

On the Spectral Radius of Bicyclic Graphs*

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Abstract

Let $\mathcal{B}(n, m)$ be the set of all bicyclic graphs on n vertices with a maximum matching of cardinality m ($m \geq 2$). In this paper, we determine the graph with the largest spectral radius among the graphs in $\mathcal{B}(n, m)$, when $m \geq 5$.

1. Introduction

In quantum chemistry the skeletons of certain non-saturated hydrocarbons are represented by graphs. By Hückel molecular orbital (HMO) theory, energy levels of electrons in such a molecule are, in fact, the eigenvalues of the corresponding graph, which are closely connected with the stability of the molecule as well as other chemically relevant facts [7, 17]. Lovász and Pelikán [23], Cvetković and Gutman [9] proposed that the spectral radius of the molecular graph (of a saturated hydrocarbon) be used as a measure of branching of the underlying molecule. This direction of research was eventually further elaborated, with emphasis on acyclic polyenes [19], alkanes [18], and benzenoid hydrocarbons [15, 16, 24]. To our best knowledge, the spectral radius of bicyclic graphs was, so far, not considered in the chemical literature. On the other hand, bicyclic graphs represent important classes of molecules, and their spectral radius was much studied in graph spectral theory (see, e.g., [5, 27]). The evaluation of graph eigenvalues were the topic of numerous papers (see, e.g., [3]–[6], [8]–[11], [13]–[16], [18]–[27]). Here we are concerned with bicyclic graphs.

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In order to describe our results, we need some graph-theoretic notation and terminology. Other undefined notation may refer to [2].

We consider only finite undirected simple graphs. Let $G = (V(G), E(G))$ be a graph with vertex set $V(G)$ and edge set $E(G)$. A graph $G' = (V(G'), E(G'))$ is a subgraph of G (written $G' \subseteq G$) if $V(G') \subseteq V(G)$ and $E(G') \subseteq E(G)$; if $G' \neq G$, G' is called a proper subgraph of G and written as $G' \subset G$; if $V(G') = V(G)$, G' is called a spanning subgraph of G . If $W \subset V(G)$, we denote by $G - W$ the subgraph of G obtained by deleting the vertices of W and the edges incident with them. Similarly, if $E' \subset E(G)$, we denote by $G - E'$ the subgraph of G obtained by deleting the edges of E' . If $W = \{v\}$ and $E' = \{xy\}$, we write $G - v$ and $G - xy$ instead of $G - \{v\}$ and $G - \{xy\}$, respectively. For a vertex of G , we denote by $N_G(v)$ the set of vertices adjacent to v in G and denote $\delta(G) = \min \{|N_G(v)| : v \in V(G)\}$.

Two edges of a graph are said to be independent if they are not adjacent. An m -matching M of G is a set of m mutually independent edges. A vertex v is said to be M -saturated, if some edge of M is incident with v ; otherwise, v is M -unsaturated. If every vertex of G is M -saturated, the matching M is perfect. If G has no matching M' with $|M'| > |M|$, then M is a maximum matching; clearly, every perfect matching is maximum. We call the number of edges in a maximum matching of G the edge-independence number and denote it by $\alpha'(G)$. An M -alternating path in G is a path whose edges are alternately in $E \setminus M$ and M . An M -augmenting path is an M -alternating path whose origin and terminus are M -unsaturated.

We denote by K_n , C_n and P_n the complete graph, the cycle and the path, respectively, each on n vertices, and denote by rG the disjoint union of r copies of the graph G . If a graph G has components G_1, G_2, \dots, G_t , then G is denoted by $\bigcup_{i=1}^t G_i$.

Let $A(G)$ be the adjacency matrix of G , then $\det(\lambda I - A(G))$ is called the characteristic polynomial of G and denoted by $\phi(G; \lambda)$. Since $A(G)$ is real and symmetric, its eigenvalues are real. These eigenvalues of $A(G)$ are independent of the ordering of the vertices of G , so they are also called the eigenvalues of G . The largest eigenvalue of G is called the spectral radius of G and denoted by $\lambda_1(G)$. In particular, if G is connected, $A(G)$ is irreducible and so $\lambda_1(G)$ has multiplicity one and there exists a unique positive unit eigenvector corresponding to $\lambda_1(G)$ by the Perron-Frobenius theory of non-negative matrices.

