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Sharp lower bounds for the general Randić index of trees with a given size of matching

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Abstract

The general Randić index $w_\alpha(G)$ of a graph G is the sum of the weights $(d(u)d(v))^\alpha$ of all edges uv of G, where α is a real number and $d(u)$ denotes the degree of the vertex u. Let $\mathscr{T}_{n,m}$ be the set of all trees on n vertices with a maximum matching of cardinality m. Denote by $T^0_{n,m}$ the tree on n vertices obtained from the star graph S_{n-m+1} by attaching a pendant edge to each of some $m-1$ non-central vertices of S_{n-m+1}. In this paper, we first prove that $T^0_{n,m}$ has the minimum general Randić index among the trees in $\mathscr{T}_{n,m}$ for $-\frac{1}{2} \le \alpha < 0$. Also we obtain lower bounds for the general Randić index among trees in $\mathscr{T}_{n,m}$ $(2m \le n \le 3m+1)$ for $\alpha > 0$, and the corresponding extremal graphs.

1 Introduction

For a (molecular) graph $G = (V, E)$, the *general Randić index* $w_\alpha(G)$ is defined in [1] as

$$w_\alpha(G) = \sum_{uv \in E} [d(u)d(v)]^\alpha,$$

where α is a real number.

It is well known that the Randić index $w_{-\frac{1}{2}}(G)$ was proposed by Randić [17] in 1975 and Bollobás and Erdős [1] generalized the index by replacing $-\frac{1}{2}$ with any real number α in 1998. The research background of Randić index together with its generalization appears in chemical field and can be found in the literature (see [4, 11, 12, 15, 17]).

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Recently, finding bounds for the general Randić index of graphs, in particular, of trees, as well as related problem of finding the graphs having maximum or minimum general Randić index, attracted the attention of many researchers and many results are obtained (see [1,3-6,8-10,13-16,18,19]). Among them the following results are of interest for trees. Yu [19] gave a sharp upper bound of $w_{-\frac{1}{2}}$ for trees of order n. Later, Caporossi *et al* [3] obtained the same result using an alternative approach. Clark and Moon [4] gave bounds for w_{-1} of trees with order n. Rautenbach [18] gave sharp upper bounds for w_{-1} of trees with some restrictions on degree of vertices. In [14], X. Li and Y. Yang gave best lower and upper bounds for w_{-1} of chemical trees. Y. Hu *et al* discussed trees with minimum and maximum general Randić index in [10] and [9], respectively. Y. Hu *et al* [8] studied two unsolved questions on the best upper bounds for w_{-1} of trees.

In this paper, we give sharp lower bounds for the general Randić index w_α among trees of order n with an m-matching and the corresponding extremal graphs, where $n \geq 2m$ for $-\frac{1}{2} \leq \alpha < 0$ and $2m \leq n \leq 3m+1$ for $\alpha > 0$, respectively.

The proofs of our results are in Section 3, and some terminologies, notations and lemmas are given in Section 2.

2 Notations and lemmas

In order to discuss the general Randić index of molecular graphs, we first introduce some terminologies and notations of graphs. Other undefined terminologies and notations may refer to [2].

The number of vertices, $|V|$, is called *order* of the graph. For a vertex x of a graph G, we denote the neighborhood and the degree of x by $N(x)$ and $d(x)$, respectively. We denote by $\Delta(G)$ the maximum degree of vertices of G. Let $V' \subset V$, we will use $G - V'$ to denote the graph obtained from G by deleting the vertices in V' together with their incident edges. If $V' = \{v\}$, we write $G - v$ for $G - \{v\}$. A tree is a connected acyclic graph. A *pendant vertex* is a vertex of degree 1 and a pendant edge is an edge incident to a pendant vertex. Denote by PV the set of pendant vertices of T. Let T be a tree and $P_s = v_0v_1\cdots v_s$ a path of T with $d(v_1) = d(v_2) = \cdots = d(v_{s-1}) = 2$ (unless $s = 1$). If $d(v_0), d(v_s) \geq 3$, then P_s is called an *internal chain* of T.

A subset $M \subseteq E$ is called a *matching* in G if its elements are edges and no two are adjacent in G. A matching M *saturates* a vertex v, and v is said to be M-*saturated*, if some edge of M is incident with v. If every vertex of G is M-saturated, the matching M is *perfect*. A matching M is said to be an m-matching, if $|M| = m$ and for every matching M' in G, $|M'| \leq m$. Denote $\mathscr{T}_{n,m} = \{T \ : \ T$ is a tree of order $n \geq 2m$ with an m-matching$\}$.

Let n and m be positive integers with $n \geq 2m$. We define $T^0_{n,m}$ (shown in Fig. 1) as a tree of order n obtained from the star graph S_{n-m+1} by attaching a pendant edge to each of some $m-1$ non-central vertices of S_{n-m+1}. Clearly, $T^0_{n,m}$ is a tree of order n with an m-matching and $T^0_{2m,m}$ (shown in Fig. 1) is a tree of order $2m$ with a perfect matching.

Lemma 2.1 [7]**.** *Let T be a tree of order n $(n > 2)$ with a perfect matching. Then T has at least two pendant vertices such that they are adjacent to vertices of degree 2, respectively.*

Lemma 2.2 [7]**.** *Let T be a tree of order n with an m-matching. If $n = 2m+1$, then T has a pendant vertex which is adjacent to a vertex of degree 2.*

Lemma 2.3 [7]**.** *Let T be a tree of order n with an m-matching, where $n > 2m$. Then there is an m-matching M and a pendant vertex v such that M does not saturate v.*

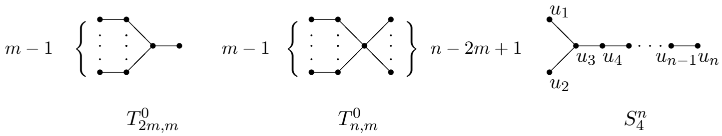

Fig. 1

The proof of the following lemma is very trivial, so we will omit it here.

Lemma 2.4. *Let G be a graph of order $2m$ with a perfect matching. If $PV \neq \emptyset$, then for any vertex $u \in V(G)$, $|N(u) \cap PV| \leq 1$.*

Lemma 2.5. *Let $f(x) := (x+1)^\alpha \cdot (x \cdot 2^\alpha + 1),\ x \geq 1$. Then*

(i) the function $f(x) - f(x+1)$ is monotonously increasing for $-\frac{1}{2} \leq \alpha < 0$ in $x \geq 1$;

(ii) the function $f(x+1) - f(x)$ is monotonously increasing for $\alpha > 0$ in $x \geq 1$.

Proof. Note that

$$\frac{d^2 f(x)}{dx^2} = \alpha(x+1)^{\alpha-2} \cdot \Big[(\alpha+1) \cdot x \cdot 2^\alpha + 2^{\alpha+1} + \alpha - 1\Big]. \tag{1}$$

(i) Let $g(x) := (\alpha+1) \cdot x \cdot 2^\alpha + 2^{\alpha+1} + \alpha - 1$. Then for $-\frac{1}{2} \leq \alpha < 0$, $\frac{dg(x)}{dx} = 2^\alpha \cdot (\alpha+1) > 0$. Since $-\frac{1}{2} \leq \alpha < 0$, by (1), we get

$$\begin{aligned}(\alpha+1) \cdot x \cdot 2^\alpha + 2^{\alpha+1} + \alpha - 1 &\geq (\alpha+1) \cdot 2^\alpha + 2^{\alpha+1} + \alpha - 1 \\ &\geq 4 \cdot 2^\alpha - \frac{3}{2} \cdot 2^\alpha - \frac{3}{2} = \frac{5 \cdot 2^\alpha - 3}{2} > 0.\end{aligned}$$

Hence $\frac{d^2 f(x)}{dx^2} < 0$ for $-\frac{1}{2} \leq \alpha < 0$. Thus the function $f(x) - f(x+1)$ is monotonously increasing for $-\frac{1}{2} \leq \alpha < 0$ in $x \geq 1$.

