MATCH
Communications in Mathematical and in Computer Chemistry

no. 51, April 2004
MATCDY (51) 213-240 (2004)

ISSN 0340 - 6253

The Number Two and the Intriguing Dualism of Nature. Scientific Curiosity and Everyday Concepts

Lionello Pogliani,* Douglas J. Klein,‡ and Alexandru T. Balaban¥

* Dipartimento di Chimica, Università della Calabria, 87030 Rende (CS), Italy, †
lionp@unical.it

‡Texas A & M University at Galveston, Galveston, Texas, 77553-1675, USA,
kleind@tamug.tamu.edu

¥ Texas A & M University at Galveston, Galveston, Texas, 77551, USA,
balabana@tamug.tamu.edu

(Received November 7, 2003)

Abstract. The concept of duality together with the meanings, origins, and uses of number two, are thoroughly discussed, in an attempt to show how everyday concepts have much broader implications, and can be the source of an unlimited curiosity. After an introduction about the general characteristics and history of the number two and related 'twoness', some dualistic systems, which originated with the dawn of the civilization all over the world, are examined. The relevance of number two is examined in some scientific and non-scientific fields. Starting with the specificity of the number two in mathematics, the discussion examines the specific importance of some binary realities in physics, chemistry, and biology, including a paragraph on the symmetry-asymmetry 'twoness', and ends up with a discussion on less scientific dualisms. The binary theme becomes, thus, the guiding theme of a reality, which seems to justify the suspect that the universe is in fact a *biverse*.

1. Introduction

It is well-known that everything can be divide by two. [1] Long before mechanical computers were invented, the Egyptian multiplied by doubling, as many times as necessary, adding the results. To multiply by $11=2\cdot2\cdot2+2+1$, the Egyptians doubled thrice, added twice and once, it was essentially a binary multiplication $(2^3+2^1+2^0)$. They somehow sensed that every number can be expressed as the sum of powers of two, and practically they were very close to unveil a binary number system.

The first proved 'twoness' in mathematics can be ascribed to Thales of Miles (c. 600-500 BC) who is credited to have given the first proof of the mathematical theorem that a diameter divides a circle into two equal halves [2, 3]. Nearly in the same period one of the fathers of atomism (Leucippus was the other c. 450 BC) Democritus of Abdera (c. 460 – c. 370 BC) asserted the first physical model of nature reducing the variety of the universe into a 'twoness' with the famous statement reported by the Latin poet Lucretius (96-55 BC) in his scientific poem, 'De Rerum Natura': *'apparently there is color, apparently sweetness, apparently bitterness, actually there are only atoms and void'*. But also the unknown writer of the bible tried to put forward a model of nature based on a 'twoness' when in the Genesis 1:2 he says: ".... the earth was without form and void....". But the Bible goes further and tells us that it was from nothing that God created the Cosmos, stating, thus, again a 'twoness': God and nothing.

Very few people know that number two, together with number one, were not considered numbers by old Ages mathematicians and philosophers, while zero was for a long time ignored, especially in Europe. [4-9] Thus, while Aristotle (384-322 BC) considered two the smallest number, the Greek mathematician Nichomachus (c. 100) regarded numbers one and two as generators of the number system: number one assuming the masculine role and number two the feminine role. The number system started, practically, with number three. Now, number two did not have such a precarious and adventurous life like one or zero, nevertheless it occupies a key position in the history of human thought.

Regarding the cited proof of Thales about the possibility of halving the circle, it is intriguing to note that one of the oldest esoteric symbols of humanity, the symbol of Ouroboros [5, 10, 11], the self-consuming snake, the great world serpent encircling the universe, which symbolizes concepts such as perfection, eternity, the unending cycle of nature, soon developed from the form of a single self-biting snake into the form of two snakes biting each other's tail (Figure 1). Has this evolution of an esoteric symbol something to do with the fundamental proof of Thales ? Are they both consequences of a perception of the dualistic character of nature ?

Why did the first mathematical proof concern the circle ? As it was and it is the form of the sun and moon, of the orbits of the stars (around the celestial pole), it symbolized the sky, the divinity. It is something that has not beginning and no ending, no orientation, and no direction. Drawn with a center it symbolized the sun, and if radii are added to it became the symbol of a wheel, a dynamic symbol. Stonehenge in England has the form of a circle. In Chinese symbolism the circle circumscribes the ying-yang dualism, which is inscribed in a circle. The old problem of squaring a circle with geometrical means has underlying it the desire of encompassing the universe (the circle) in human dimensions. In fact, the square is a terrestrial, material and human symbol. The halving of the circle by Thales can be seen as the first attempt to humanize the divinity (square the circle). The attempt to square the circle was fated to failure, but, in the meantime, and also thanks to these kind of attempts, the impressive development of Greek mathematics grew up.

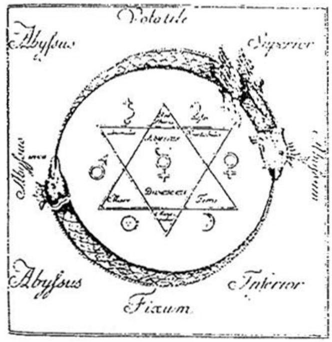

Figure 1. Left: an Aztec (?) Ouroboros; Right: a XVIII century Ouroboros by A.J. Kirchweger, Annulus Platonis (Aurea Catena Homeris) 1781. [11]

We could also wonder how Leucippus and Democritus [a] arrived at their original and rather modern model of nature. Where did the atoms come from ? The answer seems evident if we just know the history of numbers and the importance that the teachings of Pythagoras (c. 582 – c. 496) on the subject had assumed during old Greek times. Democritean atoms can be regarded as the physical counterpart of Pythagorean numbers. And the void ? The void is surely one of the oldest problems and questions of humanity, it stands, in fact, at the beginning of creation in the bible, and it traversed the entire early Greek philosophy till Aristotle's authority banished it from physics. Greek mathematics is, thus, credited, by many scholars, to have infanted philosophy and most of modern religion [12, 13].