An important topic in theory of graph spectra is to determine the graphs with maximal or minimal spectral radius in a given class of graphs. Let $\mathcal{H}(n, t)$ be the set of all connected graphs with n vertices and $n + t$ edges, where $-1 \leq t \leq \frac{1}{2} n(n-1) - n$. Then $\mathcal{H}(n, -1)$ is the set of all trees on n vertices. It is well known that the path P_n alone has the smallest spectral radius and the star $K_{1, n-1}$ alone has the largest spectral radius among the trees on n vertices [6, 7, 23].

The maximal spectral radius problem for $\mathcal{H}(n, t)$ has been solved by Brualdi, Solheid and Simić [3, 27, 28] for $0 \leq t \leq 2$ and by Bell [1] for t of the form $\binom{d-1}{2} - 1$ ($5 \leq d \leq n-1$).

In this paper, we consider the maximal spectral radius problem on the connected graphs with the given size of maximum matching, i.e., with the given edge-independence number. We denote

$$\mathcal{H}(n, t, m) = \{G : G \in \mathcal{H}(n, t) \text{ and } \alpha'(G) = m\},$$

where $-1 \leq t \leq \frac{1}{2}n(n-1) - n$. For $\mathcal{H}(n, -1, m)$, Guo and Tan have obtained the following result.

Theorem 1.1 [13]. *Let T_m^* be the tree as shown in Fig. 1. Then for each $T \in \mathcal{H}(n, -1, m)$,*

$$\lambda_1(T) \leq \sqrt{\frac{1}{2} \left(n - m + 1 + \sqrt{(n - m + 1)^2 - 4(n - 2m + 1)} \right)}$$

and equality holds if and only if $T \cong T_m^$.*

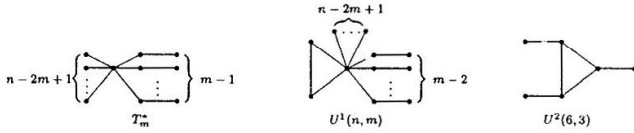


Fig. 1

When $t = 0$, $\mathcal{H}(n, t, m)$ is the set of all unicyclic graphs on n vertices with $\alpha' = m$. For short, we denote it by $\mathcal{U}(n, m)$. On the basis of the work of Chang and Tian [4], we have proved the following result.

Theorem 1.2 [29]. *Among the graphs in $\mathcal{U}(n, m)$, $U^1(n, m)$ has the largest spectral radius, except when $n = 6$ and $m = 3$, where $U^1(n, m)$ is the graph on n vertices obtained from C_3 by attaching $n - 2m + 1$ pendant edges and $m - 2$ paths of length 2 together to one of three vertices of C_3 . When $n = 6$ and $m = 3$, $U^2(6, 3)$ has the largest spectral radius among the graphs in $\mathcal{U}(6, 3)$, where $U^2(6, 3)$ is the graph obtained by attaching three pendant edges to three vertices of C_3 , respectively. ($U^1(n, m)$ and $U^2(6, 3)$ are shown in Fig. 1.)*

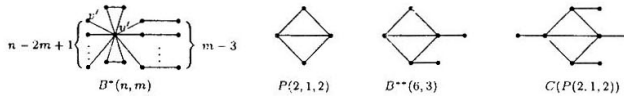


Fig. 2

When $t = 1$, $\mathcal{H}(n, t, m)$ is the set of all bicyclic graphs on n vertices with $\alpha' = m$. For

short, we denote it by $B(n, m)$. Let $B^*(n, m)$, $P(2, 1, 2)$, $B^{**}(6, 3)$ and $C(\phi(2, 1, 2))$ are the graphs shown in Fig. 2. When $n = 2m$, i.e., when the bicyclic graphs considered have a perfect matching, Chang and Tian have proved the following result.

