{4},$$

......

$$\sum_{k=q}^{p}[\frac{k+q-2}{2}] = \frac{1}{4}((p+q-2)^2-(2q-2)^2)+(q-1),$$
$$\sum_{k=q+1}^{p}[\frac{k+q-1}{2}] = \frac{1}{4}((p+q-1)^2-(2q)^2)+\frac{1}{4}(4q-1),$$
$$\sum_{k=q+2}^{p}[\frac{k+q}{2}] = \frac{1}{4}((p+q)^2-(2q+2)^2)+(q+1) = \frac{1}{4}p^2-q+\frac{1}{2}pq-\frac{3}{4}q^2.$$

$$\sum_{l=1}^{q-1}\sum_{k=l+2}^{p}[\frac{k+l}{2}] = \frac{1}{4}(\sum_{l=1}^{q-1}((p+l)^2-(2(l+1))^2)+\sum_{t=1}^{\frac{q-1}{2}}2t+\frac{1}{4}\sum_{r=3}^{q}(4r-1)$$
$$= -\frac{7}{4}-\frac{p^2}{4}+\frac{q}{8}-\frac{pq}{4}+\frac{p^2q}{4}+\frac{q^2}{8}+\frac{pq^2}{4}-\frac{q^3}{4}$$

By Lemma 6,

$$\begin{aligned} d_G(x_{11}) = & \tfrac{16}{3}q^3+4p^2q+8q^2+\tfrac{8}{3}q+8\sum_{k=2}^{p}[\tfrac{k}{2}]+8\sum_{k=q+2}^{p}[\tfrac{k+q}{2}] \\ & +16\sum_{l=1}^{q-1}\sum_{k=l+1}^{p}[\tfrac{k+l}{2}] \\ = & \tfrac{16}{3}q^3+4p^2q+8q^2+\tfrac{8}{3}q+(2p^2-2)+(2p^2-8q+4pq-6q^2) \\ & +(-28-4p^2+2q-4pq+4p^2q+2q^2+4pq^2-4q^3) \\ = & -30-\tfrac{10}{3}q+8p^2q+4q^2+4pq^2+\tfrac{4}{3}q^3 \end{aligned}$$

By Lemma 1, we have
$W(G)=4pqd_G(x_{11})=-120pq-\frac{40}{3}pq^2+32p^3q^2+16pq^3+16p^2q^3+\frac{16}{3}pq^4.$

In the following, we compute the Wiener index of $G=T_{p,q}[C_4,C_8]$ for $p<q+2$. Let $C(t)$ denote the sum of the elements of column t in $A(p,q)$.
$C(t)=\sum_{r=1}^{4q}a_{rt}=\sum_{r=1}^{2q+1}a_{rt}+\sum_{r=2q+2}^{4q}a_{rt}.$

Lemma 8. If $p\le q+1$, then

$$d_G(x_{11})=-8+\frac{32}{3}p-\frac{8}{3}p^3+4p^2q+8pq^2-8\sum_{t'=2}^{p}[\frac{t'+2}{2}]+16\sum_{t=2}^{p}\sum_{k=t}^{p}[\frac{t+k}{2}].$$

Proof. Using Theorem 1, we first compute $C(t)$.
Case I. $1\le t\le 2$.

$$\begin{aligned} C(1)= & \sum_{r=1}^{4q}a_{r1}=\sum_{r=1}^{2q+1}d_1(x_{11},x_{r1})+\sum_{r'=2}^{2q}d_3(x_{11},x_{r'1}) \\ = & \sum_{r=1}^{2q+1}(r-1)+\sum_{r'=2}^{2q}(r'-1)=4q^2. \\ C(2)= & \sum_{r=1}^{4q}a_{r2}=\sum_{r=1}^{2q+1}d_1(x_{11},x_{r2}+\sum_{r'=2}^{2q}d_3(x_{11},x_{r'2}) \\ = & \sum_{r=1}^{2q+1}r+\sum_{r'=2}^{2q}r'=4q+4q^2. \end{aligned}$$

Case II. $3\le t\le p+1$.