2. Some Dualistic Systems

Speaking about two it is impossible not to mention the impressive construction of the so-called 'dualistic systems'. They are symbolic structures behind which a bipolar order hides, and that derive their meaning from a dynamic between two components, which, if taken alone would lose any importance. It should be noticed that the two components can in many cases be complementary not just antithetic. Can you imagine an 'up' without a 'down' or an 'open' without a 'closed' ? A dualistic view can be traced back to the ancient cave drawings of the glacial era, where pictures of wild horses and bulls are the two main motifs. It has been suggested that these two species of animals made up a dualistic system. [10]. Notice that horses (and bulls or cows) have been tamed only some thousand years after the glacial era in eastern Europe or central Asia, and that eastern hordes mounting horses terrorized the Mediterranean folk for a while. The centaurs of Greek mythology were not that mythological. The couples that are at the origin of such bipolarism are endless, and embrace nearly every field of human culture, and must have surely impressed the early humans: BC (before Christ) / AD (after Christ), PM / AM (in English language), night / day, man / woman, life / dead, animals / people, sky / earth, God / demon, good / evil, body / soul, mind / body, odd / even, real / ideal, matter / spirit, cause / effect, order / chaos, master / slave, right / left, up / down, fault / innocence, Sun / Moon, sulphur / mercurius (in alchemic systems), dry / wet, ying / yang (in old and present China), entrance / exit.

Based on the entrance / exit or forward / backward dualism, whose material representation is a door, which is always two-faced, the Romans had created their only original God, the two-faced Janus. They believed that not only doors, which were protected by this God, but even every action started and ended with a sort of way-in ↔ way-out process, like entering and exiting through a door. There is surely a connection between the two-faced Janus and the fact that Romans had two consuls, and further, that they chose as the first month of the year, the way-in of the year, *Januarius* (January). But not from the very beginning, as it should not be forgotten that before 154 BC the Roman calendar started with March 1, on which day the two new consuls were inaugurated. In 154 BC a rebellion broke out in Spain at the end of the year 154, and to avoid a change of command on March 1, 153 BC, the year was allowed to last only ten months and the new year 153 BC was begun on January 1, and has remained so until the present day. With this shift the previous numerical names of the months became false and lost their meaning. October (the 8^{th} month) should now have been December (the 10^{th}) and others to correspond.

Faith in a world or universe grounded on a dualistic system seems one of the oldest *archetypes* of humanity, common to many different cultures of Earth. The intriguing evolution of the symbol of Ouroboros from a single to a double snake representation underlines the strength of the 'dualistic way of thinking'. The origin of this 'dualistic thinking' is not evident, but it can be assumed that from mankind's origin the experience of an ego and an external world created a cultural shock leading to a bipolar vision of the human experience. It remains to check the role that sex division played in such a bipolar cultural vision, even if some authors believe that it played a minor role. [10] Another primeval 'twoness' of mankind was surely the one's self facing the other self, in its individuality, as with Descartes (1596-1650) "I think, therefore I am", where conscious thought divides one from the whole.

Already the Greek mathematicians discovered the importance of the *true / false* dualism in proving theorems, in fact, one of their main tools was the 'proof per absurdum', where a certain proposition of a theorem is temporarily assumed to be false in order to show that such an assumption brings nonsense, thereon proving that the starting proposition is instead true.

Let us now spend some words about the Chinese ying-yang complementary dualism, which led to a very strange mathematical enigma that is presented in the next section. This dualism is an old Chinese representation of a cosmic dualistic system, where *ying* symbolizes the feminine, the north, the cold, the shade, the earth, the passivity, the wet, while *yang* symbolizes the sky, the south, the light, the activity, the dryness, and, further, the emperor. Strange enough, even if the feminine element had some negative characters, the ying-yang dualism was never expressed as yang-ying. Actually, the two elements were on an equal footing, and they were not seen as opposing principles but complementary to each other. Their symbolic representation is based on the circle, which symbolizes the original unity, from which the bipolar ying-yang emerged (see Figure 2).

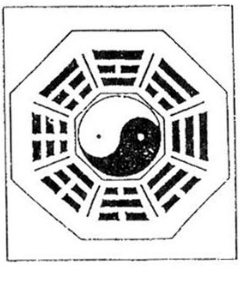

Figure 2. The Chinese ying-yang

The father of the ying-yang seems to have been the Chinese philosopher, Chu Hsi (1130-1200). The Symbolic division of the two poles is represented by a partition of the circle into two S-shaped halves around the common center, each with its center (one center gives rise to two centers). The dark side is the ying, and the white side the yang. From this polarity are born the five elements, the eight trigrams, and from the variety, the ten thousand things, as ten thousand was synonym of uncountable.

3. Numbers and number 2

In medieval schools and convents numbers were highly worshipped if we credit the words of the German monk, Hrabanus Maurus (776-869), considered at his time a specialist in this field: '*in many holy writings numbers cover secrets that should remain mysterious to everybody who ignores numbers, and it is for this reason that everybody who wants to reach the highest form of knowledge has to master arithmetics*'. [10] These words remind strongly what stood written at the entrance of Plato's Academy: '*let no one ignorant of geometry enter my door*'. For the Pythagoreans (around VI –V BC) numbers were the key to understanding the universe, like the modern physical laws, and for this reason they ended assuming a metaphysical and religious meaning among the same Pythagoreans, who believed that numbers are 'divine archetypes' existing in the universe to obey a pre-established plane and to follow the laws of the harmony. Numbers became, thus with Pythagoras and his school, the unity for every measurement and '*the main bonding that controls the essence of everything*' (Philolaus of Croton, 500 - 400 BC). Later, Plato (428 – 347 BC) substituted numbers with the more general concept of idea. It is not difficult to trace the birth of Plato's world of ideas from numbers from Plato's commentary: '*arithmetic has a very great and elevating effect compelling the soul to reason about abstract numbers, and rebelling against the introduction of visible or tangible objects..*' (Plato, Republic, VII, 525, see also [12]).

The Pythagoreans soon uncovered the unpleasant truth that numbers were not perfect and hid a stack 'twoness'. Hippasus of Metapontus, a Pythagorean (c. 582 – c. 496 BC) made the terrible discovery of the existence of another kind of numbers, the irrationals. The irrationals cannot be measured by the help of the rationals, [b] and thus the concept of number as a simple unified whole was lost forever, only shortly after the discovery of their importance in universe's picture. The commensurability [c] problem became thus a dual insoluble problem simply by the aid of the rather trivial example of the diagonal of a square that cannot be measured with the same rational units which measure its sides. Greeks further discovered that square roots from 3 up to 17 (excepting 4, 9, and 16) are irrationals.

Two means diversity as opposed from identity, and divides unity from plurality, which in many old cultures started with three. [3-5, 7, 8] In common language this is recognized when the antonyms for "coupled", "double", and "paired" are quoted as "single" and "singular" (along with separate). It is not at all odd that some monotheistic religions ended up in some way allowing forms of polytheism: the fascination for diversity seems to suffer no limits. Two is manifested in a very deep way in several fundamental *dualities* or *complementarities*, and in such dualities the two components are often not exactly logical opposites of one another, but rather are two fundamental characteristics which are both necessary in forming a common whole, not unlike say the front and back of a coin or of a door.