Theorem 1.3 [5]. *Among the graphs in $B(2m, m)$, $B^*(2m, m)$ has maximal spectral radius when $m \geq 5$, and $P(2, 1, 2)$, $B^{**}(6, 3)$ and $C(P(2, 1, 2))$ are the graphs with maximal spectral radius when $m = 2, 3, 4$, respectively.*

In this paper, we show that $B^*(n, m)$ has the largest spectral radius among the graphs in $B(n, m)$, when $m \geq 5$.

2. Preliminaries

Since the spectral radius of G is the largest root of the equation $\phi(G; \lambda) = 0$, we have $\phi(G; \lambda) > 0$ for all $\lambda > \lambda_1(G)$. Then we immediately get the following results.

Lemma 2.1. *Let G_1 and G_2 be two graphs.*

- (1) [7, 11, 22] *If $\phi(G_1; \lambda) < \phi(G_2; \lambda)$ for $\lambda \geq \lambda_1(G_2)$, then $\lambda_1(G_1) > \lambda_1(G_2)$.*
- (2) *If $\phi(G_1; \lambda) < \phi(G_2; \lambda)$ for $\lambda \geq \lambda_1(G_1)$, then $\lambda_1(G_1) > \lambda_1(G_2)$.*

Proof. (2) We prove it by contradiction. It is easy to see that $\phi(G_1; \lambda) \geq 0$ for $\lambda \geq \lambda_1(G_1)$ and $\phi(G_2; \lambda_1(G_2)) = 0$. If $\lambda_1(G_2) \geq \lambda_1(G_1)$, then $\phi(G_1; \lambda_1(G_2)) \geq \phi(G_2; \lambda_1(G_2))$, which contradicts that $\phi(G_1; \lambda) < \phi(G_2; \lambda)$ for $\lambda \geq \lambda_1(G_1)$. Therefore, $\lambda_1(G_1) > \lambda_1(G_2)$. ■

It is well known that if G' is a proper spanning subgraph of a connected graph G , then $\lambda_1(G) > \lambda_1(G')$. Moreover, we have the following results.

Lemma 2.2 [12, 21, 22].

- (1) *Let G be a connected graph and G' a proper spanning subgraph of G . Then*

$$\phi(G'; \lambda) > \phi(G; \lambda) \quad \text{for } \lambda \geq \lambda_1(G).$$

- (2) *Let G' , H' be spanning subgraphs of connected graphs G and H , respectively, and $\lambda_1(G) \geq \lambda_1(H)$, and G' is a proper subgraph of G , then*

$$\phi(G' \cup H'; \lambda) > \phi(G \cup H; \lambda) \quad \text{for } \lambda \geq \lambda_1(G).$$

Lemma 2.3 [7, 26]. *Let v be a vertex of G and $\mathcal{C}(v)$ the set of all cycles containing v . Then the characteristic polynomial of G satisfies*

$$\phi(G; \lambda) = \lambda \phi(G - v; \lambda) - \sum_u \phi(G - \{u, v\}; \lambda) - 2 \sum_{Z \in \mathcal{C}(v)} \phi(G \setminus V(Z); \lambda),$$

where the first summation extends over all vertices adjacent to v .

In particular, when v is a vertex of degree one in the graph G and u is the unique vertex adjacent to v , $\phi(G; \lambda) = \lambda\phi(G - v; \lambda) - \phi(G - \{u, v\}; \lambda)$.

Lemma 2.4 [7, 26]. *Let $e = uv$ be an edge of G and $C(e)$ the set of all cycles containing e . The characteristic polynomial of G satisfies*

$$\phi(G; \lambda) = \phi(G - e; \lambda) - \phi(G - \{u, v\}; \lambda) - 2 \sum_{Z \in C(e)} \phi(G \setminus V(Z); \lambda).$$

Lemma 2.5 [7]. *If G_1, G_2, \dots, G_t are the components of a graph G , we have*

$$\phi(G; \lambda) = \prod_{i=1}^t \phi(G_i; \lambda).$$

Lemma 2.6 [2]. *A matching M in G is a maximum matching if and only if G contains no M -augmenting path.*

Lemma 2.7 [29]. *Let G be a graph in $\mathcal{U}(n, m)$ and $G \not\cong C_n$, where $n > 2m$. Then there are an m -matching M and a pendant vertex v such that M does not saturate v .*

According to the proof of Theorem 1.3, we have the following result.