(ii) If $\alpha \geq 1$, then by (1), we have $\frac{d^2 f(x)}{dx^2} > 0$; if $0 < \alpha < 1$, then by (1), we have $2^{\alpha+1} > 2 > 1 - \alpha$, and hence $\frac{d^2 f(x)}{dx^2} > 0$. Thus the function $f(x+1) - f(x)$ is monotonously increasing for $\alpha > 0$ in $x \geq 1$. ■

Lemma 2.6. *The function* $(x+1)^\alpha \cdot (x+2^\alpha) - x^\alpha \cdot (x-1+2^\alpha)$ *is monotonously increasing for* $\alpha > 0$ *in* $x \geq 1$.

Proof. Let $f(x) := x^\alpha \cdot (x-1+2^\alpha),\ x \geq 1$. Then

$$\frac{d^2 f(x)}{dx^2} = \alpha \cdot x^{\alpha-2} \cdot [(\alpha+1)\cdot x + (2^\alpha - 1)\cdot(\alpha-1)].$$

If $\alpha \geq 1$, then $\frac{d^2 f(x)}{dx^2} > 0$; if $0 < \alpha < 1$, then $1+\alpha > 1 > 2^\alpha - 1$, and hence $\frac{d^2 f(x)}{dx^2} > 0$. Thus the function $f(x+1) - f(x)$ is monotonously increasing for $\alpha > 0$ in $x \geq 1$. ■

Lemma 2.7. *Let* $g(x) := x^\alpha,\ x \geq 1$. *Then*

(i) the function $g(x+1) - g(x)$ *are monotonously increasing in* $x \geq 1$ *for* $\alpha < 0$ *and* $\alpha > 1$, *respectively;*

(ii) the function $g(x) - g(x+1)$ *is monotonously increasing for* $0 < \alpha < 1$ *in* $x \geq 1$.

Proof. Note that $\frac{d^2 g(x)}{dx^2} = \alpha \cdot (\alpha-1) \cdot x^{\alpha-2}$. Then $\frac{d^2 g(x)}{dx^2} > 0$ if $\alpha < 0$ or $\alpha > 1$ and $\frac{d^2 g(x)}{dx^2} < 0$ if $0 < \alpha < 1$. The lemma follows. ■

Lemma 2.8. *Let* x, y *be positive integers with* $0 \leq y \leq x-1$. *Denote* $h(x,y) := x^\alpha \cdot [y + (x-y)\cdot 2^\alpha]$. *Then the function* $h(x-1,y) - h(x,y+1)$ *are monotonously increasing for* $-\frac{1}{2} \leq \alpha < 0$ *in* $x \geq 2$ *and* $y \geq 0$, *respectively.*

Proof. (i) Since $-\frac{1}{2} \leq \alpha < 0$ and $x \geq 2$, we have

$$\frac{\partial [h(x-1,y) - h(x,y+1)]}{\partial y} = (1-2^\alpha)\cdot[(x-1)^\alpha - x^\alpha] > 0.$$

Thus $h(x-1,y) - h(x,y+1)$ is monotonously increasing in $y \geq 0$ for $-\frac{1}{2} \leq \alpha < 0$.

(ii) Note that

$$\begin{aligned} & \frac{\partial [h(x-1,y) - h(x,y+1)]}{\partial x} \\ = \ & 2^\alpha \cdot [(x-1)^\alpha - x^\alpha] \\ & +\alpha \cdot \Big\{(x-1)^{\alpha-1}\cdot[y + (x-1-y)\cdot 2^\alpha] - x^{\alpha-1}\cdot[y+1+(x-1-y)\cdot 2^\alpha]\Big\}. \end{aligned}$$

Since $y \leq x-1$, we have

$$\begin{aligned} & \frac{\partial [h(x-1,y) - h(x,y+1)]}{\partial x} \\ \geq \ & 2^\alpha \cdot [(x-1)^\alpha - x^\alpha] + \alpha \cdot [(x-1)^{\alpha-1}\cdot(x-1) - x^{\alpha-1}\cdot x] \\ = \ & (2^\alpha + \alpha)\cdot[(x-1)^\alpha - x^\alpha] > 0. \end{aligned}$$

Hence $h(x-1,y) - h(x,y+1)$ is monotonously increasing in $x \geq 2$ for $-\frac{1}{2} \leq \alpha < 0$. ■

Lemma 2.9. *(i) If* $-\frac{1}{2} \leq \alpha < 0$, *then* $3\cdot 4^\alpha - 2\cdot 6^\alpha - 3^\alpha > 0$;

(ii) If $\alpha > 0$, *then* $6^\alpha + 2\cdot 3^\alpha - 2\cdot 4^\alpha - 2^\alpha > 0$, $2\cdot 6^\alpha + 3^\alpha - 3\cdot 4^\alpha > 0$, $2\cdot 3^\alpha + 9^\alpha - 6^\alpha - 4^\alpha - 2^\alpha > 0$ *and* $9^\alpha + 3^\alpha - 2\cdot 4^\alpha > 0$;

(iii) If $\alpha>0$, then $5\cdot4^\alpha+8^\alpha-3\cdot3^\alpha-3\cdot6^\alpha>0$, $2^\alpha+4\cdot4^\alpha+8^\alpha-4\cdot3^\alpha-2\cdot6^\alpha>0$ and $2^\alpha+4\cdot5^\alpha+10^\alpha-4\cdot3^\alpha-2\cdot6^\alpha>0$.

Proof. (i) If $-\frac{1}{2}\leq\alpha<0$, then, by Lemma 2.7 (i), we have $3\cdot4^\alpha-3^\alpha-2\cdot6^\alpha=4^\alpha-3^\alpha+2^{\alpha+1}\cdot(2^\alpha-3^\alpha)>3^\alpha-2^\alpha+2^{\alpha+1}\cdot(2^\alpha-3^\alpha)=(2^\alpha-3^\alpha)\cdot(2^{\alpha+1}-1)>0$.

(ii) If $\alpha=1$, then $6^\alpha+2\cdot3^\alpha-2\cdot4^\alpha-2^\alpha=2>0$. If $\alpha>1$, by Lemma 2.7 (i), then $6^\alpha+2\cdot3^\alpha-2\cdot4^\alpha-2^\alpha>6^\alpha+3^\alpha-4^\alpha-2^\alpha+(4^\alpha-5^\alpha)=6^\alpha-5^\alpha+3^\alpha-2^\alpha>0$. If $0<\alpha<1$, by Lemma 2.7 (ii), then $6^\alpha+2\cdot3^\alpha-2\cdot4^\alpha-2^\alpha>6^\alpha+3^\alpha-4^\alpha-2^\alpha+(2^\alpha-3^\alpha)=6^\alpha-4^\alpha>0$. Thus $6^\alpha+2\cdot3^\alpha-2\cdot4^\alpha-2^\alpha>0$ if $\alpha>0$.

Note that $2\cdot6^\alpha+3^\alpha-3\cdot4^\alpha=6^\alpha+2\cdot3^\alpha-2\cdot4^\alpha-2^\alpha+(2^\alpha-1)\cdot(3^\alpha-2^\alpha)>6^\alpha+2\cdot3^\alpha-2\cdot4^\alpha-2^\alpha>0$, $2\cdot3^\alpha+9^\alpha-6^\alpha-4^\alpha-2^\alpha=6^\alpha+2\cdot3^\alpha-2\cdot4^\alpha-2^\alpha+(2^\alpha-3^\alpha)^2>0$ and $9^\alpha+3^\alpha-2\cdot4^\alpha=2\cdot6^\alpha+3^\alpha-3\cdot4^\alpha+(2^\alpha-3^\alpha)^2>2\cdot6^\alpha+3^\alpha-3\cdot4^\alpha>0$, where $\alpha>0$.

