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Towards an empirical relation between the elementary polygonal circuit area and the topological form index, *l*, in the polyhedra and 2- and 3-dimensional structures

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Abstract-This paper begins with a review of the Euler relation for the polyhedra and presents the corresponding Schläfli relation in *n*, the polygonality, and *p*, the connectivity of the polyhedra. The use of the Schläfli symbols to organize the mapping of the polyhedra and its extension into the 2D and 3D networks is described. The topological form index, represented by *l*, is introduced and is defined as the ratio of the polygonality, *n*, to the connectivity, *p*, in a structure. Next a discussion is given of establishing a conventional metric of length in order to compare topological properties of the polyhedra and networks in 2D and 3D. A fundamental structural metric is assumed for the polyhedra. The metric for the polyhedra is, in turn, used to establish a metric for tilings in the Euclidean plane. The metrics for the polyhedra and 2D plane are used to establish a metric for networks in 3D. Once the metrics have been established, a conjecture is introduced that the area of the elementary polygonal circuit in the polyhedra and 2D and 3D networks is proportional to the topological form index, *l,* for these structures. Data of the form indices and the corresponding elementary polygonal circuit areas, for a selection of polyhedra and 2D and 3D networks, is tabulated and the results of a least squares regression analysis of the data plotted in a Cartesian space are reported. From the regression analysis it is seen that a quadratic in *l* successfully correlates the topological form indices with the corresponding elementary polygonal circuit area data of the polyhedra and 2D and 3D networks. A brief discussion of the evident rigorousness of the Schläfli indices over all the polyhedra and 2D and 3D networks, based upon the correlation of the topological form index with elementary polygonal circuit area, and the suggestion that an Euler-Schläfli relation for the 2D and 3D networks in terms of the Schläfli indices is possible, concludes the paper.

1. Introduction

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Euler's relation between the number of vertices, *V*, edges, *E,* and faces, *F,* of convex polyhedra was developed in the middle of the 18th century [1] and its discovery marks the origin of the discipline of topology [2]. This relation is shown in Equation 1 below:

(1) $$V - E + F = 2$$

From this equation it is said that the Euler characteristic for the sphere is 2. This simply, and elegantly, means that any division of a sphere into vertices, edges and faces will have that combination so specified in Equation 1. It happens that the convex polyhedra, with all their inherent symmetry and internal beauty, are the idealized divisions of the sphere into the topology suggested first by Euler in his 1758 paper [1].

Some time after this, in a paper due to Schläfli, [3] the identities shown in Equation 2 and Equation 3 were discovered:

(2) $$nF = 2E$$

(3) $$pV = 2E$$

Schläfli identified the polygonality of convex polyhedra; or any division of the sphere into vertices, edges and faces, as the averaged number of sides of the polygonal faces in the object. He determined the relation shown in 2, that the averaged polygonality in the object, *n*, multiplied by its number of faces, *F,* is equal to twice its number of edges, *E*. Because each edge, *E,* is shared by two faces (i.e. adjacent faces share a common edge) this relationship is rigorous.

Similarly in 3 we see the Schläfli relation between the connectivity of convex polyhedra, *p*, and the number of vertices, *V*, and edges, *E*. The connectivity, *p*, is identified as the averaged number of edges meeting at each vertex of a polyhedron. Because each of the edges terminates at two vertices, one can see that this Schläfli relation is rigorous. One speaks of averaged numbers for *n* and *p*, because unless the polyhedron is regular (meaning all faces are identical polygons) there are differing numbers of edges to each polygonal face and/or differing numbers of edges meeting at each polygonal vertex. One can therefore identify the semi-regular polyhedra, these are the Archimedean (polyhedra with more than one type of polygonal face) and Catalan (polyhedra with more than one type of polygonal vertex) polyhedra [4]. There are, in addition, innumerable irregular polyhedra, these are polyhedra in which there is more than one type of polygonal face *and* more than one type of polygonal vertex. Some of the irregular polyhedra have been reported as recently as 2001.

Schläfli substituted Equations 2 and 3 into the Euler relation, as shown in Equation 4, to obtain a relation between *V, E* and *F*; known as the primary topological indices; and *n* and *p*; known as the secondary topological indices.

$$\frac{1}{n} - \frac{1}{2} + \frac{1}{p} = \frac{1}{E} \tag{4}$$

This latter Schläfli relation is important from the perspective of the Schläfli symbols (*n, p*) that can be identified for any structure. All polyhedra have rigorously determined values of *n* and *p*, just like they have rigorously determined values of *V, E* and *F*. The ordered pair thus formed of *n* and *p*, the polyhedral Schläfli symbol, represents a location in a Cartesian-like space, called a Schläfli space, in which the polyhedral object can be mapped.