Lemma 2.8 [5]. *Let G be a graph in $\mathcal{B}(2m, m)$ ($m \geq 5$) and $G \not\cong B^*(2m, m)$. Then $\phi(G; \lambda) > \phi(B^*(2m, m); \lambda)$ for $\lambda \geq \lambda_1(B^*(2m, m))$. Therefore, $\lambda_1(G) < \lambda_1(B^*(2m, m))$.*

3. Main results

A cycle in a graph is said a minimal cycle if no other cycle is contained in it. Each bicyclic graph G in $\mathcal{B}(n, m)$ has exactly two minimal cycles. Furthermore, $\mathcal{B}(n, m)$ consists of the following two types of graphs. One type, denoted by $\mathcal{B}_1(n, m)$, are those graphs whose two minimal cycles have at least one vertex in common. The other type, denoted by $\mathcal{B}_2(n, m)$, are those graphs whose two minimal cycles have no vertex in common. Obviously, $\mathcal{B}(n, m) = \mathcal{B}_1(n, m) \cup \mathcal{B}_2(n, m)$.

In order to prove our main result, we first present two useful lemmas.

Lemma 3.1. *Let G be a graph in $\mathcal{B}(n, m)$ ($n > 2m$, $m \geq 3$) and $\delta(G) = 2$, then there exists a graph G' in $\mathcal{B}(n, m)$ satisfying the following three conditions:*

- (1) $\delta(G') = 1$;
- (2) there are a maximum matching M of G' and a pendant vertex v of G' such that v is M -unsaturated;
- (3) $\phi(G'; \lambda) < \phi(G; \lambda)$ for $\lambda \geq \lambda_1(G')$.

Proof. Let G be a graph in $\mathcal{B}(n, m)$ ($n > 2m$, $m \geq 3$) and $\delta(G) = 2$. Denote by C_{n_1} and C_{n_2} the two minimal cycles of G , respectively. Without loss of generality, we assume

$n_1 \geq n_2$. Obviously, G must belong to one of the following three types of graphs shown in Fig. 3. Furthermore, $n = 2m + 1$. (Otherwise, $n \geq 2m + 2$. Since P_n is a proper spanning subgraph of G , $\alpha'(G) \geq \alpha'(P_n) \geq m + 1$, which contradicts that $G \in \mathcal{B}(n, m)$.) We distinguish the following two cases:

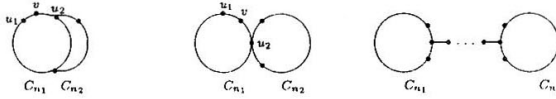


Fig. 3

Case 1. $G \in \mathcal{B}_1(n, m)$.

Then G belongs to one of the first two types of graphs shown in Fig. 3. Since $n = 2m + 1$ and $m \geq 3$, G is neither G_1 nor G_2 (as shown in Fig. 4). Hence $n_1 \geq 4$.

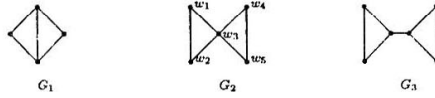


Fig. 4

Take a common vertex u_2 of C_{n_1} and C_{n_2} such that there is a vertex $v \in V(C_{n_1}) \setminus V(C_{n_2})$ adjacent to u_2 . Let u_1 be the vertex of C_{n_1} adjacent to v (as shown in Fig. 3). Since C_{n_1} is a minimal cycle of G , $u_1 u_2 \notin E(G)$. Denote $G' = G + u_1 u_2 - vu_1$. Noting that $n_1 \geq 4$, we have $G' \in \mathcal{B}(n, m)$. Furthermore, G' satisfies (1) and (2). Now we show that G' also satisfies (3).