(iii) Let $f(x):=5\cdot4^x+8^x-3\cdot3^x-3\cdot6^x$, then $f^{(n)}(x)=\frac{d^nf(x)}{dx^n}=5\cdot(\ln4)^n\cdot4^x+(\ln8)^n\cdot8^x-3\cdot(\ln3)^n\cdot3^x-3\cdot(\ln6)^n\cdot6^x$. Since $5\cdot(\ln4)^8-3\cdot(\ln3)^8>0$ and $(\ln8)^8-3\cdot(\ln6)^8=30.9188>0$, $f^{(8)}(x)>0$. Then, by $f^{(7)}(0)=33.6671>0$, $f^{(7)}(x)>0$. And, by $f^{(6)}(0)=11.7994>0$, $f^{(5)}(0)=4.27868>0$, $f^{(4)}(0)=1.87423>0$, $f^{(3)}(0)=1.07794>0$, $f^{(2)}(0)=0.681085>0$ and $f^{(1)}(0)=0.339798>0$, $f^{(1)}(x)>0$. Therefore, by $f(0)=0$, $f(\alpha)=5\cdot4^\alpha+8^\alpha-3\cdot3^\alpha-3\cdot6^\alpha>0$ if $\alpha>0$.

Note that $2^\alpha+4\cdot4^\alpha+8^\alpha-4\cdot3^\alpha-2\cdot6^\alpha=5\cdot4^\alpha+8^\alpha-3\cdot3^\alpha-3\cdot6^\alpha+(2^\alpha-1)(3^\alpha-2^\alpha)>0$ and $2^\alpha+4\cdot5^\alpha+10^\alpha-4\cdot3^\alpha-2\cdot6^\alpha>2^\alpha+4\cdot4^\alpha+8^\alpha-4\cdot3^\alpha-2\cdot6^\alpha>0$, where $\alpha>0$. ■

Let n and m be positive integers with $m\geq2$. Let $P_{2m+1}=x_1x_2\cdots x_{2m}x_{2m+1}$ be a path of order $2m+1$. Let S be internal vertices set of P_{2m+1}. Let $\mathscr{S}_{n,m}=\{S'\subset S: x_2,x_{2m}\in S', |S'|=n-2m-1$ and for every pair of vertices u and v in S', the distance between u and v in P_{2m+1} is even $\}$. Let $\mathscr{T}^{**}_{n,m}$ denote the set of trees created from P_{2m+1} by attaching a pendant edge to each vertex in $S'\in\mathscr{S}_{n,m}$. The graph T_1 shown in Fig. 2 is an element of $\mathscr{T}^{**}_{13,4}$. Let $\mathscr{T}^{*}_{n,m}$ denote the set of trees T of order n with $\Delta(T)=3$ and $n-2m+1$ pedant vertices such that each pedant vertex is adjacent to a vertex of degree 3, the length of each internal chain in T is even and the sum of length of all internal chains is $2m-2$. It is easy to see $\mathscr{T}^{**}_{n,m}\subset\mathscr{T}^{*}_{n,m}$. In Fig. 2, we have drawn $T_2\in\mathscr{T}^{*}_{13,4}\setminus\mathscr{T}^{**}_{13,4}$.

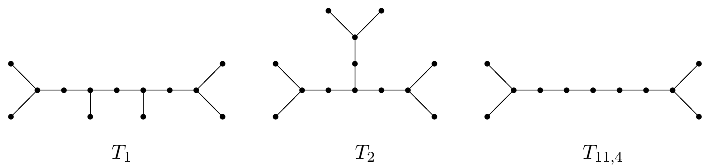

Fig. 2

Lemma 2.10. *Let $T\in\mathscr{T}^{*}_{n,m}$ $(m\geq2)$. Then $2m+3\leq n\leq 3m+1$, T has an m-matching, and $w_\alpha(T)=\phi(n,m)$, where $\phi(n,m)=3^\alpha[n-2m+1+2^{\alpha+1}(n-2m-2)]+2^{2\alpha+1}(3m-n+1)$.*

Proof. If $m = 2$, then $T \cong T_{7,2}$ by $T \in \mathscr{T}^*_{n,m}$. It is easy to check that the lemma holds clearly for $m = 2$.

We now suppose that $m \geq 3$ and proceed by induction on m. Let $P = u_0u_1u_2\cdots u_l$ be a longest path in T. Then $|N(u_1) \cap PV| = 2$. Denote $N(u_1) \cap PV = \{u_0, v\}$. Let $P' = u_1u_2\cdots u_t$ $(t < l)$ be an internal chain. Then t is odd and denote $t = 2h + 1$ $(h \geq 1)$. If $t = l - 1$, then $T \cong T_{2m+3,m}$. It is easy to check that the lemma holds in this case. Otherwise, $t \leq l - 3$. Let $T' = T - \{v, u_0, u_1, \cdots, u_{2h-1}\}$. Then $T' \in \mathscr{T}^*_{n-2h-1,m-h}$ with $m - h \geq 2$. By the induction, $2(m - h) + 3 \leq n - 2h - 1 \leq 3(m - h) + 1$, T' has an $(m - h)$-matching and $w_\alpha(T') = \phi(n - 2h - 1, m - h)$. It is not difficult to see that $2m + 4 \leq n \leq 3m - h + 2 \leq 3m + 1$, T has an m-matching and $w_\alpha(T) = w_\alpha(T') + 2 \cdot 3^\alpha + 2 \cdot 6^\alpha + (2h - 2) \cdot 4^\alpha - 3^\alpha = \phi(n, m)$. Hence the proof of Lemma 2.10 is complete. ■

Lemma 2.11 [10]. *Among trees with n $(n \geq 5)$ vertices, the path P_n has the minimum general Randić index for $\alpha > 0$.*

3 The cases for α in different intervals

In this section, we deal with our problem by considering the real number α in different intervals.

Case I. $-\frac{1}{2} \leq \alpha < 0$

Denote $\psi(n, m) = (n - m)^\alpha \cdot [n - 2m + 1 + (m - 1) \cdot 2^\alpha] + (m - 1) \cdot 2^\alpha$, where n, m are positive integers with $n \geq 2m$.

Theorem 3.1. *Let $T \in \mathscr{T}_{2m,m}$. If $-\frac{1}{2} \leq \alpha < 0$, then*

$$w_\alpha(T) \geq \psi(2m, m) \tag{2}$$

and equality in (2) holds for every particular value of α, $-\frac{1}{2} \leq \alpha < 0$, if and only if $T \cong T^0_{2m,m}$.

Proof. First we note that if $T \cong T^0_{2m,m}$, then the equality in (2) holds obviously.

Now we prove that if $T \in \mathscr{T}_{2m,m}$, then (2) holds and the equality in (2) holds only if $T \cong T^0_{2m,m}$.

If $m = 1, 2$, then the theorem holds clearly as $T \cong P_{2m}$ $(P_{2m} \cong T^0_{2m,m})$ for $m = 1, 2$. If $m = 3$, then $T \cong P_6$ or $T \cong T^0_{6,3}$. By Lemma 2.9 (i), $w_\alpha(P_6) - w_\alpha(T^0_{6,3}) = 3 \cdot 4^\alpha - 2 \cdot 6^\alpha - 3^\alpha > 0$. Thus the theorem holds for $m = 3$.