The beginnings of this topological mapping for the regular polyhedra have been shown elsewhere [3(m)]. The map is outlined, as a Cartesian-like space, by increasing connectivity, *p*, running as an axis from left to right, and by increasing polygonality, *n*, running as an axis from top to bottom of the map. The five regular polyhedra are called the Platonic solids for their role as elements in Plato's philosophical treatise known as the Timeas [4]. The Platonic solid with the highest topology is the tetrahedron (meaning it has 4 equivalent faces) with the Schläfli symbol (3, 3). It marks the origin of this map. Similarly, there is the cube (4, 3), and the pentagonal dodecahedron (5, 3), lining the column underneath the tetrahedron. To the right of (3, 3), are the octahedron (3, 4) and the icosahedron (3, 5). One can immediately see the power of the Schläfli relation as an organizing principle in its usefulness as a mapping tool for determining the identity and relative location of all of the various polyhedra. One could extend this mapping to include the semi-regular and irregular polyhedra as well. The Archimedean polyhedra have fractional polygonalities, *n*; the Catalan polyhedra have fractional connectivities, *p*; and the irregular polyhedra have both fractional polygonality and fractional connectivity.

In the 1950's Wells began his enumerative work on 2- and 3-dimensional networks and novel crystal structures [3(a)]. He labeled his novel networks with the Schläfli symbols (*n, p*) to identify them. For while he did not determine a Schläfli-like relation for crystal structures (that is collections of vertices, edges and faces filling 3-dimensional space, and not constrained to the surface of a sphere) he nonetheless discovered that both the polygonality, *n,* and the connectivity, *p,* could be rigorously calculated within the unit of pattern of extended structures in both 2- and 3-dimensions [3(m)]. He concluded that the topology map for the polyhedra could be extended in Schläfli-space (the space of *n* and *p*) by a simple augmentation of the ordered pairs of numbers (*n, p*), the Schläfli symbols, to the right of those for the polyhedra. Such an augmentation of this topology map involved moving into frontier that included the 2-dimensional tesselations, like the regular 2D extended structures of the honeycomb net (6, 3), the square net (4, 4), and the closest packed net (3, 6); and also the semi-regular and irregular tilings of the Euclidean plane; on into the territory of the 3-dimensional networks. The extension of the topology map, due to Wells, has been shown elsewhere [5]. Note that to the right of the 2D networks, the frontier of the 3-dimensional nets, the Schläfli symbol (*n, p*) may represent more than one way of filling space with a network of the specified topology.

Early on, work by Wells involved the enumeration of the regular networks, that is networks in which the polygonality of circuits in the net is a uniform number, and the connectivity of the vertices in the networks is a uniform number. Such networks, represent structures with some of the highest topologies possible. This work included such topologies as that represented by the Schläfli symbol (7, 3) in which the polygonality of the circuits in the network, n, is uniform at 7, and the connectivity of the vertices in the network is uniform at 3 [3]. In this work, according to Wells, he was attempting to extend the topology map from the index (5, 3), the polyhedron called the pentagonal dodecahedron, to (6, 3), the 2D tesselation which is known as the honeycomb net, onto (7, 3) which represents a continuation of this sequence into 3-dimensional space. He eventually determined 4 distinct structures that possessed the Schläfli symbol (7, 3). He did other similar elegant work on 3D networks of topology (8, 3), (9, 3), (10, 3) and (12, 3) [3]. Later on, Wells turned to networks whose topology was reduced, these were the semi-regular and irregular 3D networks, [3] as well as continuing his research on regular networks.

For the present discussion, the theme is to establish a relation between these topological Schläfli indices, introduced and described above, and the elementary polygonal circuit area in a structure, labelled *area*(n, p), (whether it be a polyhedron, a 2D tesselation or a 3D network). The reasons for choosing elementary polygonal circuit area to establish a geometrical-topological correlation in structures will be discussed more fully below in connection with the concept of a structural metric. It has been discovered, in the present work, that one can formulate a topological index derived from n and p that in fact, correlates with the elementary polygonal circuit area of structures, to include the polyhedra and the 2D and 3D patterns. This new index, first described in 1997 [5], is defined as the ratio of the polygonality to the connectivity in a given structure, l. This is shown in Equation 5:

(5) $$\mathrm{l} = \frac{\mathrm{n}}{\mathrm{p}}$$

Such a topological index of structures is a measure of what is termed the compactness of a structure, it is hereafter called the Schläfli topological form index.