By Lemma 2.4, we have

$$\begin{aligned} \phi(G; \lambda) &= \phi(G - u_1 v; \lambda) - \phi(G - \{u_1, v\}; \lambda) - 2 \sum_{Z \in \mathcal{C}(u_1 v)} \phi(G - V(Z); \lambda) \\ \phi(G'; \lambda) &= \phi(G' - u_1 u_2; \lambda) - \phi(G' - \{u_1, u_2\}; \lambda) - 2 \sum_{Z \in \mathcal{C}(u_1 u_2)} \phi(G' - V(Z); \lambda). \end{aligned}$$

where $\mathcal{C}(u_1 v)$ is the set of all cycles in G containing $u_1 v$ and $\mathcal{C}(u_1 u_2)$ is the set of all cycles in G' containing $u_1 u_2$. It is easy to see that $G - u_1 v \cong G' - u_1 u_2$ and $G' - \{u_1, u_2\}$ is a proper spanning subgraph of $G - \{u_1, v\}$. So we have

$$\phi(G - u_1 v; \lambda) = \phi(G' - u_1 u_2; \lambda)$$

and

$$\phi(G' - \{u_1, u_2\}; \lambda) > \phi(G - \{u_1, v\}; \lambda) \quad \text{for } \lambda \geq \lambda_1(G - \{u_1, v\}).$$

It is not difficult to see that $\sum_{Z \in \mathcal{C}(u_1, u_2)} \phi(G' - V(Z); \lambda) = \lambda \sum_{Z \in \mathcal{C}(u_1, v)} \phi(G - V(Z); \lambda)$. Since $\lambda_1(G) > 2$,

$$\sum_{Z \in \mathcal{C}(u_1, u_2)} \phi(G' - V(Z); \lambda) > \sum_{Z \in \mathcal{C}(u_1, v)} \phi(G - V(Z); \lambda) \quad \text{for } \lambda \geq \lambda_1(G).$$

Noting that $\lambda_1(G) > \lambda_1(G - \{u_1, v\})$, we have

$$\phi(G'; \lambda) < \phi(G; \lambda) \quad \text{for } \lambda \geq \lambda_1(G).$$

By Lemma 2.1 (1), $\lambda_1(G') > \lambda_1(G)$. Therefore $\phi(G'; \lambda) < \phi(G; \lambda)$ for $\lambda \geq \lambda_1(G')$.

Case 2. $G \in \mathcal{B}_2(n, m)$. (G belongs to the third type of graphs shown in Fig. 3.)

If $n_1 \geq 4$, let u_2 be the vertex of C_{n_1} with degree three, v the vertex of C_{n_1} adjacent to u_2 and u_1 the vertex of C_{n_1} adjacent to v .

If $n_1 = n_2 = 3$, then G is not isomorphic to G_3 (as shown in Fig. 4), since $n = 2m + 1$ and $m \geq 3$. Let u_2 be the vertex of C_{n_1} with degree three, v the vertex of $V(G) \setminus V(C_{n_1})$ adjacent to u_2 , and u_1 the vertex of G adjacent to v .

Denote $G' = G + u_1 u_2 - u_1 v$. Similarly to Case 1, we can show that G' satisfies (1), (2) and (3). ■

Lemma 3.2. *Let G be a graph in $\mathcal{B}(n, m)$ ($n > 2m$) and $\delta(G) = 1$, then there exists a graph G' in $\mathcal{B}(n, m)$ satisfying the following two conditions:*

(1) $G' \cong G$ or $\phi(G'; \lambda) < \phi(G; \lambda)$ for $\lambda \geq \lambda_1(G')$;

(2) there are a maximum matching M of G' and a pendant vertex v of G' such that v is M -unsaturated.

Proof. Let G be a graph in $\mathcal{B}(n, m)$ ($n > 2m$) with $\delta(G) = 1$ and M be an m -matching of G . If there is a pendant vertex v of G such that v is M -unsaturated, the result holds immediately. So we suppose each pendant vertex of G is M -saturated.