The first step beyond the One goes probably back to the awakening of conscience of the I as opposed to and distinct from what is not I, the thou, the other, and, linguistically it is not at all unlikely that the Indo-European number word *duuo* is in some way related to the German *du*, to the Latin *tu*, and the English *thou*. In the Sumerian number sequence 'one' (as) and 'two' (min) have the meaning 'man' and 'woman', respectively. [8] To man the Two is first another man, I and you, and if the essence of number is to bind many together into one, then in the Two we experience the very essence of number. Thus, our mind in every duality senses a strong unity, as has been powerfully exemplified in the Hegelian dialectics (thesis, anti-thesis, and syn-thesis). Many languages express their number 2 in the grammatical dual. Thus the Indo-European *duuo* and *ambho* gave rise to the Greek *dýo* and *ámpho*, and to the Latin *duo* and *ambo*. In this case the *duo* expresses not so much the abstract number word Two as the inclusive Both, which excludes counting beyond, whereas Two implies it. In Irish the number word Two is *da*; in Gothic: *twai*, *twos*, *twa*; in Indian: *dvi*, *dve*; in Lithuanian: *du*, *dvi*; in Dutch: *twee*; in Swedish: *tva*; in Russian: *dva*, *dve* ; in French, Italian, Portuguese, and Spanish: *deux*, *due*, *dois*, *dos*, respectively, in Rumanian: *doi*, *doua*. In India the custom arose of expressing numbers by symbols and it is said that Pythagoras brought this symbolism from a journey of his in the far East, thus 1 is 'moon', and 2 is 'eyes' or 'arms'. Collecting all the symbols that have been imagined for 1 and 2 it seems that there are more than three hundred different symbols for these two numbers, among which even the 'rhinoceros'. Well before those early times when the number system developed, there was, surely, a time when people did not know how to count. [7] It seems that in such times the number concept was a concrete reality inseparable from objects and that it was manifested only in direct perception of physical plurality. Being unable to conceive numbers by themselves, as abstractions, one had nevertheless a rudimentary number sense similar to that possessed by birds or dogs, which seem to be able 'to

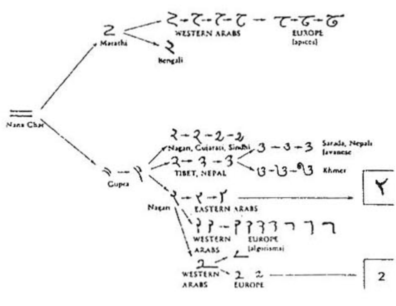

Figure 3. Graphical evolution of number 2.

count' till four. In fact, early in this century there were still peoples in Africa, Oceania, and America, who could not clearly perceive or express numbers greater than four. The Aranda tribe in Australia had only two basic number words: *ninta* (one) and *tara* (two). For three and four they said *tara-ma-ninta* (two-and-one), and *tara-ma-tara* (two-and-two), and beyond this last number they used a word meaning many. [8] In Figure 3 are collected the origins and some evolutions of the numeral Two.

4. Two in Mathematics

The most interesting aspect of the number two is that it is the smallest prime number, and the first even one (for our general purposes we normally consider positive two). Already the Greeks pointed out that $2+2 = 2\cdot2$, while $1\cdot1<1+1$ and $3\cdot3>3+3$, and that any number multiplied by two is equal to the same number added to itself. Since they expected multiplication to do more than summation, as for number 3, they insisted in considering 2 and 1 quite strange sorts of numbers. Division into two parts, dichotomy, is a most used classification. In all European languages the words for 3 and 1/3, 4 and 1/4, 5 and 1/5, ..., are related. There is evidently no connection between the Germanic words two and half, there is none in the Romance languages (Italian: 'due' and 'metà'), or in Slavic languages (Russian: 'dva' and 'pol'), and in Hungarian, which is not an Indo-European language, the words are 'bettö' and 'fil'. Powers of two appear more frequently in mathematics than those of any other number, and also in physics powers of two are fundamental in many laws. An intriguing result with two as power and as basis is the following: $2^5 9^2 = 2592$. The geometrical origin of this power, as well as the powers of three is underlined by the fact that r^2 is r square, r^3 is r cube, but r^4, r^5,...., are not geometrically related. A fascinating two was discovered by L. Euler (1707-1783) in simple polyhedra, where the number of vertices (v) plus the number of faces (f) exceeds the number of edges (e) by two: $v + f = e + 2$. A formally similar law has been discovered by J.W. Gibbs (1839-1903), nearly a century later, to be valid in chemical thermodynamics, where the variance of the system (V) plus the number of phases (F) exceeds the number of components (C) by two: $V + F = C + 2$. These are surely the most intriguing two laws involving the number two in the history of science. In the next section we will see how this strange similarity between mathematics and thermodynamics can be represented by the aid of dual graphs. Near the rational-irrational number dualism, two other number dualisms can be noted, and they are the *odd-even* dualism and the *prime-non prime* dualism. Prime numbers show another dualism, which up to day is still unresolved: the *twin primes*, i.e,. two prime numbers that differ by two (3-5, 5-7, 11-13, 17-19, ...). It has not still been solved the problem of whether they are infinite in number, nevertheless an interesting result on twin primes seems to suggest that they are finite in number. It is well-known that $\sum(1/p)$ diverges if *p* runs over all primes. In 1921 it has been proved that $\sum(1/p)$ converges if *p* runs over the twin primes only. This seems to indicate enough difference between the density of the primes and the density of the twin primes, a fact that might suggest that the number of twins is finite. The problem stays, nevertheless, unsolved. [14] Strictly related to the concept of prime numbers is the famous Goldbach (1690-1764) conjecture that every even number greater than two is the sum of two prime numbers. This conjecture that has never been proved has never failed. Two, instead, has been proved to be the only number that allow solutions in positive integers of Fermat's (1601-1665) "last" theorem: $x^n + y^n = z^n$. The solution for $n = 2$ with $x = 3$, $y = 4$, and $z = 5$, is evident. The solutions for $n = 2$ are then the sides of a right-angled Pythagorean triangle. Some mathematicians had noticed that $n^x + n^y = n^z$, has the only solution in positive integers when $n = 2$. [9] This last finding can be extended in an interesting way to the Golden ratio or divine proportion, $\phi = (\sqrt{5} + 1)/2 = 1.61803...$, i.e., $\phi^0 + \phi^1 = \phi^2 = 1 + \phi$, where $n = \phi$, and $x = 0$, $y = 1$, and $z = 2$. Surely, the most interesting development in modern number systems, for the importance that it has acquired in our era of personal computers, is the binary number system, based on 0 and 1 digits only, and that can be coded as 'off' and 'on' signals in electronics. The binary system seems to have first been devised by the German philosopher and mathematician G.W. Leibnitz (1646-1716) in connection with metaphysical purposes. Actually, about the real originator of the binary number system there is an enigma, more precisely a Chinese enigma originated around 1200-1100 BC. This enigma is centered around the strange ying-yang dualism. [3] Ying and yang were also drawn by Chinese philosophers as '- -' and '—' respectively. From these two figures they derived the eight trigrams or eight permutations of the two figures taken three at time, repetitions being allowed, and it is what we can see in Figure 2, where the eight trigrams are disposed around the central circle encompassing the ying-yang dualism. Although there is no historical evidence that the Chinese looked upon the eight trigrams, as built upon base two, it is true that if we take '-' for one and '- -' for zero (vice-versa also) the successive trigrams of Figure 3, beginning at the bottom and going around anti-clockwise (the other way round also), till the third element and then going back to the bottom left and up again, this time clockwise (building an horizontal S complementary to the vertical S of the ying-yang), have values which may represent our numerals as, 000, 001, 010, 011, 100, 101, 110, and 111. If these are considered as numbers written in base two, their respective values are, 0, 1, 2, 3, 4, 5, 6, and 7. Notice also the inverted symmetry of two trigrams that are 180° from each other. The problem in proving this enigma is that trigrams have normally been used from a very early period and until present days for divination purposes and to them were assigned various virtues. Further, it seems very improbable that Chinese were conscious of the mathematical meaning of zero, whose first appearance as a mathematical symbol is to be found in India around V-VI century of our era. Even further, the ying-yang symbols resemble more 1 and 2 symbols, i.e., what ancients believed to be the originators of the whole numbers system. Thus, the eight trigrams seem to constitute either a strange coincidence or an unconscious discovery (or both). Amusingly the golden ratio ϕ makes a related appearence, as a base-ϕ number system consisting of sequences of 0 and 1 such that no two successive digits are both 1. That is, counting proceeds: 1, 10, 100, 101, 1000, 1001, 1010, ... Notably the sequence of 1 followed by n 0s is the n^{th} Fibonacci number, and the lenght, l, of symbols represents a number that asymptotically grows as $\sim\phi^{l}$. This number system turns out to have algorithmic advantages.