2. Identification of a Geometrical Standard

In order to establish a correlation between a geometrical structural parameter and a topological structural parameter, in patterns, it is necessary to define a standard of length, called a metric, amongst which all structures in the same class; be they the polyhedra or the 2D tesselations or the 3D networks; possess the metric commonly. Establishing such a metric of length is essential to identify a property correlation across structures in all classes, and it provides an internal consistency in the correlation analysis. In this Section, we will postulate a metric for the polyhedra, called the Wells polyhedra metric [6], and from which the metric for the 2D structures and the metric for the 3D structures are derived.

Before moving on to the discussion of metrics, it is important to clarify why the geometrical-topological structural correlation being described in this paper involves the structural parameter of elementary polygonal circuit area. In the course of this investigation, the problem arose as to how one could establish the applicability of the Schläfli symbols to the 2D and 3D networks. As has been discussed in the previous Section, Wells found that he could calculate the Schläfli indices (n, p) for any 2D or 3D pattern, but the Schläfli relation given in Equation 4 was not rigorous for these ordered pairs associated with patterns in higher dimension than the polyhedra.

It is the purpose of the present communication to establish a different relation involving the Schläfli indices and another property of structures (this being the geometrical structural property of elementary polygonal circuit area) in order to demonstrate that these topological indices have applicability to the analysis of properties of the 2D and 3D networks. This may have importance with respect to the eventual formulation of an Euler-Schläfli relation for the 2D and 3D structures. In addition, such a study as the present one has as its goal to show the reader topological indices of structures have a bearing on, and are related to, geometrical properties of structures.

In a separate sense, the choice of elementary polygonal circuit area as a structural property to establish as a geometrical-topological correlation was made on the basis that 2D patterns have polygonal circuit area but, technically, no volume, and this structural property of polygonal circuit area is shared with the polyhedra and the 3D structures. Also, there are additional reasons, connected with the problem of establishing a suitable metric, for not employing structural volume in a correlation with topological structural parameters. These will not be discussed here. At any event, in the polyhedra and 2D and 3D patterns one can determine, even if this involves an averaging process, as in the semi-regular and irregular structures, the elementary polygonal circuit area, labeled as $area_{(n,\ p)}$, of a structure.

Turning to the identification of a fundamental geometrical structural parameter, a metric of length, in order to provide a basis for a geometrical-topological correlation, the original work of Euler is considered [1]. Euler envisioned the inscription of the polyhedra inside the sphere in order to establish the relation shown in Equation 1 in the previous Section. In the interest of establishing suitable metrics for the 2D and 3D patterns, we begin with the assumption that the polyhedra are inscribed in the unit sphere. Therefore, from the center of the sphere, and the corresponding polyhedra, there exist radii, of length unity, that point in all directions about the sphere (polyhedra), including into the vertices of the various polyhedra. Such an assumption is the basis for the calculation of the edge lengths and face areas of the polyhedra, and the results of this analysis are later used to establish metrics for the 2D and 3D patterns. Therefore, the assumption that the polyhedra are inscribed in the unit sphere is called the Wells fundamental polyhedra metric [6].

The analysis of edge lengths and face areas, to eventually be used in the geometrical-topological correlation, begins with the inscription of the regular tetrahedron (3, 3) in the unit sphere. It is an easy matter to calculate the corresponding edge of this polyhedron, one uses plane geometry and the fact that the unit radii pointing into a pair of tetrahedral vertices form a triangle in which the obtuse angle is ideal at 109.47°. From this one gets an edge of $\frac{2\sqrt{2}}{\sqrt{3}}$ and a corresponding face area of $\frac{2}{\sqrt{3}}$. Turning next to the cube (4, 3), unit radii pointing into adjacent vertices form a right triangle possessing a hypotenuse of length 2, comprised of the corresponding face diagonal, leading to an edge length of $\frac{2}{\sqrt{3}}$ and a face area of $\frac{4}{3}$. Turning to the octahedron (3, 4), unit radii pointing to an axial and an equatorial pair of vertices define an isosceles right triangle that leads to an octahedral edge of $\sqrt{2}$ and an octahedral face area of $\frac{\sqrt{3}}{2}$.