Since G is a graph in $\mathcal{B}(n, m)$ ($n > 2m$) and $\delta(G) = 1$, G has a proper connected subgraph H such that H is a bicyclic graph and $\delta(H) = 2$. So G can be seen as a graph obtained by attaching some trees to the vertices of H . If a tree is attached to a vertex u of H , we denote it by T_u and call u the root of the tree T_u or the root-vertex of G . Let w be a vertex of H with $d_H(w) \geq 3$, then w must be a vertex of a minimal cycle. We denote this cycle by C_G . Among two edges in $E(C_G)$ incident with w , there must be one edge belonging to $E(G) \setminus M$. We denote this edge by ww_1 , then $G - ww_1$ is a n -vertex unicyclic graph with an m -matching M , where $n > 2m$. Furthermore, $\alpha'(G - ww_1) = \alpha'(G) = m$. (Since $G - ww_1 \subset G$, $\alpha'(G - ww_1) \leq \alpha'(G) = m$.

Noting that M is also an m -matching of $G - ww_1$, we have $\alpha'(G - ww_1) \geq m$. Therefore, $\alpha'(G - ww_1) = m$.) So $G - ww_1 \in \mathcal{U}(n, m)$ and $G - ww_1 \not\cong C_n$, where $n > 2m$. By Lemma 2.7, there are an m -matching M' of $G - ww_1$ and a pendant vertex v' of $G - ww_1$ such that v' is M' -unsaturated.

If $v' \neq w_1$, then v' is also a pendant vertex of G . Noting that M' is also an m -matching of G , let $G' = G$, then G' and M' satisfy the requirements.

If $v' = w_1$, we distinguish the following two cases:

Case 1. There is a vertex v'' of some tree T_u such that v'' is M' -unsaturated.

If v'' is a pendant vertex of G , then $G' = G$ and M' satisfy the requirements. Otherwise, we can find a maximal M' -alternating path P which starts from v'' and terminates at a pendant vertex v of G . Obviously, v is M' -saturated. (Otherwise, P is an M' -augmenting path of G , by Lemma 2.6, which contradicts $\alpha'(G) = m$.) Then the symmetric difference $M' \triangle P$ is an m -matching M'' of G and v is an M'' -unsaturated pendant vertex of G . So $G' = G$ and M'' satisfy the requirements.

Case 2. Each vertex of T_u is M' -saturated for any root-vertex u of G .

Let w_2 ($w_2 \in V(C_G)$) be the unique vertex of $G - ww_1$ adjacent to w_1 , then w_2 must be M' -saturated. (Otherwise, $M' \cup \{w_1 w_2\}$ is an $(m+1)$ -matching of $G - ww_1$, which contradicts $\alpha'(G - ww_1) = m$.) So we always can find a maximal M' -alternating path $P = w_1 w_2 w_3 \cdots w_{2t} w_{2t+1}$ of $G - ww_1$, obeying the principal as follows: for each i ($1 \leq i \leq t$), if $w_{2i}, w_{2i+1} \in V(H)$ and $N_G(w_{2i+1}) \setminus V(H) \neq \emptyset$, we choose a vertex from $N_G(w_{2i+1}) \setminus V(H)$ as w_{2i+2} . Obviously, w_{2t+1} is M' -saturated. (Otherwise, P is an M' -augmenting path of $G - ww_1$, by Lemma 2.6, which contradicts $\alpha'(G - ww_1) = m$.)

If w_{2t+1} is a pendant vertex of G , let $M'' = M' \triangle P$ and $G' = G$. Then G' and M'' satisfy the requirements. Otherwise, P is a spanning subgraph of H . Then w_1 is the unique M' -unsaturated vertex of G and w_{2i+1} ($0 \leq i \leq t$) is not the root-vertex of G . Then, except when $H = G_2$ (see Fig. 4), similar to the proof of Lemma 3.1, we can choose an appropriate vertex w_{2j+1} ($0 \leq j \leq t$) and get a graph $G' = G - w_{2j+1} w_{2j+1}^+ + w_{2j+1}^+ w_{2j+1}^-$, such that $G' \in \mathcal{B}(n, m)$ ($n > 2m$) and G' satisfies conditions (1) and (2), where w_{2j+1}^+ and w_{2j+1}^- are two vertices of H adjacent to w_{2j+1} . If $H = G_2$, let $G' = G$ and then it is very easy to get an m -matching M'' and an M'' -unsaturated pendant vertex. This completes the proof of Lemma 3.2. ■

Now we show our main result.