$$\begin{aligned} & \sum_{r=1}^{2q+1}a_{rt}=\sum_{r=1}^{2q+1}d_1(x_{11},x_{rt})=\sum_{r=1}^{2t-5}d_1(x_{11},x_{rt})+\sum_{r=2t-4}^{2q+1}d_1(x_{11},x_{rt}) \\ & =((t+2[\tfrac{2t+1+1}{4}]-3)+(t+2[\tfrac{2t+3+1}{4}]-3)+\cdots+(t+2[\tfrac{2t+(2t-5)+1}{4}]-3)) \\ & +((t+2[\tfrac{2t+2+2}{4}]-4)+(t+2[\tfrac{2t+4+2}{4}]-4)+\cdots+(t+2[\tfrac{2t+(2t-6)+2}{4}]-4)) \\ & +\sum_{r=2t-4}^{2q+1}(t+r-2) \\ & =-3-q+2q^2+7t+2qt-2t^2-2[\tfrac{t+1}{2}]+4\sum_{k=1}^{t-2}[\tfrac{t+k}{2}] \end{aligned}$$

$$\sum_{r=2q+2}^{4q} a_{rt} = \sum_{r'=2}^{2q} d_3(x_{11}, x_{r't}) = \sum_{r'=2}^{2t-4} d_3(x_{11}, x_{r't}) + \sum_{r'=2t-3}^{2q} d_3(x_{11}, x_{r't})$$
$$= ((t + 2[\tfrac{2t+3+1}{4}] - 3) + (t + 2[\tfrac{2t+5+1}{4}] - 3) + \cdots + (t + 2[\tfrac{2t+(2t-5)+1}{4}] - 3))$$
$$+((t + 2[\tfrac{2t+2}{4}] - 2) + (t + 2[\tfrac{2t+4}{4}] - 2) + \cdots + (t + 2[\tfrac{2t+(2t-4)}{4}] - 2))$$
$$+ \sum_{r'=2t-3}^{2q} (t + r' - 2)$$
$$= -1 - 3q + 2q^2 + 5t + 2qt - 2t^2 - 2[\tfrac{t+1}{2}] + 4\sum_{k=1}^{t-2} [\tfrac{t+k}{2}]$$

$$C(t) = \sum_{r=1}^{4q} a_{rt} = \sum_{r=1}^{2q+1} a_{rt} + \sum_{r=2q+2}^{4q} a_{rt}$$
$$= -4 - 4q + 4q^2 + 12t + 4qt - 4t^2 - 4[\tfrac{t+1}{2}] + 8\sum_{k=1}^{t-2} [\tfrac{t+k}{2}].$$

$$\begin{aligned}\sum_{t=1}^{p+1} C(t) = {} & 4q + 8q^2 + \sum_{t=3}^{p+1}(-4 - 4q + 4q^2 + 12t + 4qt - 4t^2 \\ & -4[\tfrac{t+1}{2}] + 8\sum_{k=1}^{t-2}[\tfrac{t+k}{2}]) \\ = {} & -4 + \tfrac{16}{3}p - \tfrac{4}{3}p^3 + 2pq + 2p^2q + 4q^2 + 4pq^2 \\ & -4\sum_{t=3}^{p+1}[\tfrac{t+1}{2}] + 8\sum_{t=3}^{p+1}\sum_{k=1}^{t-2}[\tfrac{t+k}{2}].\end{aligned}$$

Case III. $p + 2 \le t \le 2p$, i.e., $2 \le t' \le p$.

$$\begin{aligned}\sum_{r=1}^{2q+1} a_{rt} = {} & \sum_{r=1}^{2q+1} d_2(x_{11}, x_{rt'}) = \sum_{r=1}^{2t'-3} d_2(x_{11}, x_{rt'}) + \sum_{r=2t'-2}^{2q+1} d_2(x_{11}, x_{rt'}) \\ = {} & ((t' + 2[\tfrac{2t'+1+3}{4}] - 3) + (t' + 2[\tfrac{2t'+3+3}{4}] - 3) + \cdots \\ & +(t' + 2[\tfrac{2t'+(2t'-3)+3}{4}] - 3)) + ((t' + 2[\tfrac{2t'+2+4}{4}] - 4) \\ & +(t' + 2[\tfrac{2t'+4+4}{4}] - 4) + \cdots + (t' + 2[\tfrac{2t'+(2t'-4)+4}{4}] - 4)) \\ & + \sum_{r=2t'-2}^{2q+1} (t' + r - 2) \\ = {} & 1 - q + 2q^2 + 3t' + 2qt' - 2t'^2 - 2[\tfrac{t'+2}{2}] + 4\sum_{k=2}^{t'} [\tfrac{t'+k}{2}]\end{aligned}$$