Greek geometric constructions were practically based on 'binary tools', i.e., they could only be performed with compass and straight-edge. It was the attempt to solve with this strict 'binary code' the problem of the (i) duplication of a cube, (ii) the squaring of a circle, and (iii) trisection of an angle, that brought Greek geometry into troubles, even if it allowed some interesting discoveries, like the conic sections, the conchoid, and the cissoid curves. Similar troubles have been caused in logic by the true-false dualism. It is to be noticed that to solve the problem of the duplication of a cube (solved by Archytas of Tarentum, c. 400 BC) the cubic root of two should be derived. But before speaking of this last dualism let us notice that these geometrical 'binary tools' were shown to be redundant in 1797, when an Italian poet and mathematician (Lorenzo Mascheroni, 1750-1800) demonstrated all that Euclidean geometric constructions needed only a compass. [2, 3, 4-6] The squaring the circle problem had a counterpart in the newly born Logic. As soon as Greek logicians, especially Aristotle, discovered the laws of logic, a series of unsoluble dilemmas started to plague it. The most famous is surely the paradox of the person from Crete asserting that "Cretans always lie". A modern version of this dilemma is given by the story of the master soothsayer who in examining his student asks him if he can guess the result of the examination, and the candidate answers that he will not get through the examination, bringing an enigma for his master. Di-lemmas continued to plague many logico-mathematical constructions till the XX^{th} century. The true & false dilemma was based on the Aristotelean law of the excluded middle "*tertium non datur*" precluding other possibilities, but modern developments recognize paradoxical problems, especially with self-referential statements, as with "this statement is false". There is in fact a quite active area of multivalued logic, which is even argued by some to be relevant to quantum mechanics. On their hand the irrationals (non-repeating decimals infinite in number), gave rise in modern mathematics to a new dualism, as soon it was noticed that there are two types of irrational numbers: *algebraic irrational numbers*, that are roots of polynomial equations with rational coefficients, i.e, $\sqrt{5}$ = 5.2360...., which is a root of $x^2 - 5 = 0$, and *transcendental irrational numbers* that are not a root of any polynomial equations with rational coefficients. Such transcendental numbers include π and e, and their total number is a higher infinity then that of the rational or the algebraic irrationals. In coincidence with irrational numbers some interesting results should be mentioned: $2^{\sqrt{2}} = (\sqrt{2}^{\sqrt{2}})^2$. The irrational series $\sqrt{2}$ = 1.4142.., $\sqrt{2+\sqrt{2}}$ = 1.8478.., $\sqrt{2+\sqrt{2+\sqrt{2}}}$ = 1.9616.., $\sqrt{2+\sqrt{2+\sqrt{2+\sqrt{2}}}}$ = 1.9904..., is seen to converge towards 2 fairly rapid.

The 'binary theme' is central also in boolean algebras (G. Boole, 1815-1864), which are applied in logic design, in switching theory, and in computer science. These algebras consist of a set of elements S together with two binary operations , denoted by '·' (the Boolean product) and '+' (the Boolean sum) obeying certain axioms, and where 0 is the identity element for sums, and 1 for products. There are two common examples of Boolean algebras : (i) the algebra of classes in which '+' is union of sets ('∪'), '·' is intersection ('∩'), 0 is the null set, and 1 is the universal set; (ii) the algebra of propositions in which '·' is 'and' (&) '+' is 'or' (∨) and 0 and 1 are truth values. There are many identities in Boolean algebra, i.e, $x + x = x$, $x{\cdot}x = x$ (idempotent laws); $x + 1= x$, $x{\cdot}0 = 0$ (dominance laws), $x + 0 = x$, $x{\cdot}1 = x$ (identity laws)...., normally these identities come in pairs. To explain the relationship between the two identities in each pair the concept of *dual* is used: the *dual* of a Boolean expression is obtained by interchanging Boolean sums and Boolean products and interchanging 0s and 1s.

