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Unicyclic graphs with minimum general Randić index

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Abstract

The general Randić index $R_\alpha(G)$ of a graph G is defined as the sum of the weights $(d(u)d(v))^\alpha$ of all edges uv of G, where $d(u)$ denotes the degree of a vertex u in G and α is an arbitrary real number. In this paper, we show that among unicyclic graphs with n vertices, the cycle C_n for $\alpha > 0$ and S_n^+ for $-1 \le \alpha < 0$, respectively, has the minimum general Randić index, where S_n^+ denotes the unicyclic graph obtained from the star S_n with n vertices by joining its two vertices of degree one. For $\alpha < -1$, we also give the structure of graphs with minimum general Randić index.

1 Introduction

Let $G = (V(G), E(G))$ be a graph. The degree and the neighborhood of a vertex $u \in V(G)$ is denoted by $d_G(u)$ and $N_G(u)$ (or simply by $d(u)$ and $N(u)$), respectively. In 1975, Randić [8] proposed an important topological index of a (molecular) graph in his research on molecular structures, which is closely related with many chemical properties. Now it is called Randić index or is known as the connectivity index [1,3]. Given two adjacent vertices u and v of a graph G, the Randić weight of the edge uv is $R(uv) = (d(u)d(v))^{-\frac{1}{2}}$, and the Randić index of a graph G, $R(G)$, is the sum of the Randić weights of its edges. Later, Bollobás and Erdös [2] generalized this index by replacing $-\frac{1}{2}$ by any real number α, which was called the general Randć index. More formally, fixing $\alpha \in R - \{0\}$, the

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general Randić index is defined as $R_\alpha(G) = \sum_{uv\in E(G)} R_\alpha(uv) = \sum_{uv\in E(G)}(d(u)d(v))^\alpha$. Hence, $R_{-\frac{1}{2}}(G)$ is the ordinary Randić index of G.

Bollobás and Erdös [2] gave a sharp lower bound of R_α for $-1 \le \alpha < 0$ in restricted to the graphs (may have isolated vertices) of given size. Clark and Moon [4] gave several extremal and probabilistic results of R_α for certain families of trees. Hu, Li and Yuan [7] showed that among trees with n vertices, the path P_n for $\alpha > 0$ and the star S_n for $\alpha < 0$, respectively, has the minimum general Randić index.

A simple connected graph G is called unicyclic graph if it contains exactly one cycle. From this definition, one can see that a unicyclic graph has the same number of vertices and edges, and it is a cycle or cycle with trees attached to it. For $n \ge 3$, let S_n^+ denote the unicyclic graph obtained from the star S_n with n vertices by joining its two vertices of degrees one. Gao and Lu [6] showed that for a unicyclic graph G, $R_{-\frac{1}{2}}(G) \ge (n-3)(n-1)^{-\frac{1}{2}} + 2(2n-2)^{-\frac{1}{2}} + \frac{1}{2}$, and the equality holds if and only if $G \cong S_n^+$. In this paper, we investigate the general Randić index R_α of unicyclic graphs for arbitrary real number α. Let G be a unicyclic graph with n vertices. We show that for $\alpha > 0$, $R_\alpha(G) \ge n \cdot 4^\alpha$ and the the equality holds if and only if G is the cycle C_n, and for $-1 \le \alpha < 0$, $R_\alpha(G) \ge (n-3)(n-1)^\alpha + 2(2n-2)^\alpha + 4^\alpha$ and the equality holds if and only if $G \cong S_n^+$. For $\alpha < -1$, we also give the structure of graphs with minimum general Randić index.

For convenience, we need some additional notations and terminology. For a graph $G = (V(G), E(G))$, $|V(G)|$ and $|E(G)|$ are called the order and the size of G, respectively. $d(u, v)$ denotes the length of the shortest path connecting u and v in G. A vertex of degree one of a graph is called a pendent vertex. In S_n $(n \ge 3)$, we call the vertex of degree greater than one as the center of S_n. The path of order n is denoted by P_n.