$$\begin{aligned}\sum_{r=2q+2}^{4q} a_{rt} = {} & \sum_{r'=2}^{2q} d_4(x_{11}, x_{r't'}) \\ = {} & \sum_{r'=2}^{2t'-2} d_4(x_{11}, x_{r't'}) + \sum_{r'=2t'-1}^{2q} d_4(x_{11}, x_{r't'}) \\ = {} & ((t' + 2[\tfrac{2t'+3+3}{4}] - 3) + (t' + 2[\tfrac{2t'+5+3}{4}] - 3) + \cdots \\ & +(t' + 2[\tfrac{2t'+(2t'-3)+3}{4}] - 3)) + ((t' + 2[\tfrac{2t'+2+2}{4}] - 2) \\ & +(t' + 2[\tfrac{2t'+4+2}{4}] - 2) + \cdots + (t' + 2[\tfrac{2t'+(2t'-2)+2)}{4}] - 2)) \\ & + \sum_{r=2t'-1}^{2q} (t' + r - 2) \\ = {} & 3 - 3q + 2q^2 + t' + 2qt' - 2t'^2 - 2[\tfrac{t'+2}{2}] + 4\sum_{k=2}^{t'} [\tfrac{t'+k}{2}]\end{aligned}$$

$$C(t) = \sum_{r=1}^{4q} a_{rt} = \sum_{r=1}^{2q+1} a_{rt} + \sum_{r=2q+2}^{4q} a_{rt}$$

$$= 4 - 4q + 4q^2 + 4t' + 4qt' - 4t'^2 - 4[\tfrac{t'+2}{2}] + 8\sum_{k=2}^{t'}[\tfrac{t'+k}{2}].$$

$$\begin{aligned}\sum_{t=p+2}^{2p} C(t) = & \sum_{t'=2}^{p}(4 - 4q + 4q^2 + 4t' + 4qt' - 4t'^2 - 4[\tfrac{t'+2}{2}] + 8\sum_{k=2}^{t'}[\tfrac{t'+k}{2}]) \\ = & -4 + \tfrac{16}{3}p - \tfrac{4}{3}p^3 - 2pq + 2p^2q - 4q^2 + 4pq^2 - 4\sum_{t'=2}^{p}[\tfrac{t'+2}{2}] \\ & +8\sum_{t'=2}^{p}\sum_{k=2}^{t'}[\tfrac{t'+k}{2}].\end{aligned}$$

Therefore, we have

$$\begin{aligned}d_G(x_{11}) = & \sum_{t=1}^{2p}\sum_{r=1}^{4q} a_{rt} = \sum_{t=1}^{2p} C(t) = \sum_{t=1}^{p+1} C(t) + \sum_{t=p+2}^{2p} C(t) \\ = & -4 + \tfrac{16}{3}p - \tfrac{4}{3}p^3 + 2pq + 2p^2q + 4q^2 + 4pq^2 - 4\sum_{t=3}^{p+1}[\tfrac{t+1}{2}] \\ & +8\sum_{t=3}^{p+1}\sum_{k=1}^{t-2}[\tfrac{t+k}{2}] - 4 + \tfrac{16}{3}p - \tfrac{4}{3}p^3 - 2pq + 2p^2q - 4q^2 \\ & +4pq^2 - 4\sum_{t'=2}^{p}[\tfrac{t'+2}{2}] + 8\sum_{t'=2}^{p}\sum_{k=2}^{t'}[\tfrac{t'+k}{2}] \\ = & -8 + \tfrac{32}{3}p - \tfrac{8}{3}p^3 + 4p^2q + 8pq^2 - 8\sum_{t'=2}^{p}[\tfrac{t'+2}{2}] + 16\sum_{t=2}^{p}\sum_{k=2}^{t}[\tfrac{t+k}{2}] \\ = & -8 + \tfrac{32}{3}p - \tfrac{8}{3}p^3 + 4p^2q + 8pq^2 - 8\sum_{t'=2}^{p}[\tfrac{t'+2}{2}] + 16\sum_{t=2}^{p}\sum_{k=t}^{p}[\tfrac{t+k}{2}]\end{aligned}$$

Theorem 3.

If $p \le q + 1$, then the Wiener index of $G = T_{p,q}[C_4, C_8]$ is $W(G)$

$$= \begin{cases} -96pq - \frac{136}{3}p^2q + 16p^3q + \frac{16}{3}p^4q + 16p^3q^2 + 32p^2q^3, & \text{if } p \text{ is even;} \\ -\frac{16}{3}p^2q + \frac{16}{3}p^4q + 16p^3q^2 + 32p^2q^3, & \text{if } p \text{ is odd.}\end{cases}$$