One should not forget some well-known dual operations like: addition / subtraction, multiplication / division (of non-zero numbers), integration / differentiation, trigonometric / inverse trigonometric functions ($y = \tan x$ / $x = \tan^{-1}y$), where each operation is the inverse of the other. As we just introduced the concept of operation and function and their inverse, it is not at all odd to say a few words about the idea of mapping, which is a basic dualistic concept that establishes a correspondence between two sets of variables. A mapping of a set A onto a set B is defined by a rule or operation which assigns to every element of A a definite element of B (which need not be different than A). It is commonplace to refer to mappings also as transformations or functions, and to denote a mapping f of A onto B by: $f : A \rightarrow B$. If x is an element of the set A, the element of B which is assigned to x by the mapping f is denoted by $f(x)$

and is called the image of x. A special mapping is the identity mapping: $f: A \rightarrow A$, such that $x = f(x)$. The identity mapping is usually denoted by I. Given a one-to-one mapping f an inverse mapping f^{-1} can always be defined, which undoes the work of f. Thus, if $y = f(x)$, then $x = f^{-1}(y)$ and $y = f[f^{-1}(y)] = ff^{-1}(y) = I(y)$. Let, $y = f(x) = e^x$, then the inverse mapping f^{-1} is clearly that mapping which assigns to each element its logarithm (to base e) since $\ln e^x = x$ and $e^{\ln x} = x$.

A dualism is also at the core of topology where each *graph* has its dual, the *dual graph*, already cited in correspondence with Euler's and Gibbs' laws. A graph G = (V, E) consists of V, a nonempty set of vertices, and E, a set of pairs of distinct elements of V called edges. When multiple edges and loops are present, the concept of pseudograph is used. A *dual* graph is something that has to do with a map in the plane. To set up a correspondence between a graph and a map, each region of the map is represented by a vertex. Edges connect two vertices if the regions represented by these vertices have a common border (a common point is not sufficient). The resulting graph is called the *dual* graph of the map. In Figure 4 is an example of a map, a phase diagram, and its dual. Graph theory is more and more used in chemistry, as organic compounds can easily be represented by graphs. i.e., by two sets of objects: a set of vertices and a set of edges. [15-19]

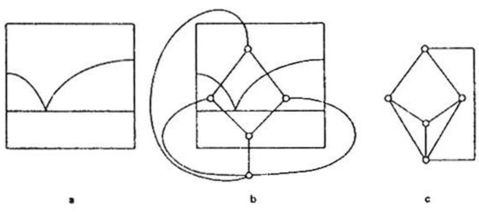

Figure 4. A phase diagram (a), together with its dual graph (b), and the dual graph alone (c).

The duals of Plato's regular polyhedra are the same Plato's regular polyhedra: tetrahedron is self-dual, the cube and octahedron are a dual couple, like the dodecahedron and the icosahedron.

The binary theme encompasses the so-called amicable numbers. In amicable number pairs each number is the sum of the aliquot parts of the other. The first pair of amicable number pair, 220, and the result of its aliquots , 1, 2 , 4, 5, 10, 11, 20, 22, 44, 55, and 110, i.e., 284 whose aliquots are 1, 2, 4, 71, and 142 (totally 220) was known by Pythagoras. The second pair of amicable numbers, 1184 and 1210, has a curious history: it was discovered in 1866 by an Italian 16-year-old-schoolboy, Niccolò Paganini (do not confound with the homonym Italian componist and violinist), having been missed by Descartes, Fermat, Euler, Gauss, among others. [9]

In mathematical axiom systems for numbers the method of distinction between 1 & 2 extends to all the additional natural numbers. For instance Peano's (1858-1932) postulates for the natural numbers $\mathbb{N}$ may be stated:

(i) There is a natural number 1 $\mathbb{N}$.

(ii) For every x $\mathbb{N}$ there is a unique element x' $\mathbb{N}$, called the *successor* of x .

(iii) 1 is not a successor.

(iv) If x'=y', then x=y .

(v) If $\mathbb{T}$ $\mathbb{N}$ such that 1 $\mathbb{T}$ and also t' $\mathbb{T}$ for all t $\mathbb{T}$, then $\mathbb{T}=\mathbb{N}$.

The distinction between 1 & 2 as it arises in postulate (ii) then leads to all subsequent differentiation between all numbers upon iteration. Indeed this general idea is also the foundation of mathematical induction: if we prescribe (1) and a rule *f* for obtaining (n+1) from (n), then (2)= f((1)), (3) = f((2)), *etc*, so that (n) for all natural numbers n are determined. G. Gamow's book *One, Two, Three ..., Infinity* [20] emphasizes that the extension of a mathematical idea from the unitary case to the binary case frequently needs only to be repeated to go on to any higher-order case, so long as the order is finite. An example is the extension of the idea of a (real) number to the 2-dimensional case of a vector space, where after one is rather readily led to n-dimensional vector spaces, for all n $\mathbb{N}$, and it is only at infinity that one can expect something non-trivially different to happen again. The idea has relevance beyond mathematics.

5. Two in Physics

Time occupies a central position in Physics, and it is the only important dimension that is not bipolar, in that it is unidirectional and in classical descriptions has no natural origin or 0. The uni-directionality of the arrow of time was first proposed by the Greek philosopher Heraclitus, known as the *obscure*, who two-and-a-half millennia ago having observed that the world is in eternal flux, and wrote that one can never step in the same river twice. Thus time is one of the few objects in physics that is not bipolar, even if many physical laws are invariant under a theoretical time inversion. This time inversion seems to be forbidden by a fundamental postulate of thermodynamics, the evergrowing entropy of an isolated system. But even when locally entropy diminishes, time locally does not go back. [21-23]

Modern Physics is 'plagued' by bipolar antithetic or complementary dual objects, like positive / negative charges, wave / particle, matter / anti-matter, fermions / bosons (permutation symmetry / antisymmetry), order / disorder, energy / entropy, coordinate / momentum representations of quantum mechanics, particles / holes (as in quantum mechanics and they seem to be related to the dualities in projective geometries); an energy / anti-energy dualism has been suggested. All these dualisms would strongly suggest that our universe is not at all an '*uni-verse*' but much more a '*bi-verse*'.

Let us start discussing about what has become to be the most important dualism in thermodynamics, in physics (some view thermodynamics as a field independent from physics) and in every field of science: the energy-entropy dualism. Behind this complementary dualism hides the first detected scientific asymmetry of the universe, the asymmetry of the energy: though energy is conserved, a postulate already figured out by Greek philosophers, the form of energy is not. At the middle of the 19^{th} century, W. Thomson, Lord Kelvin (1824-1907) and R.J.E. Clausius (1822-1888) discovered the impossibility of a *perpetuum mobile* of the second kind (impossibility of utilizing continuously thermal energy of the same temperature), i.e., of the evergrowing entropy (Clausius) or of the degradation of energy (Kelvin). This discovery came shortly after the postulation of the conservation of energy and of the impossibility of a *perpetuum mobile* of the first kind (impossibility of creating energy). [24, 25] This discovery seems to reintroduce an old dualism in science, the *matter / form* dualism, i.e., the material constancy of energy and its formal variety. The universe of thermodynamics is subdivided into two parts, i.e., the environment, and the system under examination. The thermodynamic definition of absolute temperature, $(\partial U/\partial S)_V = T$ (v is volume), allows for a new dualism: the positive and the negative temperatures. To obtain negative temperatures all is needed is a decreasing entropy with growing energy, and this can be achieved with what is known as a *population inversion*, that is, an equilibrium (or near-equilibrium) state in which there are more particles in upper energy level than in the lower one.