2 The case for $\alpha > 0$

The following theorem is due to Hu, Li and Yuan [7].

Theorem 2.1. ([7]) Among trees with n $(n \ge 5)$ vertices, the path P_n has the minimum general Randić index for $\alpha > 0$.

Note that the result of Theorem 2.1 is not true for $n = 4$ since $R_\alpha(S_4) < R_\alpha(P_4)$ when α is large enough.

Theorem 2.2. Among unicyclic graphs of order n, the cycle C_n has the minimum general Randić index for $\alpha > 0$.

Proof. Suppose G is a unicyclic graph which has the minimum value of Randić index, but is not the cycle C_n. We will derive contradiction. Assume C is the unique cycle in G and v is a vertex of C with $d(v) = d \geq 3$. Let u, w be the two neighbors of v on the cycle C with $d(u) = d_1$, $d(w) = d_2$ and T be the subtree of G attached to v (i.e. it is the component containing v in $G - \{u, w\}$). We consider several cases.

Case 1. T is a star with v as its center.

Let G' be the graph obtained from $G - v$ by connecting u and w with a new path of length d. Then $R_\alpha(G) - R_\alpha(G') = ((dd_1)^\alpha + (dd_2)^\alpha + (d-2)d^\alpha) - ((2d_1)^\alpha + (2d_2)^\alpha + (d-2)4^\alpha)$. If $d \geq 4$, it is easy to see that $R_\alpha(G) > R_\alpha(G')$. If $d = 3$, then

$$\begin{aligned} R_\alpha(G) - R_\alpha(G') &> ((3d_1)^\alpha - (2d_1)^\alpha) - (4^\alpha - 3^\alpha) \\ &\geq (6^\alpha - 4^\alpha) - (4^\alpha - 3^\alpha) = 6^\alpha + 3^\alpha - 2 \cdot 4^\alpha \\ &> 2\sqrt{18^\alpha} - 2\sqrt{16^\alpha} > 0. \end{aligned}$$

Case 2. T is not a star and every vertex in $V(T) \setminus \{v\}$ has degree not greater than two.

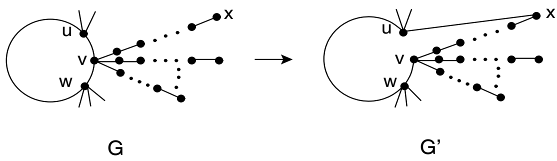

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Fig. 1

Then there must be a path $P(v, x)$ of length $l \geq 2$, where x is a pendent vertex of T. Let $G' = G - uv + ux$ as shown in Fig. 1. If $d \geq 4$, then $R_\alpha(G) - R_\alpha(G') \geq (d_1 d)^\alpha - (2d_1)^\alpha + 2^\alpha - 4^\alpha > d^\alpha - 2^\alpha + 2^\alpha - 4^\alpha \geq 0$. If $d = 3$, then T is just the path $P(v, x)$, and then

$$\begin{aligned} R_\alpha(G) - R_\alpha(G') &= ((3d_1)^\alpha + (3d_2)^\alpha + 6^\alpha + (l-2)4^\alpha + 2^\alpha) \\ &\quad -((2d_1)^\alpha + (2d_2)^\alpha + 4^\alpha + (l-2)4^\alpha + 4^\alpha) \\ &> (3^\alpha - 2^\alpha) + (6^\alpha + 2^\alpha) - 2 \cdot 4^\alpha \\ &= 3^\alpha + 6^\alpha - 2 \cdot 4^\alpha > 0. \end{aligned}$$

Case 3. There exists a vertex in $V(T)\setminus\{v\}$ with degree at least three.

First we consider $\alpha \geq 1$.