The other two regular (Platonic) polyhedra, the pentagonal dodecahedron (5, 3) and the triangular icosahedron (3, 5) present less straightforward geometrical problems, and they are not essential to further establish the 2D and 3D metrics, so their analysis will be left to a separate paper. From the preceding paragraph, all the information required for the derivation of the 2D and 3D metrics is available, upon positing a few further assumptions. One should bear in mind that the metric for the polyhedra is provided through the assumption that they are inscribed in the unit sphere. This leads to different edge lengths and different face areas in each of the polyhedra, however they share their inscription on the unit sphere, which is the metric of length for them. They, in fact, must have different face areas, and the following relation holds: $area(5, 3) > area(4, 3) > area(3, 3) > area(3, 4) > area(3, 5)$ due to the equation between the form index, ℓ, and the elementary polygonal circuit area, called $area(n, p)$, which will be described below.

To identify the metric for the 2D tesselations one looks to the Schläfli indices in 2D and in the polyhedra to see if there are any identical form indices. If two structures have the same topological form index, *l,* they will have the same elementary polygonal circuit area, according to the relation assumed to hold for structures in the development of this paper. Therefore this relationship, called the Wells structural correspondence principle [6], that the identity of the metric in 2D structures is based upon, represents a second assumption introduced in this paper. The regular square net (4, 4) has a form index of unity, which is the same as the form index in the tetrahedron (3, 3). The regular square net (4, 4) has been illustrated elsewhere [3(m)]. Therefore, the 2D metric is established as the corresponding edge length of the square face of the square net, which has the same face area as the tetrahedron inscribed in the unit sphere. Therefore, the following relation, shown in Equation 6, holds:

(6) $$\text{area}(3, 3) = \text{area}(4, 4) = \frac{2}{\sqrt{3}}$$

And the corresponding 2D metric is just the edge of the square in (4, 4), or $\sqrt{\frac{2}{\sqrt{3}}}$.

To get the edge metric in 3D we turn to the related morphologies of the cube (4, 3), the square net (4, 4) and the primitive cubic net (4, 6), these have been discussed and illustrated elsewhere [3(m)]. It is a third assumption, introduced in this paper, that structures of related morphologies in different structural classes have face areas that are proportional. This is called the Wells morphological principle [6]. The cube, with the Schläfli symbol (4, 3), the square net (4, 4), and the primitive cubic net (NaCl structure) (4, 6), all share perfectly square faces, as a common morphological theme, in their structures. So, on the basis of the morphological principle we can write the following proportionality expression down:

(7) $$\frac{\text{area}(4,3)}{\text{area}(4,4)} = \frac{\text{area}(4,4)}{\text{area}(4,6)}$$

By substitution the unknown in 7, *area*(4,6) can be solved for as in 8.

(8) $$\text{area}(4,6) = \text{area}(4,4)\,\frac{\text{area}(4,4)}{\text{area}(4,3)} = \text{unity}$$

It is therefore established in this scheme; developed out of the fundamental assumptions of inscription of the polyhedra on the unit sphere, the Wells polyhedra metric, and the Wells structural correspondence principle described above, and finally the Wells morphological principle, introduced in this paragraph; that the metric for all of the 3D networks is unit edge length. This is derived from the fact that the primitive cubic net (4, 6) has unit face area and therefore unity for its edge lengths. Therefore all the edges of all of the circuits in the 3D nets share edge length unity for the purposes of providing a geometrical-topological analysis of structures that is internally consistent.