Everybody during its physics studium was certainly amazed by the strange similarity between Newton's law of gravitation and Coulomb's law for charged particles. Apart from their formal similarity these two laws have in common the inverse dependence on the distance, r^2, rather than $r^{1.9}$ or $r^{2.1}$. Nature seems here to have strange preferences. Now, Newton, in the Book I of his Principia, proved that had the universal law of gravitation been an inverse cubic law, one possible orbit of the planets around the sun would be a logarithmic spiral, and the hyperbolic spiral $r = k / \theta$ would be another possible orbit.

There is no book in physics, chemistry, and even biology that does not speak about the wave-particle dualism in some detail. Here we will only underline some interesting 'twonesses' that originated with this fundamental dualism. The wave/particle dualism is the fundamental duality of quantum mechanics, [26] where the traditional ideas of wave and particle break down in such a way that each type of entity partakes of some aspects of the other - the prototypical manifestation of this duality occurs with a double-slit interference experiment at low intensities, either of photons or of electrons. But associated with this dualism there were soon found other important 'twonesses', e.g., the spins components of Fermions, early recognized by W. Pauli (1900-1958) when he referred to a "twoness" characteristic of electrons. A Fermion is an elementary particle having half-integer spin, which obeys the Fermi-Dirac statistics. All fundamental particles are either Fermions or Bosons (these last follow the Bose-Einstein statistics), a further dualism that was early detected in atomic physics. Underlying the wave/particle one finds Bohrian complementarity, which in quantum theory included complementarity between position and momentum, angle and angular momentum, and time and energy. But, Bohr (1885-1962) imagined that complementarity extended beyond physics to other sciences and to philosophy, so that complementary concepts might also include identity and distinction, life and death, freedom and security, as well as truth and clarity. For complementary pairs Bohr imagined that there is a minimum value for the product of the uncertainties for two complementary quantities. For two complementary quantities, especially position and momentum, a quantitative manifestation is found in Heisenberg's (1901-1976) uncertainty principle. Other dualisms engendered by the fundamental dualism on which has been built the whole construction of quantum mechanics are Dirac's the particle/hole pairs, where the whole universe is imagined to be filled with a sea of negative-energy particles (e.g. electrons), and a hole in this sea results from a negatively-charged electron being excited from its negative-energy state to a positive-energy state. These holes travel acting like positive charges with a mass nearly the same as the mass of the initial particle, and should such an electron hole encounter an electron, the electron can fall into the lower energy state available as a hole, thereby releasing much energy – a result characterized as matter-antimatter annihilation.

6. Two in Chemistry

Number two caused a revolution in chemistry, a revolution that has to do with the simplest molecule, the hydrogen molecule, H_2. [27-29] It was the proposition by S. Cannizzaro (1826-1910) at the First International Chemistry Congress held in Karlsruhe, Germany, in 1860, based on the forgotten molecular theory put forward by A. Avogadro (1776-1856) half a century before, that the molecular theory started to reshape chemistry. Cannizzaro suggested, following a reasoning used by Avogadro in 1811, that a molecule of hydrogen consists of two atoms of hydrogen, and thus weights and molecular formulas of a wide variety of compounds should be rewritten following this new molecular paradigma. Up to then J. Dalton's (1766-1844) atomic theory was not only rather controversial in scientific milieus, but in the explanation of many chemical phenomena it originated a great confusion, and multitude of formulas for the same compound were in use, and various atomic weights, e.g., in 1861 F.A. Kekulé (1829-1896) gave nineteen different formulas for acetic acid. In many cases the chemical language was a rather personal matter, and many chemists used to develop their own chemical language. Thus, seminal reflection on a bi-atomic object brought the introduction of a more exact concept of molecule.

In chemistry there are other very important two's: the atom of Helium with atomic number two, and with two protons, two neutrons and two electrons. Helium was discovered by J. Norman Lockyer (1836-1920), the founder in 1869 of the renowned scientific journal, *Nature*, of which he was the editor for the first 50 years. Lockyer held an appointment at the Solar Physics Observatory, in London, and when observing in 1868 with a spectroscope the solar prominences, he detected a yellow line in the prominence spectrum of Sun, which he attributed to an unknown element, to which he gave the name Helium (He), and predicted its existence on Earth. Twenty-one years later, in 1889, W. Hillebrand of the US Geological Survey, came across this gas in the rare mineral Deveite. It was not until 1895 that Sir William Ramsay (1852-1916) working with a Deveite mineral isolated this gas and proved practically its existence on Earth. One year later H.G. Kayser announced the presence of He in very minute amounts (1/185000) in the Earth's atmosphere. But with He the story goes on with Sir William Ramsay and Frederick Soddy (1877-1956) who together discovered the Radium emanation (1895) by the spontaneous generation of He (1904). With Lord Rayleigh (1842-1919), Ramsay had discovered the other rare (noble) gases, starting with Argon. In 1911 Soddy pointed out that the radioactive generation of an α-ray (He ion) gave a product falling into a group of the periodic table two places lower than that of the parent element (Jaffe 1976, Partington 1989). Further, He has two isotopes boiling at different points at atmospheric pressure: 4.224 K for ^{4}He and 3.195 K for ^{3}He. [25] Deuterium, D, the second heavier isotope of hydrogen with one proton and one neutron, i.e, with two nucleons, was discovered by H.C. Urey of Columbia University in 1932, [27, 28] and since then has become of fundamental importance in physics and engineering. Because it has the largest ratio to the mass of the more abundant isotope (protium), deuterium gives rise to the largest isotope effects, and deuterium oxide (heavy water) is toxic when ingested by higher organisms such as mammals.