We suppose that $m \geq 4$ and proceed by induction on m. Let $T \in \mathscr{T}_{2m,m}$. By Lemma 2.1, T has a pendant vertex v which is adjacent to a vertex w of degree 2. Thus $vw \in E(T)$ and there is a unique vertex $u \neq v$ such that $uw \in E(T)$. Denote $N(u) \cap PV = \{v_1, \cdots, v_r\}$ and $N(u) \setminus PV = \{x_1, \cdots, x_{t-r} = w\}$. Then $t \leq m$ and all $d(x_j) = d_j \geq 2$.

Let $T' = T - v - w$. Then $T' \in \mathscr{T}_{2(m-1),m-1}$. By the induction, we have

$$\begin{aligned} w_\alpha(T) &= w_\alpha(T') + 2^\alpha + 2^\alpha \cdot t^\alpha + r \cdot [t^\alpha - (t-1)^\alpha] + \sum_{i=1}^{t-r-1} d_i^\alpha \cdot [t^\alpha - (t-1)^\alpha] \\ &\geq \psi(2m-2, m-1) + 2^\alpha + 2^\alpha \cdot t^\alpha \\ &\quad + r \cdot [t^\alpha - (t-1)^\alpha] + (t-r-1) \cdot 2^\alpha \cdot [t^\alpha - (t-1)^\alpha] \\ &= \psi(2m, m) + (m-1)^\alpha + (m-2) \cdot 2^\alpha \cdot (m-1)^\alpha - m^\alpha - (m-1) \cdot 2^\alpha \cdot m^\alpha \\ &\quad + r \cdot [t^\alpha - (t-1)^\alpha] + (t-r-1) \cdot 2^\alpha \cdot [t^\alpha - (t-1)^\alpha] + 2^\alpha \cdot t^\alpha. \end{aligned} \tag{3}$$

Note that $r \leq 1$ by Lemma 2.4. We consider the following two cases.

Case 1. $r = 1$.

In this case, by (3), we have

$$\begin{aligned} w_\alpha(T) \geq\ & \psi(2m, m) + (m-1)^\alpha \cdot [(m-2) \cdot 2^\alpha + 1] - m^\alpha \cdot [(m-1) \cdot 2^\alpha + 1] \\ & + t^\alpha \cdot [(t-1) \cdot 2^\alpha + 1] - (t-1)^\alpha \cdot [(t-2) \cdot 2^\alpha + 1]. \end{aligned}$$

Let $f(x) := (x+1)^\alpha \cdot (x \cdot 2^\alpha + 1)$. Then

$$w_\alpha(T) \geq \psi(2m, m) + [f(m-2) - f(m-1)] - [f(t-2) - f(t-1)] \geq \psi(2m, m).$$

The last inequality follows by Lemma 2.5 as $m \geq t$.

In order for the equality to hold, all inequalities in the above argument should be equalities. Thus we have $w_\alpha(T') = \psi(2(m-1), m-1)$, $m = t \geq 4$, $r = 1$ and $d_1 = \cdots = d_{t-1} = 2$. By the induction hypothesis, $T' \cong T^0_{2m-2,m-1}$. Note that $T^0_{2m-2,m-1}$ has a unique vertex of degree greater than 2, and hence $T \cong T^0_{2m,m}$.

Case 2. $r = 0$.

In this case, by (3), we have

$$\begin{aligned} w_\alpha(T) &\geq \psi(2m, m) + (2^\alpha - 1) \cdot [t^\alpha - (t-1)^\alpha] \\ &\quad + [f(m-2) - f(m-1)] - [f(t-2) - f(t-1)] \\ &\geq \psi(2m, m) + (2^\alpha - 1) \cdot [t^\alpha - (t-1)^\alpha] > \psi(2m, m). \end{aligned}$$

The last second inequality follows by Lemma 2.5 as $m \geq t$.

Hence the proof of Theorem 3.1 is complete. ■

Theorem 3.2. *Let $T \in \mathscr{T}_{n,m}$ $(n \geq 2m,\ m \geq 2)$. If $-\frac{1}{2} \leq \alpha < 0$, then*

$$w_\alpha(T) \geq \psi(n, m) \tag{4}$$

and equality in (4) holds for every particular value of $\alpha, -\frac{1}{2} \leq \alpha < 0$, if and only if $T \cong T^0_{n,m}$.

Proof. First we note that if $T \cong T^0_{n,m}$, then the equality in (4) holds by an elementary calculation.

Now applying induction on n, we prove that if $T \in \mathscr{T}_{n,m}$, then (4) holds and the equality in (4) holds only if $T \cong T^0_{n,m}$.

If $n = 2m$, then the theorem holds by Theorem 3.1. Therefore we assume that $n > 2m$ and the result holds for smaller values of n. By Lemma 2.3, T has an m-matching M and a pendant vertex v such that M does not saturate v. Let $uv \in E(T)$ with $d(u) = t$. Denote $N(u) \cap PV = \{v_1, \cdots, v_{r-1},\ v_r = v\}$ and $N(u) \setminus PV = \{x_1, \cdots, x_{t-r}\}$. Then all $d(x_j) = d_j \geq 2$.

Let $T' = T - v$. Then $T' \in \mathscr{T}_{n-1,m}$. By the induction, we have

$$\begin{aligned}
w_\alpha(T) &= w_\alpha(T') + r \cdot t^\alpha - (r-1) \cdot (t-1)^\alpha + \sum_{i=1}^{t-r} d_i^\alpha \cdot [t^\alpha - (t-1)^\alpha] \\
&\geq \psi(n-1, m) + r \cdot t^\alpha - (r-1) \cdot (t-1)^\alpha + (t-r) \cdot 2^\alpha \cdot [t^\alpha - (t-1)^\alpha] \\
&= \psi(n, m) + [r + 2^\alpha \cdot (t-r)] \cdot t^\alpha + [n - 2m + 2^\alpha \cdot (m-1)] \cdot (n-m-1)^\alpha \\
&\quad -[r - 1 + 2^\alpha \cdot (t-r)] \cdot (t-1)^\alpha - [n - 2m + 1 + 2^\alpha \cdot (m-1)] \cdot (n-m)^\alpha \\
&= \psi(n, m) + [h(n-m-1, n-2m) - h(n-m, n-2m+1)] \\
&\quad -[h(t-1, r-1) - h(t, r)],
\end{aligned} \tag{5}$$

where $h(x, y)$ is defined in Lemma 2.8. Since T has an m-matching, $n-m \geq t$ and $n-2m \geq r-1$. Then, by (5) and Lemma 2.8, we have

$$\begin{aligned}
w_\alpha(T) &\geq \psi(n, m) + h(n-m-1, n-2m) - h(n-m, n-2m+1) \\
&\quad -[h(n-m-1, r-1) - h(n-m, r)] \\
&\geq \psi(n, m).
\end{aligned}$$

In order for the equality to hold, all inequalities in the above argument should be equalities. Thus we have $w_\alpha(T') = \psi(n-1, m)$, $n - m = t$, $r - 1 = n - 2m$ and $d_1 = \cdots = d_{t-r} = 2$. By the induction hypothesis, $T' \cong T^0_{n-1,m}$. Then it is not difficult to see $T \cong T^0_{n,m}$.

Hence the proof of Theorem 3.2 is complete. ∎

Case II. $\alpha > 0$

Denote $\phi_0(m) = 2^{\alpha+1} + 4^\alpha \cdot (2m-3)$, $\phi_1(m) = 2^{\alpha+1} + 4^\alpha \cdot (2m-2)$, $\phi_2(m) = 2^\alpha + 6^\alpha + 2 \cdot 3^\alpha + 4^\alpha \cdot (2m-3)$ and $\phi_3(m) = 4 \cdot 3^\alpha + 2 \cdot 6^\alpha + 4^\alpha \cdot (2m-4)$, where m is a positive integer.