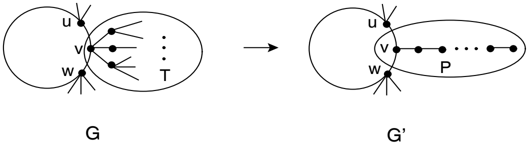

Fig. 2

Let G' be a unicyclic graph obtained from G by replacing T with the path P with the same size of T as shown in Fig. 2. Denote $d_3, d_4, \cdots, d_d$ the degrees of $N_G(v)\setminus\{u,w\}$ and $W=\sum_e R_\alpha(e)$, where $e \in E(G)\setminus E(T)$ is not incident with v. Then
$R_\alpha(G) = W + d^\alpha(d_1^\alpha + d_2^\alpha) + (d^\alpha - (d-2)^\alpha)(d_3^\alpha + \cdots + d_d^\alpha) + R_\alpha(T)$, and
$R_\alpha(G') = W + 3^\alpha(d_1^\alpha + d_2^\alpha) + (3^\alpha - 1)\cdot 2^\alpha + R_\alpha(P)$. Note that $d^\alpha \geq 3^\alpha$. If $d=3$ and T is just the star S_4, it is straightforward to check that $R_\alpha(G) - R_\alpha(G') = 3^\alpha + 3^\alpha + 9^\alpha - (2^\alpha + 4^\alpha + 6^\alpha) > (9^\alpha - 6^\alpha) - (4^\alpha - 3^\alpha) > 0$. Otherwise, we have $R_\alpha(T) > R_\alpha(P)$ by Theorem 2.1, and $d^\alpha - (d-2)^\alpha \geq 3^\alpha - 1$ by the Lagrange's mean-value theorem. Thus, $R_\alpha(G) > R_\alpha(G')$.

Now assume $0 < \alpha < 1$.

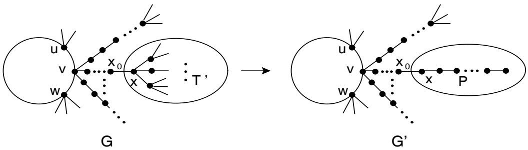

Fig. 3

Let $x \in V(T)$ be a vertex such that $d_G(x) = s+1 \geq 3$ and $d(x,v)$ is minimum. Denote $t_0, t_1, \cdots, t_s$ the degrees of the neighbors of x, where t_0 is the degree of the neighbor x_0 on the path between x and v. Let T' be the component of $G - xx_0$ containing x and G' be the graph obtained from G by replacing T' with a path P' as the same size of T' as shown in Fig. 3. If T' is the star S_4, one can easily check that $R_\alpha(G) > R_\alpha(G')$. Otherwise, by Theorem 2.1, we then

$$\begin{aligned} R_\alpha(G) - R_\alpha(G') &\geq t_0^\alpha(s+1)^\alpha + R_\alpha(T') - s^\alpha(t_1^\alpha + \cdots + t_s^\alpha) + (s+1)^\alpha(t_1^\alpha + \cdots + t_s^\alpha) \\ &\quad -(t_0^\alpha 2^\alpha + R_\alpha(P') - 1\cdot 2^\alpha + 2^\alpha\cdot 2^\alpha) \\ &> t_0^\alpha((s+1)^\alpha - 2^\alpha) - (4^\alpha - 2^\alpha). \end{aligned}$$

It is easy to see that $R_\alpha(G) - R_\alpha(G') > 0$ for $s \geq 3$. If $s = 2$, then

$$\begin{aligned} R_\alpha(G) - R_\alpha(G') &\geq t_0^\alpha 3^\alpha + R_\alpha(T') - 2^\alpha(t_1^\alpha + t_2^\alpha) + 3^\alpha(t_1^\alpha + t_2^\alpha) \\ &\quad -(t_0^\alpha 2^\alpha + R_\alpha(P') - 2^\alpha + 4^\alpha) \\ &\geq (t_1^\alpha + t_2^\alpha + t_0^\alpha)(3^\alpha - 2^\alpha) - (4^\alpha - 2^\alpha) \\ &= (t_1^\alpha + t_2^\alpha + t_0^\alpha - 1)(3^\alpha - 2^\alpha) - (4^\alpha - 3^\alpha). \end{aligned}$$

It is easy to see that $R_\alpha(G) - R_\alpha(G') > 0$ by $0 < \alpha < 1$ and the Lagrange's mean-value theorem.