3. Consequences of the Metrics

A representative sampling of 12 structures has been analyzed topologically and in terms of the elementary polygonal face areas of the structures, for use in establishing a geometrical-topological correlation. The set includes 3 regular polyhedra, the 3 regular 2D tesselations, 3 regular 3D nets, one Archimedean 3D net, one Catalan 3D net and one irregular (Wellsean) 3D net. This sampling provides a broad base of possible topological varieties of structure from which to determine if a correlation exists between the topological form index, *l,* of Equation 5, and the elementary polygonal circuit area, labeled $area_{(n,\, p)}$. Table 1 provides a compilation of the data for these 12 structures, note that the metric for the polyhedra is inscription on the unit sphere, the resulting edge metric for the 2D tesselations is just $\sqrt{\frac{2}{\sqrt{3}}}$, and the edge metric of the 3D networks is just unity. In Table 1 $ThSi_2$ (10, 3) [7], diamond (6, 4) [3(m)] and the primitive cubic net (4, 6) [3(m)] are the regular structures, and they possess ideal bond angles. The Cooperite structure ($6^{2/5}$, 4) [8] is Archimedean, and has ideal tetrahedral angles and distorted square planar angles assumed in the calculation of its polygonal circuit area. The Waserite structure (8, 3.4285) [9] is Catalan and has ideal bond angles, and the glitter structure with the Schläfli index (7, $3^{1/3}$) [10] is irregular and has ideal tetrahedral angles and distorted trigonal planar angles assumed in the calculation of its polygonal circuit area.

Table 1: Geometrical-Topological Data for 12 Structures

name	(n, p)	$l = n/p$	$area(n,p)$
ccp network	(3, 6)	1/2	1/2
primitive cubic	(4, 6)	2/3	1
octahedron	(3, 4)	3/4	$\sqrt{3}/2$
tetrahedron	(3, 3)	1	$2/\sqrt{3}$
square net	(4, 4)	1	$2/\sqrt{3}$
cube	(4, 3)	$1^{1/3}$	$1^{1/3}$
diamond	(6, 4)	$1^{1/2}$	$2/3\pi$
Cooperite (PtS)	$(6^{2/5}, 4)$	$1^{3/5}$	$2\sqrt{2}\pi/3$
honeycomb net	(6, 3)	2	3
glitter	$(7, 3^{1/3})$	$2\pi/3$	$\sqrt{\frac{2}{\sqrt{3}}}\ \pi$
Waserite (Pt_3O_4)	$(8, \frac{2}{5}\ e\pi)$	$2^{1/3}$	$\sqrt{2}e$
$ThSi_2$	(10, 3)	$3^{1/3}$	$7\sqrt{3}/2$

One can see immediately that the form indices, *l,* and the polygonal circuit areas, called $area(n,\ p)$, are all expressible in closed form as factors of whole numbers, fractions, square roots and the mathematical constants π and *e*. The honeycomb network (6, 3), the structure of the graphene sheet, with an edge length of $\sqrt{\frac{2}{\sqrt{3}}}$, has a hexagonal face area of 3. The diamond network (6, 4), illustrated elsewhere [3(m)], with unity edge length and tetrahedral bond angles, has elementary polygonal face area of $\frac{2\pi}{3}$. The Waserite network (8, 3.4285), illustrated and discussed previously [9], a Catalan network in 3D, has octagonal elementary polygonal circuits in its structure which come out with face area $\sqrt{2}e$. Finally, the glitter network with topological index (7, $3^{1/3}$), an irregular network illustrated elsewhere [10], has a form index, *l,* of $\frac{2\pi}{3}$ and an elementary polygonal circuit area (weighted average of 6-gon and 8-gon areas) consisting of a factor that is the edge metric determined for the 2D tesselations, and the mathematical constant π, it is given as $\sqrt{\frac{2}{\sqrt{3}}}\ \pi$.

The occurrence of closed form numbers, and especially the occurrence of the mathematical constants π and *e* in the computation of some of the polygonal circuit areas, is mysterious. Such apparent coincidences are termed Wells coincidences [6]. They suggest that the polygonal circuit area of the chair hexagons in the diamond lattice, for instance, is just a scaling of π. They suggest that the area of the octagonal circuitry in the real Waserite phase, Pt_3O_4, is just a scaling of e. They suggest that the structure of crystalline matter is an approximation to Platonic archetypes, that indeed all the polyhedra, 2D tesselations and 3D networks; perhaps numbering in the hundreds in terms of those observed as pure forms in various structural-types; have an eternal, separate existence as Platonic archetypes. The diamond structural-type exists in a perfect form as a Platonic archetype, in which its chair substructures possess unity edge length and have an area of $\frac{2\pi}{3}$, for example. It has not been overlooked in this context that the derivation of the metrics in 2D and 3D provided in this paper, together with the standard crystallographic description of structures in terms of the space group symmetry and the Wyckoff positions of the vertices, and the use of elementary plane geometry, provides a geometric construction of the mathematical constants π and *e* that complement the innumerable series and product representations of these ubiquitous numbers.