By Lemma 2.11, it is easy to obtain Theorems 3.3 and 3.4 immediately.

Theorem 3.3. *Let $T \in \mathscr{T}_{2m,m}$. If $\alpha > 0$, then*

$$w_\alpha(T) \geq \phi_0(m) \tag{6}$$

and equality in (6) holds for every particular value of $\alpha, \alpha > 0$, if and only if $T \cong P_{2m}$.

Theorem 3.4. *Let $T \in \mathscr{T}_{2m+1,m}$. If $\alpha > 0$, then*

$$w_\alpha(T) \geq \phi_1(m) \tag{7}$$

and equality in (7) holds for every particular value of $\alpha, \alpha > 0$, if and only if $T \cong P_{2m+1}$.

Let $S_4^n (n \geq 4)$ (shown in Fig. 1) denote a tree created from the star graph S_4 by subdividing one edge $n-4$ times, where n is a positive integer.

Theorem 3.5. *Let $T \in \mathscr{T}_{2m+2,m}$ $(m \geq 2)$. Then*

$$w_\alpha(T) \geq \phi_2(m) \tag{8}$$

and equality in (8) holds for every particular value of $\alpha, \alpha > 0$, if and only if $T \cong S_4^{2m+2}$.

Proof. Note that if $T \cong S_4^{2m+2}$ $(m \geq 2)$, then the equality in (8) holds clearly.

Now we prove if $T \in \mathscr{T}_{2m+2,m}$ $(m \geq 2)$, then (8) holds and the equality in (8) holds only if $T \cong S_4^{2m+2}$.

If $m = 2$, then there are only three trees, T_3, T_4 and S_4^6 (shown in Fig. 3), of order $2m+2$ with an m-matching. Then $w_\alpha(T_3) - w_\alpha(S_4^6) = 2 \cdot 3^\alpha + 9^\alpha - 6^\alpha - 4^\alpha - 2^\alpha > 0$ by Lemma 2.9 (ii) and $w_\alpha(T_4) - w_\alpha(S_4^6) = 2 \cdot 4^\alpha + 8^\alpha - 2 \cdot 3^\alpha - 6^\alpha > 0$. Thus the theorem holds clearly for $m = 2$.

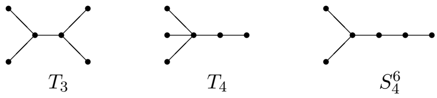

Fig. 3

We now suppose that $m \geq 3$ and proceed by induction on m. By Lemma 2.3, T has an m-matching M and a pendant vertex v such that M does not saturate v. Let $uv \in E(T)$ with $d(u) = t \geq 2$. Denote $N(u) \cap PV = \{v_1, \cdots, v_{r-1},\ v_r = v\}$ and $N(u) \setminus PV = \{x_1, \cdots, x_{t-r}\}$. Then all $d(x_j) = d_j \geq 2$.

We consider the following two cases.

Case 1. $t = 2$.

In this case, there is a unique vertex $w \neq v$ such that $uw \in E(T)$. Denote $N(w) \cap PV = \{u_1, \cdots, u_p\}$ and $N(w) \setminus PV = \{y_1, \cdots, y_{s-p} = u\}$. Then all $d(y_j) = q_j \geq 2$.

Let $T' = T - v - u$. Then $T' \in \mathcal{T}_{2m,m-1}$. By the induction, we have

$$\begin{aligned} w_\alpha(T) &= w_\alpha(T') + 2^\alpha + 2^\alpha \cdot s^\alpha + p \cdot [s^\alpha - (s-1)^\alpha] + \sum_{i=1}^{s-p-1} q_i^\alpha \cdot [s^\alpha - (s-1)^\alpha] \\ &\geq w_\alpha(T') + 2^\alpha + 2^\alpha \cdot s^\alpha + p \cdot [s^\alpha - (s-1)^\alpha] + (s-p-1) \cdot 2^\alpha \cdot [s^\alpha - (s-1)^\alpha] \\ &\geq \phi_2(m-1) + 2^\alpha + 2^\alpha \cdot s^\alpha \\ &\quad + p \cdot [s^\alpha - (s-1)^\alpha] + (s-p-1) \cdot 2^\alpha \cdot [s^\alpha - (s-1)^\alpha] \end{aligned}$$

$$
\begin{aligned}
&= \phi_2(m) - 2\cdot 4^\alpha + 2^\alpha + 2^\alpha s^\alpha \\
&\quad + p\cdot[s^\alpha - (s-1)^\alpha] + (s-p-1)\cdot 2^\alpha\cdot[s^\alpha - (s-1)^\alpha]. \qquad (9)
\end{aligned}
$$

Subcase 1.1. $p = 0$.

Let $g(x) := x^{\alpha+1}$. Then, by (9), Lemma 2.7 (i) and $s \geq 2$, we have

$$w_\alpha(T) \;\geq\; \phi_2(m) + 2^\alpha\cdot\{[g(s) - g(s-1)] - [g(2) - g(1)]\} \geq \phi_2(m).$$

In order for the equality to hold, all inequalities in the above argument should be equalities. Thus we have $w_\alpha(T') = \phi_2(m-1)$, $s = 2$, $p = 0$ and $q_1 = 2$. By the induction hypothesis, $T' \cong S_4^{2m}$. Then it is easy to see that $T \cong S_4^{2m+2}$.

Subcase 1.2. $p = 1$.

In this subcase, $s \geq 3$. By (9), Lemma 2.5 and Lemma 2.9 (ii), we have

$$
\begin{aligned}
w_\alpha(T) &\geq \phi_2(m) + 2^\alpha - 2\cdot 4^\alpha + s^\alpha\cdot[2^\alpha\cdot(s-1)+1] - (s-1)^\alpha\cdot[2^\alpha\cdot(s-2)+1] \\
&\geq \phi_2(m) + 2^\alpha - 2\cdot 4^\alpha + (2\cdot 6^\alpha + 3^\alpha - 4^\alpha - 2^\alpha) \\
&= \phi_2(m) + 2\cdot 6^\alpha + 3^\alpha - 3\cdot 4^\alpha > \phi_2(m).
\end{aligned}
$$

Subcase 1.3. $p \geq 2$.

In this subcase, $s \geq 4$. By (9), we have

$$w_\alpha(T) \;>\; \phi_2(m) - 2\cdot 4^\alpha + 2^\alpha + 2^\alpha s^\alpha \geq \phi_2(m) - 2\cdot 4^\alpha + 2^\alpha + 2^\alpha 4^\alpha > \phi_2(m).$$

Case 2. $t \geq 3$.

Let $T' = T - v$. Then $T' \in \mathscr{T}_{2m+1,m}$. By Theorem 3.4, $w_\alpha(T') \geq \phi_1(m)$. Note that

$$
\begin{aligned}
w_\alpha(T) &= w_\alpha(T') + r\cdot t^\alpha - (r-1)\cdot(t-1)^\alpha + \sum_{i=1}^{t-r} d_i^\alpha\cdot[t^\alpha - (t-1)^\alpha] \\
&\geq \phi_1(m) + r\cdot t^\alpha - (r-1)\cdot(t-1)^\alpha + (t-r)\cdot 2^\alpha\cdot[t^\alpha - (t-1)^\alpha] \\
&= \phi_2(m) + 4^\alpha + 2^\alpha - 6^\alpha - 2\cdot 3^\alpha \\
&\quad + [r + 2^\alpha(t-r)]t^\alpha - [r - 1 + 2^\alpha(t-r)](t-1)^\alpha. \qquad (10)
\end{aligned}
$$

Subcase 2.1. $t - r = 1$.