Therefore, in all cases we obtain a unicyclic graph G' with the smaller value of the index, which contradicts to the choice of G, and the proof is completed. □

3 The case for $\alpha < 0$

Lemma 3.1. Suppose the star S_n, $n \geq 2$, is disjoint from a graph G (may not be a unicyclic graph) and v is its center. For a vertex $u \in V(G)$, let $G_1 = G \cup S_n + uv$, and G_2 be the graph obtained from G by attaching a star S_{n+1} to the vertex u with u as its center as shown in Fig. 4. If u is not an isolated vertex, then $R_\alpha(G_1) > R_\alpha(G_2)$ for $\alpha < 0$.

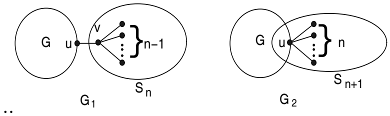

Fig. 4

Proof. Let $d_G(u) = d \geq 1$, then

$$\begin{aligned} R_\alpha(G_1) - R_\alpha(G_2) &> [(n-1)n^\alpha + (n(d+1))^\alpha] - n \cdot (n+d)^\alpha \\ &= (n-1)(n^\alpha - (n+d)^\alpha) + [(n(d+1))^\alpha - (n+d)^\alpha] \\ &= -\alpha(n-1)d\xi_1^{\alpha-1} + \alpha(n-1)d\xi_2^{\alpha-1} \\ &= \alpha(n-1)d(\xi_2^{\alpha-1} - \xi_1^{\alpha-1}) \\ &= \alpha(\alpha-1)(n-1)d\xi^{\alpha-2} > 0, \end{aligned}$$

where $\xi_1 \in (n, n+d), \xi_2 \in (n+d, nd+n)$, and $\xi \in (\xi_1, \xi_2)$. □

Using Lemma 3.1, we give a simple proof of the following result due to Hu, Li and Yuan [7].

Theorem 3.2. ([7]) Among trees with n vertices, the star S_n has the minimum general Randić index for $\alpha < 0$.

Proof. Suppose T is a tree with the minimum general Randić index, but T is not a star. Let $P = v_1 v_2 \cdots v_{l+1}$ be a longest path in T, then $l > 2$. Note that $\{v_2\} \cup N(v_2) \setminus \{v_3\}$ induce a star S_d where $d(v_2) = d$. Now we construct a new tree T' from T by contracting the edge $v_2 v_3$ and adding a new pendent vertex w as a neighbor of $v_3(v_2)$. By Lemma 3.1, $R_\alpha(T) > R_\alpha(T')$, a contradiction. □

Lemma 3.3. Assume G has the minimum general Randić index among unicyclic graphs of order n. If T is a tree attached to a vertex v of the unique cycle in G, then T must be a star with v as its center.

Proof. Let T be a tree attached to a vertex v of the unique cycle in G, and T is not a star with v as its center. Suppose $x \in V(T)$ is a pendent vertex of G such that $d(v, x)$ is maximum. Assume y is the neighbor of x and z is the neighbor of y on the path connecting v and x (z and v may be identical), then contracting the edge yz and adding a new pendent vertex w as a neighbor of $z(y)$ to obtain a new graph G'. By Lemma 3.1, $R_\alpha(G) > R_\alpha(G')$, a contradiction. Thus, T must be a star with v as its center. □

Lemma 3.4. Let G_1 and G_2 be two unicyclic graphs with the same order n, and their unique cycles are i-cycle and $(i-1)$-cycle respectively, $i \geq 4$. Further, for each graph, the vertices not on the cycle are pendent vertices adjacent to exactly one vertex of the cycle. Then $R_\alpha(G_1) > R_\alpha(G_2)$ for $\alpha < 0$.