A famous dual concept in chemistry is the acid and base concept, in any one of various chemical definitions, of Arrhenius, of Brønsted & Lowry, of Lewis, or of Usanovich. The complementary acid-base theory of Bronsted & Lowry is quantified with K_a scales for acid-base strengths. The concepts of oxidation and reduction involving loss or gain of electrons, respectively, by of some chemical species, and since electrons are conserved, oxidation and reduction occur simultaneously. Normally elementary chemical reactions are viewed as monomolecular (as with a spontaneous decay) or bimolecular (as with a binary collision), while more complicated chemical reactions are viewed as sequences of such primary ones. The concept of electronegativity and electropositivity, so relevant in characterizing the chemical behavior of different elements, has been central in the history of chemistry, [28, 29] especially in the hands of one of the founders of modern chemistry, J.J. Berzelius (1779-1848), whose misunderstanding of the homo-covalent bond was an important element in the rejection of Avogadro's molecular theory. In fact, only this last theory was able to explain the common basis of homonuclear and heteronuclear compounds, and the pairing of atoms in a molecule became the basis to understand other pairing problems in chemistry, i.e, the dimers, the enzyme-coenzyme pairing, the pairing of lipids in aqueous solutions to form lipid bilayers which enclose microsystems, where some important endoergonic phenomena can occur, and which gave rise to life. [30] In speaking of life and chemistry it is appropriate to mention the base pairing in the double DNA helix between conjugate bases (C/G and A/T) playing a fundamental role in the maintenance and reproduction of life. A pairing mechanism is also effective on a more microscopic level, i.e., the spin-pairing in valence-bond theory, that is central to the formation of covalent chemical bonds, and also has special relevance in Pauling & Wheland's theory of "resonance". [31] Concerning molecular asymmetry based on molecules which are mirror images of each other, we discuss this in the next section. A kind of molecular 'antipairing' can be seen in the formation of polar molecules, when they are built up by electropositive and electronegative groups or atoms. In this case they show a well-defined dipole moment, which is of relevance in intermolecular interactions, along with induced dipole interactions. An extreme case of 'antipairing' is the formation of ions with opposite charge in aqueous solutions, giving rise to a conduction of second kind due to these opposite ions, while the conduction of first kind in metals is only given by electrons, even if in some semiconductors holes contribute to conduction. A major breakthrough in organic chemistry, with important consequences in theoretical chemistry, was the proposition, during the second half of the XIXth century, of the two resonance structures of benzene. The first finding premonition is surely Kekule's idea of a sort of oscillation back and forth between the two classical bonding patterns, which however in later quantum-mechanical explanations are seen to occur simultaneously in each individual benzene molecule with equal C-C bond lengths, all really as a manifestation of wave/particle duality. It is told that Kekule a night in his sleep dreamed an Ouroboros, while trying to solve the problem of benzene. [32]

Central in chemical thermodynamics are the criteria for stability (or instability) of natural changes in terms of properties relating to the dual partitioning of the universe into a system and its surroundings. These criteria are expressed as a series of inequalities centered on four thermodynamic functions, the entropy, S, the internal energy U, the Helmholtz function, A, and the Gibbs function, G: $dS_{U,V} \geq 0$, $dU_{S,V} \leq 0$, $dH_{S,P} \leq 0$, $dA_{T,V} \leq 0$, and $dG_{T,P} \leq 0$ (subscripts are parameters that are held constant during the change). [33] The duality at work here being that the stability of a system rests on the zero or non-zero value of its thermodynamic function.

Recent developments in the philosophy of chemistry discovered another dualism in the current image of chemistry, which is either manifest image (the daily practice or common-sense-human-life-form) or scientific image. It is, in fact, argued, that chemistry is primarily the science of manifest substances, whereas 'micro' or 'submicro' scientific talk, though important, useful, and insightful does not change what matters, namely the properties of manifest substances. [34] This recent dualism in chemistry seems to parallel the much older dualism on which was based alchemy. In early times, alchemy was centered around the two elements, *Sulphur & Mercurius*, which build the primeval dualistic 'chemical' system. These two elements were the essence from which matter was composed, and it was only later (by Paracelsus, it seems), in the development of alchemy, that a third "element" was added, table salt. The Greek God Hermes (the Roman Mercurius) had a stab where two snakes were symmetrically wounded around it, and it seems that they symbolized mercury and sulphur. [32]

7. The Symmetry-Asymmetry Dualism

In 1844 Mitscherlich (1794-1863) observed that, although tartaric and racemic acids are isomeric, the former is optically active while the latter is inactive. Pasteur (1822-1895) in 1848 by the slow crystallization of a solution of sodium ammonium racemate obtained crystals with small facets, some on the right (d-tartaric acid, d = dextro, right) and some on the left (l-tartaric acid, l = levo, left). After having painstakingly separated them, Pasteur noted that a solution of those with facets on the right deviated the plane of polarization to the right (dextro-rotatory), and those with facets on the left deviated it to the left (levo-rotatory). One form of the molecule is the mirror image of the other (like the left and right hands), i.e, they are optical isomers or enantiomers. From wine tartar deposits he could also isolate the optically inactive meso-tartaric acid, which was indifferent towards the light. This seminal discovery by Pasteur that had to wait till 1874 to be explained in terms of chemical constitution by J.H. van't Hoff (1852-1911) and J.A. Le Bel (1847-1930) [27, 28] opened the way not only to the discovery of stereoisomers in nature, but to the even more troubling discovery of the asymmetry of nature in this field, as living matter synthesizes and uses only one form and forgets the other. Thus between L- and D-forms of aminoacids, the building blocks of proteins, to build life on earth the L-amino acids were preferred. Even for natural sugars the problem is the same, but in this case nature prefers the D-series. A famous harmful case during the sixties shows this strange preference of nature. It was the case of thalidomide, where the synthesized material marketed was a 50/50 mixture of R and S enantiomers (another way to denote optical or chiral isomers), with one exhibiting the desired effect to counteract nausea, while the other enantiomer was teratogenic, engendering severely deformed fetuses in pregnant women. In Lewis Carroll's (1832-1898) *Alice in Wonderland*, Alice, having gone through the looking glass, wonders what happens if one eats mirror image items. This dramatic case of thalidomide might be an answer. Another dualistic asymmetry in nature that is intensively studied by physics is the matter and antimatter asymmetry. Matter and antimatter are assemblies of particles and antiparticles, respectively. The members of a particle / antiparticle pair have the same mass and spin but opposite signs of charge (if they are charged) and opposite signs of other quantum numbers. When a particle meets an antiparticle the two can annihilate each other and their combined energy reappears in another form. During the annihilation of an electron with a positron energy reappears as two highly energetic γ-ray photons that fly off in different directions so as to conserve momentum. Our universe seems to consist largely of matter, rather than of antimatter. If in nature symmetry is expected these asymmetries are quite surprising, and this physical asymmetry becomes even more intriguing since in order to explain the stability of our universe the existence of some as yet undetected dark matter has been postulated. Thus with particles (excluding neutrinos and photons) a series of dual questions should always be asked: (i) Is it a fermion (with half-integer spin) or a boson (with integer or zero spin) ? (ii) Is it a lepton (on which the strong force does not act) or a hadron (on which the strong force acts) ? If the latter, is it a meson (hadron-bosons) or a baryon (hadron-fermion) ? Finally, (iii) Is it a particle or an antiparticle ?