By (10) and Lemma 2.6, we get

$$
\begin{aligned}
w_\alpha(T) &\geq \phi_2(m) + t^\alpha\cdot[t-1+2^\alpha] - (t-1)^\alpha\cdot[t-2+2^\alpha] + 4^\alpha + 2^\alpha - 6^\alpha - 2\cdot 3^\alpha \\
&\geq \phi_2(m) + 3^\alpha\cdot(2+2^\alpha) - 2^\alpha\cdot(1+2^\alpha) + 4^\alpha + 2^\alpha - 6^\alpha - 2\cdot 3^\alpha = \phi_2(m).
\end{aligned}
$$

In order for the equality to hold, all inequalities in the above argument should be equalities. Thus we have $w_\alpha(T') = \phi_1(m)$, $t = 3$ and $r = 2$. By the induction hypothesis, $T' \cong P_{2m+1}$. Therefore $T \cong S_4^{2m+2}$.

Subcase 2.2. $t-r\geq 2$.

In this subcase, $r\leq t-2$. By (10) and Lemma 2.6, we have

$$\begin{aligned} w_\alpha(T) &\geq \phi_2(m)+(2^\alpha-1)\cdot[t^\alpha-(t-1)^\alpha] \\ &\quad +t^\alpha\cdot[t-1+2^\alpha]-(t-1)^\alpha\cdot[t-2+2^\alpha]+4^\alpha+2^\alpha-6^\alpha-2\cdot 3^\alpha \\ &\geq \phi_2(m)+(2^\alpha-1)\cdot[t^\alpha-(t-1)^\alpha]>\phi_2(m). \end{aligned}$$

Therefore the proof of Theorem 3.5 is complete. ■

Let $T_{2m+3,m}$ denote a tree obtained from a path $P_{2m+1}=x_1x_2\cdots x_{2m}x_{2m+1}$ by attaching a new pendant edge to x_2 and x_{2m}, respectively. $T_{11,4}$ is shown in Fig. 2.

Theorem 3.6. *Let $T\in\mathscr{T}_{2m+3,m}$ $(m\geq 2)$. If $\alpha>0$, then*

$$w_\alpha(T)\geq\phi_3(m) \tag{11}$$

and equality in (11) holds for every particular value of $\alpha>0$ if and only if $T\cong T_{2m+3,m}$.

Proof. Note that if $T\cong T_{2m+3,m}$ $(m\geq 2)$, then the equality in (11) holds obviously.

Now we prove if $T\in\mathscr{T}_{2m+3,m}$ $(m\geq 2)$, then (11) holds and the equality in (11) holds only if $T\cong T_{2m+3,m}$.

If $m=2$, then there are only four trees, T_5, T_6, T_7 and $T_{7,2}$ (shown in Fig. 4), of order $2m+3$ with an m-matching. Then $w_\alpha(T_6)-w_\alpha(T_{7,2})=3\cdot 4^\alpha+12^\alpha-2\cdot 3^\alpha-2\cdot 6^\alpha>3\cdot 4^\alpha+9^\alpha-2\cdot 3^\alpha-2\cdot 6^\alpha=2\cdot(4^\alpha-3^\alpha)+(2^\alpha-3^\alpha)^2>0$ and, by Lemma 2.9 (iii), $w_\alpha(T_5)-w_\alpha(T_{7,2})=2^\alpha+4\cdot 5^\alpha+10^\alpha-4\cdot 3^\alpha-2\cdot 6^\alpha>0$ and $w_\alpha(T_7)-w_\alpha(T_{7,2})=2^\alpha+4\cdot 4^\alpha+8^\alpha-4\cdot 3^\alpha-2\cdot 6^\alpha>0$. Thus the theorem holds for $m=2$.

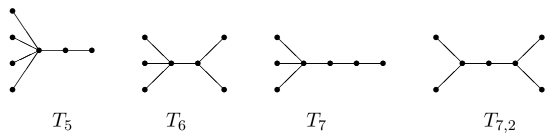

Fig. 4

We now suppose that $m\geq 3$ and proceed by induction on m. By Lemma 2.3, T has an m-matching M and a pendant vertex v such that M does not saturate v. Let $uv\in E(T)$ with $d(u)=t$. Denote $N(u)\cap PV=\{v_1,\cdots,v_{r-1},\ v_r=v\}$ and $N(u)\setminus PV=\{x_1,\cdots,x_{t-r}\}$. Then all $d(x_j)=d_j\geq 2$. We consider the following two cases.

Case 1. $t=2$.

In this case, there is a unique vertex $w\neq v$ such that $uw\in E(T)$. Denote $N(w)\cap PV=\{u_1,\cdots,u_p\}$ and $N(w)\setminus PV=\{y_1,\cdots,y_{s-p}=u\}$. Then all $d(y_j)=q_j\geq 2$. Let $T'=T-v-u$. Then $T'\in\mathscr{T}_{2m+1,m-1}$. By the induction, we have

$$w_\alpha(T) = w_\alpha(T')+2^\alpha+2^\alpha\cdot s^\alpha+p\cdot[s^\alpha-(s-1)^\alpha]+\sum_{i=1}^{s-p-1}q_i^\alpha\cdot[s^\alpha-(s-1)^\alpha]$$

$$\begin{aligned}
&\geq \phi_3(m-1) + 2^\alpha + 2^\alpha \cdot s^\alpha \\
&\quad +p \cdot [s^\alpha - (s-1)^\alpha] + (s-p-1) \cdot 2^\alpha \cdot [s^\alpha - (s-1)^\alpha] \\
&= \phi_3(m) - 2 \cdot 4^\alpha + 2^\alpha + 2^\alpha s^\alpha \\
&\quad +p \cdot [s^\alpha - (s-1)^\alpha] + (s-p-1) \cdot 2^\alpha \cdot [s^\alpha - (s-1)^\alpha]. \qquad (12)
\end{aligned}$$

Let $g(x) := x^{\alpha+1}$. If $p = 0$, by (12), Lemma 2.7 (i) and $s \geq 2$, then

$$w_\alpha(T) \geq \phi_3(m) + 2^\alpha \cdot \{[g(s) - g(s-1)] - [g(2) - g(1)]\} \geq \phi_3(m).$$

In order for the equality to hold, all inequalities in the above argument should be equalities. Thus we have $w_\alpha(T') = \phi_3(m-1)$, $s = 2$, $p = 0$ and $q_1 = 2$. By the induction hypothesis, $T' \cong T_{2m+1,m-1}$. Since each vertex of degree 1 in T' is adjacent to a vertex of degree 3, it is easy to see the equality does not hold. Otherwise, $p \geq 1$. Then the theorem holds by an argument similar to that in Subcase 1.2 and Subcase 1.3 in the proof of Theorem 3.5.

Case 2. $t \geq 3$.

Let $T' = T - v$. Then $T' \in \mathscr{T}_{2m+2,m}$. By Theorem 3.5, $w_\alpha(T') \geq \phi_2(m)$. Note that

$$\begin{aligned}
w_\alpha(T) &= w_\alpha(T') + r \cdot t^\alpha - (r-1) \cdot (t-1)^\alpha + \sum_{i=1}^{t-r} d_i^\alpha \cdot [t^\alpha - (t-1)^\alpha] \\
&\geq \phi_2(m) + r \cdot t^\alpha - (r-1) \cdot (t-1)^\alpha + (t-r) \cdot 2^\alpha \cdot [t^\alpha - (t-1)^\alpha] \\
&= \phi_3(m) + 4^\alpha + 2^\alpha - 6^\alpha - 2 \cdot 3^\alpha \\
&\quad +[r + 2^\alpha(t-r)]t^\alpha - [r - 1 + 2^\alpha(t-r)](t-1)^\alpha.
\end{aligned}$$

Hence the theorem holds by an argument similar to that in Case 2 in the proof of Theorem 3.5.