Proof. (1) If p is even, then by Lemma 7, we have

$$\sum_{t'=2}^{p}[\tfrac{t'+2}{2}] = \tfrac{1}{4}((p+2)^2 - 4^2) + 2 = -1 + p + \tfrac{1}{4}p^2,$$

$$\sum_{k=2}^{p}[\tfrac{k+2}{2}] = \tfrac{1}{4}((p+2)^2 - 4^2) + 2,$$

$$\sum_{k=3}^{p}[\tfrac{k+3}{2}] = \tfrac{1}{4}((p+3)^2 - 6^2) + \tfrac{11}{4},$$

$$\sum_{k=4}^{p}[\tfrac{k+4}{2}] = \tfrac{1}{4}((p+4)^2 - 8^2) + 4,$$

$$\sum_{k=5}^{p}[\tfrac{k+5}{2}] = \tfrac{1}{4}((p+5)^2 - 10^2) + \tfrac{19}{4},$$

$\cdots\cdots$

$$\sum_{k=p-2}^{p}[\tfrac{k+p-2}{2}] = \tfrac{1}{4}((p+(p-2))^2 - (2p-4)^2) + (p-2),$$

$$\sum_{k=p-1}^{p}[\frac{k+p-1}{2}] = \frac{1}{4}((p+(p-1))^2-(2p-2)^2)+\frac{1}{4}(4(p-1)-1),$$
$$\sum_{k=p}^{p}[\frac{k+p}{2}] = \frac{1}{4}((p+p)^2-(2p)^2)+p.$$
$$\begin{aligned}\sum_{t=2}^{p}\sum_{k=t}^{p}[\frac{t+k}{2}] = & \frac{1}{4}(\sum_{l=2}^{p}((p+l)^2-(2l)^2)+\sum_{t=1}^{\frac{p}{2}}2t+\frac{1}{4}\sum_{r=3}^{p-1}(4r-1) \\ = & -\frac{3}{2}-\frac{7p}{8}+\frac{3p^2}{8}+\frac{p^3}{4}\end{aligned}$$

By Lemma 8,

$$\begin{aligned}d_G(x_{11}) = & -8+\frac{32}{3}p-\frac{8}{3}p^3+4p^2q+8pq^2-8\sum_{t'=2}^{p}[\frac{t'+2}{2}] \\ & +16\sum_{t=2}^{p}\sum_{k=t}^{p}[\frac{t+k}{2}] \\ = & -8+\frac{32}{3}p-\frac{8}{3}p^3+4p^2q+8pq^2-8(-1+p+\frac{1}{4}p^2) \\ & +16(-\frac{3}{2}-\frac{7p}{8}+\frac{3p^2}{8}+\frac{p^3}{4}) \\ = & -24-\frac{34}{3}p+4p^2+\frac{4}{3}p^3+8pq^2+4p^2q\end{aligned}$$

By Lemma 1, we have
$W(G) = 4pqd_G(x_{11}) = -96pq-\frac{136}{3}p^2q+16p^3q+\frac{16}{3}p^4q+16p^3q^2+32p^2q^3.$

(2) If p is odd, then by Lemma 7, we have

$$\sum_{t'=2}^{p}[\frac{t'+2}{2}] = \frac{1}{4}((p+2)^2-4^2)+2-\frac{1}{4} = -\frac{5}{4}+p+\frac{1}{4}p^2,$$
$$\sum_{k=2}^{p}[\frac{k+2}{2}] = \frac{1}{4}((p+2)^2-4^2)+\frac{7}{4},$$
$$\sum_{k=3}^{p}[\frac{k+3}{2}] = \frac{1}{4}((p+3)^2-6^2)+3,$$
$$\sum_{k=4}^{p}[\frac{k+4}{2}] = \frac{1}{4}((p+4)^2-8^2)+\frac{15}{4},$$
$$\sum_{k=5}^{p}[\frac{k+5}{2}] = \frac{1}{4}((p+5)^2-10^2)+5,$$

......

$$\sum_{k=p-2}^{p}[\frac{k+p-2}{2}] = \frac{1}{4}((p+(p-2))^2-(2p-4)^2)+(p-2),$$
$$\sum_{k=p-1}^{p}[\frac{k+p-1}{2}] = \frac{1}{4}((p+(p-1))^2-(2p-2)^2)+\frac{1}{4}(4(p-1)-1),$$
$$\sum_{k=p}^{p}[\frac{k+p}{2}] = \frac{1}{4}((p+p)^2-(2p)^2)+p.$$
$$\begin{aligned}\sum_{t=2}^{p}\sum_{k=t}^{p}[\frac{t+k}{2}] = & \frac{1}{4}(\sum_{l=2}^{p}((p+l)^2-(2l)^2)+\sum_{t=1}^{\frac{p-1}{2}}(2t+1)+\frac{1}{4}\sum_{r=1}^{\frac{p-1}{2}}(8r-1) \\ = & -\frac{1}{8}-\frac{p}{4}+\frac{p^2}{8}+\frac{p^3}{4}\end{aligned}$$

