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On a Pair of Equienergetic Graphs

G. Indulal* and **A. Vijayakumar†**

Department of Mathematics, Cochin University of Science and Technology, Cochin-682 022, India.

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Abstract

In this paper we solve the problem of constructing a pair of equienergetic graphs on p vertices for $p = 6, 14, 18$ and for all $p \geq 20$.

1 Introduction

Let G be a graph on p vertices. The eigenvalues of the adjacency matrix of G, denoted by λ_i , $i = 1, 2, ...p$, are the eigenvalues of G, and form the spectrum of G [4], denoted by $spec(G)$.The eigenvalues of graphs play a very important role in many branches of science. Cvetković [4], Godsil [5], and Schwenk [15] have studied the eigenvalues of graphs in detail.

From the pioneering work of Coulson [3] there exists a continuous interest towards the general mathematical properties of the total π-electron energy E as calculated within the framework of the Hückel Molecular Orbital (HMO) model [8]. These efforts enabled one to get an insight into the dependence of E on molecular structure.

The energy $E(G)$ of a graph G with p vertices is defined as $E(G) = \sum_{i=1}^{p} |\lambda_i|$. The energy of graphs has been extensively studied by Gutman [6, 7, 8, 9] whereas some other recent works are [1, 11, 12, 16, 18, 19, 20, 21, 22, 23].

*E-mail: indulalgopal@rediffmail.com
†E-mail:vijay@cusat.ac.in

Two graphs of the same order are cospectral if they have the same spectrum. Two cospectral graphs are obviously equienergetic. It can be seen that there are no equienergetic graphs of order $p \leq 5$. For $p = 7, 9$, there are such graphs but they are cospectral [4]. In [2] a class of equienergetic graphs for $p \equiv 0(\text{mod}4)$ are constructed using the tensor product of graphs. Stevanović [17] recently constructed such graphs for $p \equiv 0(\text{mod}5)$.

In this paper we construct pairs of equienergetic graphs first for $p = 6, 14, 18$ and then for every $p \geqslant 20$.

Property 1. *If $\lambda_1, \lambda_2, \ldots, \lambda_p$ are the eigenvalues of G with adjacency matrix A, then* $\det A \prod_{i=1}^{p} \lambda_i$. *Also for any polynomial $P(x)$, $P(\lambda)$ is a characteristic value of $P(A)$ and hence* $\det P(A) = \prod_{i=1}^{p} P(\lambda_i)$.

Property 2. *If M, N, P, and Q are matrices with M being non-singular, then*

$$\begin{vmatrix} M & N \\ P & Q \end{vmatrix} = \begin{vmatrix} M \end{vmatrix} \begin{vmatrix} Q - PM^{-1}N \end{vmatrix} .$$

Definition 1. *[13] Let G be a graph on p vertices labelled as $V\{v_1, v_2, v_3, \ldots, v_p\}$. Then take another set $U = \{u_1, u_2, \ldots, u_p\}$ of p vertices. Now define a graph H with $V(H) = V \bigcup U$ and edge set of H consisting only of those edges joining u_i to neighbors of v_i in G for each i. The resultant graph H is called the identity duplication graph of G denoted by DG.*

Definition 2. *Let G be a graph on p vertices labelled as $\{v_1, v_2, \ldots, v_p\}$. Now take another copy of G with vertices labelled as $\{u_1, u_2, \ldots, , u_p\}$. Then make u_i adjacent to neighbors of v_i in G, for each i. The resultant graph is denoted by D_2G.*

Lemma 1. *[4] Let G be a connected r-regular graph on p vertices with its adjacency matrix A having m distinct eigenvalues $\lambda_1 = r$, $\lambda_2, \ldots, \lambda_m$. Then there exists a polynomial $P(x) = p \times \frac{(x-\lambda_2)(x-\lambda_3)\ldots(x-\lambda_m)}{(r-\lambda_2)(r-\lambda_3)\ldots(r-\lambda_m)}$, such that $P(A) = J$ where J is the square matrix of order p whose all entries are one, so that $P(r) = p$ and $P(\lambda_i) = 0 \; \forall \lambda_i \neq r$.*

Lemma 2. *[4] Let A and B be two matrices with spec$(A)\{\lambda_i\}$, $i = 1$ to m and spec$(B) = \{\mu_j\}$, $j = 1$ to n. Let $C = A \bigotimes B$, the tensor product of A and B. Then spec$(C) = \{\lambda_i \mu_j\}$, $i = 1$ to m and $j = 1$ to n.*

Lemma 3. *Let G be a graph with spec$(G) = \{\lambda_i\}$, $i = 1$ to p. Then spec$(DG) = \{\pm\lambda_i\}$ and spec$(D_2G) \begin{pmatrix} 2\lambda_1 & 2\lambda_2 & .. & .. & 2\lambda_p & 0 \\ 1 & 1 & .. & .. & 1 & p \end{pmatrix}$. Then DG and D_2G are non co-spectral and are equienergetic.*

Proof. By Definition 1, the adjacency matrix of DG is

$$A(DG) = \begin{bmatrix} 0 & A(G) \\ A(G) & 0 \end{bmatrix} = A \bigotimes \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}.$$

Then by Lemma 2, $spec(DG) = \{\pm\lambda_i\}$.