Therefore the proof of Theorem 3.6 is complete. ∎

Theorem 3.7. *Let $T \in \mathscr{T}_{n,m}$ $(2m+3 \leq n \leq 3m+1)$. If $\alpha > 0$, then*

$$w_\alpha(T) \geq \phi(n,m), \qquad (13)$$

where $\phi(n,m)$ is defined in Lemma 2.10, and equality in (13) holds for every particular value of $\alpha > 0$ if and only if $T \in \mathscr{T}^*_{n,m}$.

Proof. Note that if $T \in \mathscr{T}^*_{n,m}$, then the equality in (13) holds by Lemma 2.10.

Now applying induction on n, we prove if $T \in \mathscr{T}_{n,m}$, then (13) holds and the equality in (13) holds only if $T \in \mathscr{T}^*_{n,m}$.

If $n = 2m+3$, then the theorem holds by Theorem 3.6 and $\{T_{2m+3,m}\} = \mathscr{T}^*_{2m+3,m}$. Therefore we assume that $n \geq 2m+4$ and the result holds for smaller values of n.

By Lemma 2.3, T has an m-matching M and a pendant vertex v such that M does not saturate v. Let $uv \in E(T)$ with $d(u) = t$. Denote $N(u) \cap PV = \{v_1, \cdots, v_{r-1},\ v_r = v\}$ and $N(u) \setminus PV = \{x_1, \cdots, x_{t-r}\}$. Then all $d(x_j) = d_j \geq 2$. We consider the following three cases.

Case 1. $d(u) = t \geq 4$.

Let $T' = T - v$. Then $T' \in \mathscr{T}_{n-1,m}$ and $n - 1 \geq 2m + 3$. By the induction, we have

$$\begin{aligned} w_\alpha(T) &= w_\alpha(T') + r \cdot t^\alpha - (r-1) \cdot (t-1)^\alpha + \sum_{i=1}^{t-r} d_i^\alpha \cdot [t^\alpha - (t-1)^\alpha] \\ &\geq w_\alpha(T') + r \cdot t^\alpha - (r-1) \cdot (t-1)^\alpha + (t-r) \cdot 2^\alpha \cdot [t^\alpha - (t-1)^\alpha] \\ &\geq \phi(n-1,m) + r \cdot t^\alpha - (r-1) \cdot (t-1)^\alpha + (t-r) \cdot 2^\alpha \cdot [t^\alpha - (t-1)^\alpha] \\ &= \phi(n,m) - 3^\alpha - 2 \cdot 6^\alpha + 2 \cdot 4^\alpha \\ &\quad + t^\alpha \cdot [r + 2^\alpha(t-r)] - (t-1)^\alpha \cdot [r - 1 + 2^\alpha(t-r)] \\ &\geq \phi(n,m) - 3^\alpha - 2 \cdot 6^\alpha + 2 \cdot 4^\alpha + t^\alpha \cdot [t + 2^\alpha - 1] - (t-1)^\alpha \cdot [t + 2^\alpha - 2] \\ &\geq \phi(n,m) + 5 \cdot 4^\alpha + 8^\alpha - 3 \cdot 3^\alpha - 3 \cdot 6^\alpha > \phi(n,m). \end{aligned}$$

The last first and second inequalities follow by Lemma 2.9 (iii) and Lemma 2.6, respectively.

Case 2. $d(u) = 2$.

In this case, the theorem holds by an argument similar to that in Case 1 in the proof of Theorem 3.6.

Case 3. $d(u) = 3$.

Subcase 3.1. $r = 1$.

In this subcase, $N(u) \backslash \{v\} = \{x_1, x_2\}$. Let $T' = T - v$. Then $T' \in \mathscr{T}_{n-1,m}$ and $n-1 \geq 2m+3$. Thus by the induction, we have

$$\begin{aligned} w_\alpha(T) &= w_\alpha(T') + 3^\alpha + (3^\alpha - 2^\alpha) \cdot (d_1^\alpha + d_2^\alpha) \\ &\geq \phi(n-1,m) + 3^\alpha + (3^\alpha - 2^\alpha) 2^{\alpha+1} = \phi(n,m) \end{aligned}$$

and the equality holds only if $d_1 = d_2 = 2$ and $w_\alpha(T') = \phi(n-1,m)$. By the induction hypothesis, $T' \in \mathscr{T}^*_{n-1,m}$. Thus there is a vertex $u' \in V(T')$ such that $|N(u') \cap PV| = 2$. we replace u with u'. Then, in this subcase, theorem holds by an argument similar to that in the next subcase.

Subcase 3.2. $r = 2$.

In this subcase, $N(u) \cap PV = \{v_1, v_2\}$. Let $P = u_0 u_1 \cdots u_l$ $(u = u_0, x_1 = u_1)$ be an internal chain of T with $d(u_l) = d \geq 3$, where $l \geq 1$. Let $|N(u_l) \cap PV| = q$. We consider the following three subcases.

Subcase 3.2.1. $l = 1$.

Let $T' = T - \{v_1, v_2, u\}$. Then $T' \in \mathscr{T}_{n-3,m-1}$ and $n - 3 \geq 2(m-1) + 3$. Thus we have

$$\begin{aligned} w_\alpha(T) &= w_\alpha(T') + 2 \cdot 3^\alpha + 3^\alpha d^\alpha + q[d^\alpha - (d-1)^\alpha] \\ &\quad + \sum_{z \in N(u_1) \backslash (PV \cup \{u\})} (d(z))^\alpha [d^\alpha - (d-1)^\alpha] \end{aligned}$$

$$\begin{aligned} &\geq \phi(n-3,m-1)+2\cdot 3^\alpha+3^\alpha d^\alpha \\ &\quad +q[d^\alpha-(d-1)^\alpha]+(d-q-1)\cdot 2^\alpha\cdot[d^\alpha-(d-1)^\alpha] \\ &= \phi(n,m)+3^\alpha-2\cdot 6^\alpha+3^\alpha d^\alpha+q[d^\alpha-(d-1)^\alpha] \\ &\quad +(d-q-1)\cdot 2^\alpha\cdot[d^\alpha-(d-1)^\alpha]. \end{aligned} \tag{14}$$

If $d \geq 4$, by (14), then

$$w_\alpha(T) > \phi(n,m)+3^\alpha-2\cdot 6^\alpha+12^\alpha=\phi(n,m)+3^\alpha\cdot(2^\alpha-1)^2>\phi(n,m).$$

If $d=3$, then $q\leq 1$. If $q=0$, by (14) and Lemma 2.9 (ii), then

$$w_\alpha(T) \geq \phi(n,m)+3^\alpha+9^\alpha-2\cdot 4^\alpha>\phi(n,m).$$

Otherwise, if $q=1$, by (14) and Lemma 2.9 (ii), then

$$w_\alpha(T) \geq \phi(n,m)+9^\alpha+2\cdot 3^\alpha-6^\alpha-2^\alpha-4^\alpha>\phi(n,m).$$

Subcase 3.2.2. $l=2h$ $(h\geq 1)$.