Proof. One can see that $R_\alpha(G_1) = (i-2)4^\alpha + (n-i)(n-i+2)^\alpha + 2 \cdot 2^\alpha (n-i+2)^\alpha$, $R_\alpha(G_2) = (i-3)4^\alpha + (n-i+1)(n-i+3)^\alpha + 2 \cdot 2^\alpha (n-i+3)^\alpha$. Then $R_\alpha(G_1) - R_\alpha(G_2) > 4^\alpha - (n-i+3)^\alpha \geq 0$ for $\alpha < 0$. □

Let G be a unicyclic graph with the unique cycle C and S_{a+1}, S_{b+1} are the two stars attached to two vertices u and w of C, respectively. Assume the two paths between u and w on C are P_{uw} and P_{wu}. Suppose the degrees of all the vertices on P_{uw} are two (if there exists such vertex) in G, $|E(P_{uw})| = c$ and $|E(P_{wu})| \geq 3$. Now transform G into a new unicyclic graph G' as following: contracting the path P_{uw} into one vertex $u(w)$, and attaching a star $S_{a+b+c+1}$ to it (see Fig. 5). Note that the order of G' is equal to that of G. Next we show that this transformation will decrease the general Randić index of the graph for $\alpha < 0$.

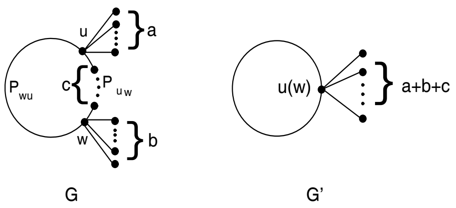

Fig. 5

Lemma 3.5. Let G and G' be the two unicyclic graphs described above. If $a, b \geq 1$, then $R_\alpha(G) > R_\alpha(G')$ for $\alpha < 0$.

Proof. In the following proof, we frequently use the Lagrange's mean-value theorem.

If $c = 1$, that is $uw \in E(G)$, then

$$\begin{aligned}
& R_\alpha(G) - R_\alpha(G') \\
> \; & a(a+2)^\alpha + b(b+2)^\alpha + (a+2)^\alpha(b+2)^\alpha - (a+b+1)(a+b+3)^\alpha \\
= \; & a((a+2)^\alpha - (a+b+3)^\alpha) + b((b+2)^\alpha - (a+b+3)^\alpha) \\
& +(a+2)^\alpha(b+2)^\alpha - (a+b+3)^\alpha \\
= \; & -\alpha\xi_1^{\alpha-1}a(b+1) - \alpha\xi_2^{\alpha-1}b(a+1) + \alpha\xi_3^{\alpha-1}((ab+2a+2b+4) - (a+b+3)) \\
= \; & -\alpha[\xi_1^{\alpha-1}a(b+1) + \xi_2^{\alpha-1}b(a+1) - \xi_3^{\alpha-1}(ab+a+b+1)] \\
> \; & -\alpha\xi_3^{\alpha-1}(ab-1) \geq 0,
\end{aligned}$$

where $\xi_1 \in (a+2, a+b+3), \xi_2 \in (b+2, a+b+3), \xi_3 \in (a+b+3, ab+2a+2b+4)$.

Now we consider the case of $c \geq 2$. Without loss of generality, let $a \geq b$, and then

$$\begin{aligned}
& R_\alpha(G) - R_\alpha(G') \\
> \; & a(a+2)^\alpha + b(b+2)^\alpha + 2^\alpha(a+2)^\alpha + 2^\alpha(b+2)^\alpha \\
& +(c-2)4^\alpha - (a+b+c)(a+b+c+2)^\alpha \\
> \; & a(a+2)^\alpha + b(b+2)^\alpha + 2^\alpha(a+2)^\alpha + 2^\alpha(b+2)^\alpha - (a+b+2)(a+b+4)^\alpha \\
= \; & a((a+2)^\alpha - (a+b+4)^\alpha) + b((b+2)^\alpha - (a+b+4)^\alpha) \\
& +((2a+4)^\alpha - (a+b+4)^\alpha) + ((2b+4)^\alpha - (a+b+4)^\alpha) \\
\geq \; & -\alpha\xi_1^{\alpha-1}a(b+2) - \alpha\xi_2^{\alpha-1}b(a+2) + \alpha\xi_3^{\alpha-1}(a-b) > 0,
\end{aligned}$$

where $\xi_1 \in (a+2, a+b+4), \xi_2 \in (b+2, a+b+4), \xi_3 \in (a+b+4, 2a+4)$. □