Also by Definition 1, the adjacency matrix of D_2G,

$$A(D_2G) = \begin{bmatrix} A(G) & A(G) \\ A(G) & A(G) \end{bmatrix} = A \bigotimes \begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix}.$$

Then by Lemma 2, $spec(D_2G) = \begin{pmatrix} 2\lambda_1 & 2\lambda_2 & .. & .. & 2\lambda_p & 0 \\ 1 & 1 & .. & .. & 1 & p \end{pmatrix}$ as the $spec(\begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix})$ is $\{2, 0\}$.

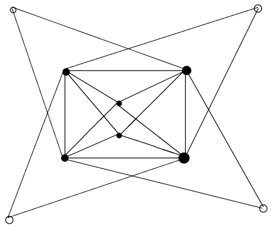
Thus by the definition of energy, DG and D_2G are equienergetic and non co-spectral.

□

We shall introduce the following operation on a regular graph G.

Definition 3. *Let G be an r-regular graph on p vertices labelled as $\{v_1, v_2, \ldots, v_p\}$. Introduce a set of p isolated vertices $\{u_1, u_2, \ldots, u_p\}$ and make each u_i adjacent to neighbors of v_i in G for every i. Then introduce a set of k, $(k \geq 0)$ isolated vertices and make all of them adjacent to all vertices of G. The resultant graph is denoted by H.*

Ex: When G= C_4 H:



Note that when $k=0$, H is the splittance graph of G [14].

Theorem 1. *Let G be a connected r-regular graph on p vertices and H be the graph obtained by the above operation. Then* $E(H)=\sqrt{5}\left[E(G)+\sqrt{r^2+\frac{4}{5}pk}-r\right]$.

Proof. Let A be the adjacency matrix of G and J the all one matrix. Then the adjacency matrix of H is given by

$$\begin{bmatrix} A & A & J_{p\times k} \\ A & 0 & 0 \\ J_{k\times p} & 0 & 0 \end{bmatrix}.$$

Therefore the characteristic polynomial of H is

$$\begin{vmatrix} \lambda I-A & -A & -J_{p\times k} \\ -A & \lambda I & 0 \\ -J_{k\times p} & 0 & \lambda I_k \end{vmatrix}=0 \qquad (1)$$

Now,

$$(1)\Rightarrow \begin{vmatrix} \lambda I_k & 0 & -J_{k\times p} \\ 0 & \lambda I & -A \\ -J_{p\times k} & -A & \lambda I-A \end{vmatrix}=0$$

$$\Rightarrow \lambda^k \left| \begin{bmatrix} \lambda I & -A \\ -A & \lambda I-A \end{bmatrix} - \begin{bmatrix} 0 \\ -J_{p\times k} \end{bmatrix} \frac{I}{\lambda} \begin{bmatrix} 0 & -J_{k\times p} \end{bmatrix} \right| = 0$$

$$\Rightarrow \lambda^{k-2p} \begin{vmatrix} \lambda^2 I & -A\lambda \\ -A\lambda & \lambda^2 I - A\lambda - kJ \end{vmatrix} = 0$$

$$\Rightarrow \lambda^k \left|\lambda^2 I - A\lambda - kJ - A^2\right| = 0$$

$$\Rightarrow \lambda^k \prod_{i=1}^{p} \left[x^2 - \lambda_i x - kP(\lambda_i) - \lambda_i^2\right] = 0$$

Thus the eigenvalues of H are

$$x = 0;\ k \text{ times}$$

$$= \frac{r \pm \sqrt{5r^2 + 4pk}}{2};\ \text{corresponding to } \lambda_1 = r \text{ by Lemma 1}$$

$$= \left(\frac{1 \pm \sqrt{5}}{2}\right) \lambda_i\ ;\ i \neq 1\ .$$

Thus $E(H) = \sqrt{5}\left[E(G) + \sqrt{r^2 + \frac{4}{5}pk} - r\right]$. □

We shall now discuss the problem of constructing pairs of equienergetic graphs on p vertices, by analyzing the various cases.

Theorem 2. *There exists a pair of equienergetic graphs for $p = 6, 14, 18$ and $p \geq 20$.*

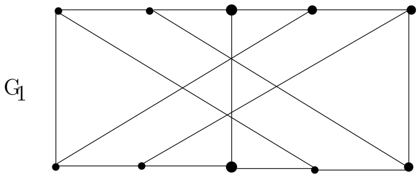
Proof.

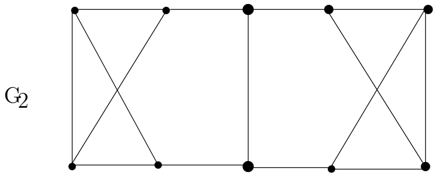
Case 1. $p = 6, 14, 18$.

For $p = 6, 14\ \ 18$, consider $G = C_3, C_7$, and C_9, respectively. Let $G_1 = DG$ and $G_2 = D_2G$ for each case. Then both G_1 and G_2 are on 6,14, and 18 vertices, respectively, connected, non-co spectral and by Lemma 2, $E(G_1) = E(G_2) = 2E(G)$.

Case 2. $p \geq 20$.

The following cubic graphs G_1 and G_2 on 10 vertices are equienergetic and non co-spectral [4].





Let H_k and H'_k be the graphs obtained from G_1 and G_2 as in Definition 3 for $k \geq 0$. Then by Theorem 1, the graphs H_k and H'_k are non co-spectral, equienergetic and are on $20 + k$, $k \geq 0$ vertices.

Hence the theorem. $\square$

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