In this subcase, $h\leq m-2$. Let $T'=T-\{v_1,v_2,u_0,\cdots,u_{2h-3},u_{2h-2}\}$. Then $T'\in\mathscr{T}_{n-2h-1,m-h}$ with $n-2h-1\geq 2(m-h)+3$ and $m-h\geq 2$. Thus we have

$$\begin{aligned} w_\alpha(T) &= w_\alpha(T')+2\cdot 3^\alpha+6^\alpha+4^\alpha\cdot(2h-2)+2^\alpha d^\alpha-d^\alpha \\ &\geq \phi(n-2h-1,m-h)+2\cdot 3^\alpha+6^\alpha+4^\alpha\cdot(2h-2)+2^\alpha d^\alpha-d^\alpha \\ &= \phi(n,m)+3^\alpha-6^\alpha+(2^\alpha-1)\cdot d^\alpha\geq\phi(n,m). \end{aligned}$$

In order for the equality to hold, all inequalities in the above argument should be equalities. Thus we have $w_\alpha(T')=\phi(n-2h-1,m-h)$, $r=2$ and $d=3$. By the induction hypothesis, $T'\in\mathscr{T}^*_{n-2h-1,m-h}$. Then it is not difficult to see $T\in\mathscr{T}^*_{n,m}$.

Subcase 3.2.3. $l=2h+1$ $(h\geq 1)$.

In this subcase, $h\leq m-3$ by T having an m-matching and $n\geq 2m+4$. Let $T'=T-\{v_1,v_2,u_0,\cdots,u_{2h-2},u_{2h-1}\}$. Then either $T'\in\mathscr{T}_{n-2h-2,m-h-1}$ or $T'\in\mathscr{T}_{n-2h-2,m-h}$. Note that $n-2h-2\geq 2(m-h-1)+4$.

If $T'\in\mathscr{T}_{n-2h-2,m-h-1}$, by Lemma 2.9 (ii), then

$$\begin{aligned} w_\alpha(T) &= w_\alpha(T')+2\cdot 3^\alpha+6^\alpha+4^\alpha\cdot(2h-1)+2^\alpha d^\alpha-d^\alpha \\ &\geq \phi(n-2h-2,m-h-1)+2\cdot 3^\alpha+6^\alpha+4^\alpha\cdot(2h-1)+2^\alpha d^\alpha-d^\alpha \\ &\geq \phi(n,m)+2\cdot 3^\alpha+6^\alpha-3\cdot 4^\alpha+(2^\alpha-1)d^\alpha>\phi(n,m). \end{aligned}$$

If $T'\in\mathscr{T}_{n-2h-2,m-h}$, then every $(m-h)$-matching of T' saturates u_{2h}. Thus $N(u_{2h+1})\cap PV=\emptyset$. Let $T''=T'-\{u_{2h}\}$. Then $T''\in\mathscr{T}_{n-2h-3,m-h-1}$. We have

$$w_\alpha(T) = w_\alpha(T'')+2\cdot 3^\alpha+6^\alpha+4^\alpha\cdot(2h-1)+2^\alpha d^\alpha$$

$$
\begin{aligned}
& + \sum_{z \in N(u_{2h+1}) \setminus \{u_{2h}\}} (d(z))^\alpha \cdot [d^\alpha - (d-1)^\alpha] \\
\geq \ & \phi(n-2h-3, m-h-1) + 2 \cdot 3^\alpha + 6^\alpha + 4^\alpha \cdot (2h-1) \\
& +2^\alpha d^\alpha + (d-1) \cdot 2^\alpha [d^\alpha - (d-1)^\alpha] \\
\geq \ & \phi(n,m) + 3^\alpha - 6^\alpha - 4^\alpha + 2^\alpha [d^{\alpha+1} - (d-1)^{\alpha+1}] \\
\geq \ & \phi(n,m) + 3^\alpha - 6^\alpha - 4^\alpha + 2^\alpha [3^{\alpha+1} - 2^{\alpha+1}] \\
= \ & \phi(n,m) + 3^\alpha + 2 \cdot 6^\alpha - 3 \cdot 4^\alpha > \phi(n,m).
\end{aligned}
$$

The last first and second inequalities follow by Lemma 2.9 (ii) and Lemma 2.7 (i), respectively.

Hence the proof of Theorem 3.7 is complete. ■

4 Remarks

In [7], Hou and Li prove that among trees of order $n \geq 2m$ with an m-matching, the maximum spectral radius is obtained uniquely at $T^0_{n,m}$. From the theorems in preceding section, we can see that among trees of order $n \geq 2m$ with an m-matching, the minimum general Randić index for $-\frac{1}{2} \leq \alpha < 0$ is obtained uniquely at $T^0_{n,m}$. We do not know whether there are other classes of graphs such that in each of these classes the graph with maximum spectral radius has the minimum general Randić index for $-\frac{1}{2} \leq \alpha < 0$. Thus we would like to propose naturally the following question: Given a class of graphs $\mathscr{G}$, $G_0 \in \mathscr{G}$ and $\rho(G) \leq \rho(G_0)$ for every $G \in \mathscr{G}$, where $\rho(G)$ is the spectral radius, is it true that $w_\alpha(G) \geq w_\alpha(G_0)$ for every particular value of α, $-\frac{1}{2} \leq \alpha < 0$, and each $G \in \mathscr{G}$?

Also, it may be of interest to give sharp bound of $w_\alpha(T)$ for $\alpha < -\frac{1}{2}$ when $T \in \mathscr{T}_{n,m}$.

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References

[1] B. Bollobás and P. Erdős, Graphs of extremal weights, Ars Combin. 50 (1998) 225-233.

[2] J.A. Bondy and U.S.R. Murty, Graph Theory with Applications, Macmillan Press, London, 1976.

[3] G. Caporossi, I. Gutman, P. Hansen and L. Pavlović, Graphs with maximum connectivity index, Comput. Biol. Chem. 27(2003) 85-90.

[4] L.H. Clark and J.W. Moon, On the general Randić index for certain families of trees, Ars Combin. 54(2000) 223-235.

[5] C. Delorme, O. Favaron and D. Rautenbach, On the Randić index, Discrete Math. 257(2002), 29-38.

[6] J. Gao and M. Lu, On the Randić index of unicyclic graphs, MATCH Commun. Math. Comput. Chem. 53(2005) 377-384.

[7] Y. Hou and J. Li, Bounds on the largest eigenvalues of trees with a given size of matching, Linear Algebra Appl. 342(2002) 203-217.

[8] Y. Hu, X. Li and Y. Yuan, Solutions to two unsolved questions on the best upper bound for the Randić index R_{-1} of trees, MATCH Commun. Math. Comput. Chem. 54(2005), in press.

[9] Y. Hu, X. Li and Y. Yuan, Trees with maximum general Randić index, MATCH Commun. Math. Comput. Chem. 52(2004) 129-146.

[10] Y. Hu, X. Li and Y. Yuan, Trees with minimum general Randić index, MATCH Commun. Math. Comput. Chem. 52(2004) 119-128.

[11] L.B. Kier and L.H. Hall, Molecular Connectivity in Chemistry and Drug Research, Academic Press, San Francisco, 1976.

[12] L.B. Kier and L.H. Hall, Molecular Connectivity in Structure-Activity Analysis, Wiley, 1986.

[13] X. Li, X. Wang and B. Wei, On the lower and upper bounds for general Randić index of chemical (n, m)-graphs, MATCH Commun. Math. Comput. Chem. 52(2004) 157-166.

[14] X. Li and Y. Yang, Best lower and upper bounds for the Randić index R_{-1} of chemical trees, MATCH Commun. Math. Comput. Chem. 52(2004) 147-156.

[15] X. Li and Y. Yang, Sharp bounds for the general Randić index, MATCH Commun. Math. Comput. Chem. 51(2004) 155-166.

[16] M. Lu, H. Liu and F. Tian, The connectivity index, MATCH Commun. Math. Comput. Chem. 51(2004) 149-154.

[17] M. Randić, On characterization of molecular branching, J. Am. Chem. Soc. 97(1975) 6609-6615.

[18] D. Rautenbach, A note on trees of maximum weight and restricted degrees, Discrete Math. 271(2003) 335-342.

[19] P. Yu, An upper bound for the Randić indices of trees, J. Math. Study (Chinese) 31(1998) 225-230.