Now we define two classes $\mathcal{G}$ and $\mathcal{H}$ of graphs. $\mathcal{G}$ consists of the unicyclic graphs each of which has a triangle as its unique cycle, and the vertices not on the cycle are pendent vertices. $\mathcal{H}$ consists of unicyclic graphs each of which has a 4-cycle as its unique cycle, and the vertices not on the cycle are pendent vertices and are neighbors of two nonadjacent vertices of the cycle.

From Lemma 3.3, Lemma 3.4 and Lemma 3.5, we can conclude that

Theorem 3.6. For $\alpha < 0$, the unicyclic graph with the minimum general Randić index must be in $\mathcal{G}$ or $\mathcal{H}$.

For $-1 \leq \alpha < 0$, we can even show that S_n^+ is just the unique extremal graph with minimum Randić index. Before proving it, we need another two lemmas.

Lemma 3.7. Let G_1, G_2, G_3 be three unicyclic graphs in $\mathcal{G}$ as in Fig. 6. If $x \geq y \geq 1$, then $R_\alpha(G_1) > R_\alpha(G_3)$ for $-1 \leq \alpha < 0$.

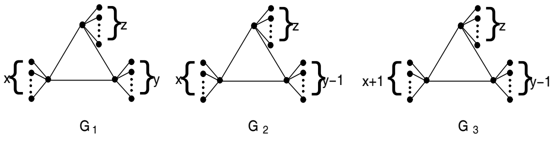

Fig. 6

Proof. From Fig. 6, first we have

$$\begin{aligned}
& R_\alpha(G_1) - R_\alpha(G_2) \\
= \; & y(y+2)^\alpha + (x+2)^\alpha(y+2)^\alpha + (y+2)^\alpha(z+2)^\alpha \\
& -((y-1)(y+1)^\alpha + (x+2)^\alpha(y+1)^\alpha + (y+1)^\alpha(z+2)^\alpha) \\
= \; & y(y+2)^\alpha - (y-1)(y+1)^\alpha + ((x+2)^\alpha + (z+2)^\alpha)((y+2)^\alpha - (y+1)^\alpha) \\
= \; & ((y+2)^{\alpha+1} - (y+1)^{\alpha+1}) + ((x+2)^\alpha + (z+2)^\alpha - 2)((y+2)^\alpha - (y+1)^\alpha) \\
= \; & (\alpha+1)\xi_1^\alpha + ((x+2)^\alpha + (z+2)^\alpha - 2)\alpha\xi_2^{\alpha-1},
\end{aligned}$$

where $\xi_1, \xi_2 \in (y+1, y+2)$. Similarly, we have

$$\begin{aligned}
& R_\alpha(G_3) - R_\alpha(G_2) \\
= \; & (x+1)(x+3)^\alpha + (x+3)^\alpha(y+1)^\alpha + (x+3)^\alpha(z+2)^\alpha \\
& -(x(x+2)^\alpha + (x+2)^\alpha(y+1)^\alpha + (x+2)^\alpha(z+2)^\alpha)
\end{aligned}$$

$$
\begin{aligned}
&= (x+1)(x+3)^\alpha - x(x+2)^\alpha + ((y+1)^\alpha + (z+2)^\alpha)((x+3)^\alpha - (x+2)^\alpha) \\
&= ((x+3)^{\alpha+1} - (x+2)^{\alpha+1}) + ((y+1)^\alpha + (z+2)^\alpha - 2)((x+3)^\alpha - (x+2)^\alpha) \\
&= (\alpha+1)\xi_3^\alpha + ((y+1)^\alpha + (z+2)^\alpha - 2)\alpha\xi_4^{\alpha-1},
\end{aligned}
$$

where $\xi_3, \xi_4 \in (x+2, x+3)$. It is easy to see that $R_\alpha(G_1) - R_\alpha(G_2) > R_\alpha(G_3) - R_\alpha(G_2)$ by $-1 \le \alpha < 0$ and $x \ge y$. Hence $R_\alpha(G_1) > R_\alpha(G_3)$. □