4. The Wells Conjecture and Geometrical-Topological Correlation

Data from Table 1 has been mapped to a graph in which the topological form index, *l,* is plotted along the horizontal axis, and the elementary polygonal circuit area is plotted along the vertical axis, for the set of structures. Such an empirical plot will be shown completely in a separate publication. The data, consisting of the geometrical-topological information on the 12 structures given in the previous Section, was fit to a quadratic function in *l*. Least squares regression analysis of the data showed a reliability factor of 0.976 (a perfect correlation has a reliability factor of 1.00). The geometrical-topological correlation equation is shown below:

$$\textbf{(9)} \qquad \mathrm{area}_{(n,\,p)} = A\, l^2 + B\, l + C$$

The parameters in Equation 9 are given as $A = 0.16208$, $B = 1.3543$ and $C = -0.21606$, these parameters will shift slightly as more geometrical-topological data for the polyhedra, 2D tesselations and 3D networks is obtained and plotted. It is not clear to the author whether the assumptions introduced earlier in the paper, have biased the data towards exhibiting a correlation. Also, it is possible, under the assumptions introduced earlier in the paper, to calculate the parameters in Equation 9 from a set of three simultaneous equations and this direction will be looked into in a separate paper.

The presence of the very strong correlation between the topological form index for structures and their elementary polygonal circuit area suggests a mathematical conjecture called the Wells conjecture [6]. It is stated below:

The elementary polygonal circuit area of a structure, be it a polyhedron, a 2D tesselation or a 3D network, under a suitable metric, is proportional to a function *of the topological form index l, which is the ratio of the structure's polygonality, n, to the structure's connectivity, p.*

There is no proof of the Wells conjecture presently. It appears that such a proof, if one exists, will be very tenuous and difficult to elucidate, as the correlation described above is only approximate. For example, the data points for the cube, represented by (4, 3), the primitive cubic net (4, 6) and the Cooperite net ($6^{2/5}$, 4) show fairly substantial deviations from Equation 9.

The presence of this strong geometrical-topological correlation is quite surprising in that one would not have expected topological parameters, like *n* and *p*, which are pure numbers, to be related to a geometrical property of the structure like elementary polygonal circuit area, which would seem to have a purely empirical value for a given arbitrary network. This empirical correlation is also fundamental from the point of view of the Schläfli symbols as it shows there is a degree of rigor, evidenced by the strong reliability index of the functional fit of the data, in the Schläfli symbols for the 2D and 3D structures. This latter result suggests it may be possible to formulate an Euler-Schläfli relation, using *n* and *p* in some functional form, to predict the number of edges occurring in the units of pattern of 2D and 3D structures.

Next, we state a note on compactness. Earlier it was thought by the author that the topological form index, *l,* was a measure of the density of the network. Density is a measure of the number of vertices in a metric of volume of a structure, at this juncture it is not clear that *l* correlates with density. In fact empirical evidence from hexagonite and the expanded hexagonites [11] suggests strongly that *l* is not a measure of density. It is suggested here that the term compactness be used with reference to *l,* compactness is a measure of how tightly connected together the circuitry in a net is, it is a measure of the compactness of area which is occupied by matter in the structure. Low l correlates with low elementary polygonal circuit area and high compactness, and vice versa.

Finally, it is important to point out the significance of Equation 9 in terms of the space of all possible networks. Equation 9 represents a set of points through the space of all possible networks and it identifies those networks with a given set of coordinates in the space ($area(n, p)$, l) that are potentially realizable in Euclidean 3-D space as actual networks. One could propose a network with a value of (n, p), its associated Schlaefli symbol, and use Equation 9 to calculate the associated value of $area(n, p)$ and the topological form index, l, and locate that point in the space represented by the graph of Equation 9. If in fact such a point doesn't fall on the curve given by Equation 9, then the proposed network will not be able to be realized in practice in crystallography. Therefore Equation 9 represents all the potential crystal structures that may be realized in model building or in actual crystallography.

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6. Dedication

This manuscript is dedicated to Joseph Bucknum late of Buckingham, PA. Joseph Bucknum passed away on Christmas Eve, 2001. He was a veteran of World War II and was my Uncle and the Godfather of one of us (M.J.B). In my childhood I remember how friendly and full of humor he was as my father Walter Bucknum would take the family from Holland to Yardley to visit with Uncle Joe and Aunt Betty. Aunt Betty is my Godmother. God Bless Joseph Bucknum as he lives out his years in Heaven.

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