Theorem 3.3. *Let G be a graph in $\mathcal{B}(n, m)$ ($m \geq 5$). Then $\lambda_1(G) \leq \lambda_1(B^*(n, m))$ and the*

equality holds if and only if $G \cong B^*(n, m)$, where $\lambda_1(B^*(n, m))$ is the largest root of the equation $\lambda^4 - (n - m + 3)\lambda^2 - 4\lambda + n - 2m + 1 = 0$.

Proof. By Lemma 2.3, it is not difficult to get that the characteristic polynomial of $B^*(n, m)$ is $\phi(B^*(n, m); \lambda) = \lambda^{n-2m}(\lambda^2 - 1)^{n-2}[\lambda^4 - (n - m + 3)\lambda^2 - 4\lambda + n - 2m + 1]$. Since $B^*(n, m)$ has a subgraph C_3 , $\lambda_1(B^*(n, m)) > 2$. So $\lambda_1(B^*(n, m))$ is the largest root of the equation $\lambda^4 - (n - m + 3)\lambda^2 - 4\lambda + n - 2m + 1 = 0$.

Let G be a graph in $B(n, m)$ ($m \geq 5$) and $G \not\cong B^*(n, m)$. By Lemma 2.1, it is sufficient to prove $\phi(G; \lambda) > \phi(B^*(n, m); \lambda)$ for $\lambda \geq \lambda_1(B^*(n, m))$. We prove it by induction on n . When $n = 2m$, the result holds by Lemma 2.8. Now we suppose $n > 2m$ and the result holds for all the graphs in $B(n - 1, m)$ which are not isomorphic to $B^*(n - 1, m)$. By Lemmas 3.1 and 3.2, we have a graph G' in $B(n, m)$ satisfying the following two conditions:

- (1) $G' \cong G$ or $\phi(G'; \lambda) < \phi(G; \lambda)$ for $\lambda \geq \lambda_1(G')$;
- (2) there are a maximum matching M of G' and a pendant vertex v of G' such that v is M -unsaturated.

If $G' \cong B^*(n, m)$, the result holds immediately. So we suppose $G' \not\cong B^*(n, m)$. Let u be the vertex of G' adjacent to v . Let $v'u'$ be a pendant edge of $B^*(n, m)$ attached to C_3 (see Fig. 2).

By Lemma 2.3, we have

$$\begin{aligned}\phi(G'; \lambda) &= \lambda\phi(G' - v) - \phi(G' - \{u, v\}; \lambda) \\ \phi(B^*(n, m); \lambda) &= \lambda\phi(B^*(n, m) - v'; \lambda) - \phi(B^*(n, m) - \{v', u'\}; \lambda).\end{aligned}$$

It is easy to see that $G' - v \in B(n - 1, m)$ and $B^*(n, m) - v' \cong B^*(n - 1, m)$. By the induction hypothesis,

$$\phi(G' - v; \lambda) \geq \phi(B^*(n, m) - v'; \lambda) \text{ for } \lambda \geq \lambda_1(B^*(n, m) - v').$$

Since $B^*(n, m) - \{v', u'\} \cong (m - 1)K_2 \cup (n - 2m)K_1$, $G' \not\cong B^*(n, m)$ and that $G' - \{u, v\}$ has an $(m - 1)$ -matching, $B^*(n, m) - \{v', u'\}$ is a proper spanning subgraph of $G' - \{u, v\}$. By Lemma 2.2, we have

$$\phi(G' - \{u, v\}; \lambda) < \phi(B^*(n, m) - \{v', u'\}; \lambda) \text{ for } \lambda \geq \lambda_1(G' - \{u, v\}).$$

So $\phi(G'; \lambda) > \phi(B^*(n, m); \lambda)$ for $\lambda \geq \lambda_1(B^*(n, m))$, since $\lambda_1(B^*(n, m)) > \lambda_1(B^*(n, m) - v') \geq \lambda_1(G' - v) > \lambda_1(G' - \{u, v\})$. Therefore,

$$\phi(G; \lambda) > \phi(B^*(n, m); \lambda) \text{ for } \lambda \geq \lambda_1(B^*(n, m)).$$

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