Lemma 3.8. Let H_1, H_2, H_3 be three unicyclic graphs as in Fig. 7. If $a \ge b \ge 1$, then $R_\alpha(H_1) > R_\alpha(H_3)$ for $-1 \le \alpha < 0$.

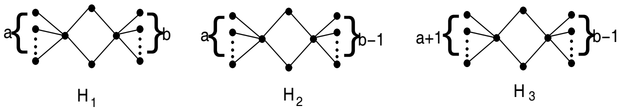

Fig. 7

Proof. Define a function $f(x) = (x + 2^{\alpha+1})(x+2)^\alpha$, $x \ge 0$ and $-1 \le \alpha < 0$. As shown in Fig. 7, we have

$R_\alpha(H_1) = a(a+2)^\alpha + b(b+2)^\alpha + 2\cdot 2^\alpha(a+2)^\alpha + 2\cdot 2^\alpha(b+2)^\alpha$,

$R_\alpha(H_2) = a(a+2)^\alpha + (b-1)(b+1)^\alpha + 2\cdot 2^\alpha(a+2)^\alpha + 2\cdot 2^\alpha(b+1)^\alpha$, and

$R_\alpha(H_3) = (a+1)(a+3)^\alpha + (b-1)(b+1)^\alpha + 2\cdot 2^\alpha(a+3)^\alpha + 2\cdot 2^\alpha(b+1)^\alpha$. Thus,

$R_\alpha(H_1) - R_\alpha(H_2) = (b+2^{\alpha+1})(b+2)^\alpha - (b-1+2^{\alpha+1})(b+1)^\alpha = f(b) - f(b-1) = f'(\xi_1)$,

$R_\alpha(H_3) - R_\alpha(H_2) = (a+1+2^{\alpha+1})(a+3)^\alpha - (a+2^{\alpha+1})(a+2)^\alpha = f(a+1) - f(a) = f'(\xi_2)$,

where $\xi_1 \in (b-1, b)$ and $\xi_2 \in (a, a+1)$. By $a \ge b \ge 1$, $-1 \le \alpha < 0$ and since $f''(x) = \alpha(x+2)^{\alpha-2}(2(x+2) - (1-\alpha)(x+2^{\alpha+1})) < 0$ for any $x > 0$, we know that $f'(\xi_1) > f'(\xi_2)$, which implies that $R_\alpha(H_1) > R_\alpha(H_3)$ for $-1 \le \alpha < 0$. □

From Lemma 3.4, Theorem 3.6, Lemma 3.7 and Lemma 3.8, one can see that

Theorem 3.9. Among all the unicyclic graphs of order n, S_n^+ has the minimum general Randić index for $-1 \le \alpha < 0$.

4 Concluding remarks

In this paper, we discuss unicyclic graphs with the minimum general Randić index. We use the following table to summarize our main results.

α	$\alpha > 0$	$-1 \leq \alpha < 0$	$\alpha < -1$
extremal unicyclic graph	C_n	S_n^+	in $\mathcal{G}$ or $\mathcal{H}$
minimum value	$n \cdot 4^\alpha$	$(n-3)(n-1)^\alpha + 2(2n-2)^\alpha + 4^\alpha$	

For $\alpha < -1$, S_n^+ cannot be the extremal graph with minimum Randić index when n is large enough. To see this, we consider the graph G_1 of $\mathcal{G}$ as shown in Fig. 6. Taking $x = y = z = \frac{n-3}{3}$, we can see that $R_\alpha(G_1) \to 0$ when $n \to \infty$. However, $R_\alpha(S_n^+) > 4^\alpha$ for any n.

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