By Lemma 8,

$$
\begin{aligned}
d_G(x_{11}) = \; & -8 + \tfrac{32}{3}p - \tfrac{8}{3}p^3 + 4p^2q + 8pq^2 - 8\sum_{t'=2}^{p}[\tfrac{t'+2}{2}] + 16\sum_{t=2}^{p}\sum_{k=t}^{p}[\tfrac{t+k}{2}] \\
= \; & -8 + \tfrac{32}{3}p - \tfrac{8}{3}p^3 + 4p^2q + 8pq^2 - 8(-\tfrac{5}{4} + p + \tfrac{1}{4}p^2) \\
& +16(-\tfrac{1}{8} - \tfrac{p}{4} + \tfrac{p^2}{8} + \tfrac{p^3}{4}) \\
= \; & -\tfrac{4}{3}p + \tfrac{4}{3}p^3 + 4p^2q + 8pq^2
\end{aligned}
$$

By Lemma 1, we have
$W(G) = 4pqd_G(x_{11}) = -\frac{16}{3}p^2q + \frac{16}{3}p^4q + 16p^3q^2 + 32p^2q^3.$

For example. From the figure 2, we have $p = 7$ and $q = 4$

$$
A(7,4) = \begin{bmatrix}
0 & 1 & 4 & 5 & 8 & 9 & 12 & 13 & 12 & 11 & 8 & 7 & 4 & 3 \\
1 & 2 & 3 & 6 & 7 & 10 & 11 & 14 & 13 & 10 & 9 & 6 & 5 & 2 \\
2 & 3 & 4 & 7 & 8 & 11 & 12 & 15 & 14 & 11 & 10 & 7 & 6 & 3 \\
3 & 4 & 5 & 6 & 9 & 10 & 13 & 14 & 13 & 12 & 9 & 8 & 5 & 4 \\
4 & 5 & 6 & 7 & 10 & 11 & 14 & 15 & 14 & 13 & 10 & 9 & 6 & 5 \\
5 & 6 & 7 & 8 & 9 & 12 & 13 & 16 & 15 & 12 & 11 & 8 & 7 & 6 \\
6 & 7 & 8 & 9 & 10 & 13 & 14 & 17 & 16 & 13 & 12 & 9 & 8 & 7 \\
7 & 8 & 9 & 10 & 11 & 12 & 15 & 16 & 15 & 14 & 11 & 10 & 9 & 8 \\
8 & 9 & 10 & 11 & 12 & 13 & 16 & 17 & 16 & 15 & 12 & 11 & 10 & 9 \\
7 & 8 & 9 & 10 & 11 & 14 & 15 & 18 & 17 & 14 & 13 & 10 & 9 & 8 \\
6 & 7 & 8 & 9 & 10 & 13 & 14 & 17 & 16 & 13 & 12 & 9 & 8 & 7 \\
5 & 6 & 7 & 8 & 11 & 12 & 15 & 16 & 15 & 14 & 11 & 10 & 7 & 6 \\
4 & 5 & 6 & 7 & 10 & 11 & 14 & 15 & 14 & 13 & 10 & 9 & 6 & 5 \\
3 & 4 & 5 & 8 & 9 & 12 & 13 & 16 & 15 & 12 & 11 & 8 & 7 & 4 \\
2 & 3 & 4 & 7 & 8 & 11 & 12 & 15 & 14 & 11 & 10 & 7 & 6 & 3 \\
1 & 2 & 5 & 6 & 9 & 10 & 13 & 14 & 13 & 12 & 9 & 8 & 5 & 4
\end{bmatrix}
$$

Then $W(T_{7,4}[C_4, C_8]) = 4pq\sum_{i=1}^{4q}\sum_{j=1}^{2p} = 234752$. By Theorem 2, we have also that $W(T_{7,4}[C_4, C_8]) = -96pq - \frac{136}{3}pq^2 + 32p^3q^2 + 16pq^3 + 16p^2q^3 + \frac{16}{3}pq^4 = 234752$.

If $p = 5$ and $q = 4$, then we can obtain $A(5,4)$ from $A(7,4)$ by deleting its columns 7,8,9 and 10. So, $W(T_{5,4}[C_4, C_8]) = 4pq\sum_{i=1}^{4q}\sum_{j=1}^{2p} = 96000$. And from Theorem 3, we have the same value $W(T_{5,4}[C_4, C_8]) = -\frac{16}{3}p^2q + \frac{16}{3}p^4q + 16p^3q^2 + 32p^2q^3 = 96000$.

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