The strong force is the force that holds protons and neutrons together in atomic nuclei, and it is about 10^{10} times greater than the electromagnetic force between charged elementary particles. The weak force, weaker than the strong force, is observable in the decay of long-lived elementary particles. Beta decay (ejection of an electron or positron from a nucleus) is the most common example of a weak interaction decay. Actually weak and electromagnetic forces have been unified by Glashow (1932-), Weinberg (1933-), and Abdus Salam (1926-1996) into a single electroweak force. Efforts are under way to unify electroweak and strong forces, thus leaving in the universe the gravitational force and the unified electro-strong-weak force, which seems to be less easy to unify. [35-37] In physics one has discovered a strange 'asymmetry' law whose importance might influence other scientific fields. The space-reflection symmetry or principle of parity invariance states that no fundamental distinction can be made between left and right, i.e, the laws of physics are the same in a right-handed system of coordinates as they are in a left-handed system. However, the weak interaction does not exhibit parity invariance and parity is not conserved in weak interactions. In beta decay, the electron is always left-polarized, instead of being 50% left- and right-polarized. Scientists are looking for a similar broken symmetry law for the matter / antimatter pair, and far away, for the R and S chemical pairs in nature. Could it not be possible that the big-bang that originated the whole universe came into being through a broken symmetry ?

Other broken symmetry examples are found between a the gas and its condensed solid. [38] The infinite gas is symmetric under a continuous translation group, but the condensed solid has a lower symmetry, as it is invariant only under a discrete translation group. In the nucleation event the symmetry of the system is suddenly and spontaneously lowered by a random event. Macroscopic sciences, such as solid-state physics or thermodynamics, are qualitatively different from microscopic sciences because of the effect of broken symmetry. The second postulate of thermodynamics practically introduces a broken symmetry near the symmetry achieved by the first postulate.

About broken symmetry one should discuss Pierre Curie's (1859-1906) principle of symmetry of an effect, *i.e.*, the symmetry of an effect is no higher than a cause. E.g. the symmetry of an animal is no higher than that of the environment, which for mobile animals has two directions down in the direction of gravitational attraction and forward in the direction of motion, so that the symmetry of the environment is bilateral, C_σ. If the symmetry of the effect is lower than the cause, one has "broken symmetry". In biology there are two interesting apparent symmetries but actually asymmetries: the apparent symmetry of the face (and of the whole body) that, when examined more in detail, reveals striking asymmetries, and the perfect symmetry of the brain that is actually broken by the asymmetry functions of each hemisphere.

Parity invariance has to do with inversion transformations, that are transformations as through a plane, through a point, or through a sphere; in connection with inversion through a sphere, there is an amusing prescription for lion hunting (H. Petard, as quoted in *Geometry Revisited* by Coxeter & Greitzer):

We place a spherical cage in the desert, enter it, and lock it.
We perform an inversion with respect to the cage.
The lion is then in the interior of the cage, and we are outside.

8. Two in Biology

In biology many dualities can be detected, and we will here review some of them, mostly in molecular biology. The chromosome pairing, which takes place during meiotic transcription and crossover neatly lead to possibilities for an astronomical diversity of genetically different offsprings; the codon (in mRNA, messenger RNA, coding triplets, complementary to those of DNA) & anticodon (in tRNA, transfer-RNA, aminoacid acceptors for protein synthesis) duality, the antigen / antibody complementarity, which is an important mechanism in every form of life for building protection against foreign bodies entering or trying to enter in the cell cycle. [30] We have already spoken about the strange asymmetry of the two hemispheres of brain, which likely has something to do with left / right broken symmetry of other biological structures. The most common symmetry is bilateral (2-sided, of order two about a single axis, or through a single plane). Our bodies are bilaterally symmetrical, they have thus adopted the simplest form of symmetry. For example, two eyes are needed for depth perception in binocular vision, two ears to help in locating origins of sounds, the two tips to snakes' tongues, which as for eyes or ears might be surmised to aid in depth perception by detecting gradients in smell or temperature. Another duality is the fundamental division in plants between monocotyledons and dicotyledons, with many herbicides favoring one or the other. Further to notice are the species that can live both in water and on earth, the amphibians.

Dichotomy, division into two parts of our psyche is very significant in psychology, and has to do with the related phenomenon of double personality, such as associated with schizophrenia. What to say of the famous biological dual paradox: which was the first, the hen or the egg ? This biological paradox could also be phrased in a more 'molecular biology' fashion: which was the first, the RNA / DNA sequence or the amino acid sequence, i.e., proteins ? It appears that the primitive life forms were centered on RNA, and DNA appeared later. The self-replicating ability of RNA was exposed to too many mutations so that DNA conferred genetic stability. The gradual development of biosynthetic pathways to today's twenty natural amino acids co-evolved with the assignment of 64 triplet codons for these amino acids.

9. Two outside science

Till now, in the preceding chapters, we have been concerned more with some scientific aspects of dualism, now we direct attention with more general aspects of dualism. There are some very important religious dualisms: god and devil, where in Christianity there is also the idea of Christ and anti-christ, and in many religions good and evil is manifested on a diminished scale by angel and demon, and in Judaism, Christianity, and Islam, at a mortal scale there are Cain and Abel. Heaven and hell in many religions form a very important dualism, though in some forms of Buddhism there is reincarnation into the present world, albeit likely at a diminished or enhanced level, while in other forms of the same religion there is just the nothingness of the Nirvana. In Hindu religion very important is the 'double' god Shiva the creator and Shiva the destroyer, where the unity of the dual possibilities is emphasized. The dualism between good and evil goes beyond religion attaining the level of general values of humanity universally recognized in all cultures. At the beginning of the 20th century classical Freudian psychoanalysis introduced the ego and id duality, evidently harkening back to the duality between good and evil. Also the idea of a *doppelgänger* as a soulless evil duplicate seems to be related to the idea of this duality. In schizophrenia one may find good and evil personalities, though often there are other aspects of personality which may clash. To the good and evil duality for a long time, and sometimes even today, often is associated the right and left duality, as well as the mind and body or soul and body duality, where left and body are the negative counterparts of the duality.

That is, in these cases (in English and in many other languages) names are provided for but two out of many more possibilities. In some cases an intermediate condition is expressed, as over and under having a third named possibility, namely "through" , or with past and future having the intermediate of present. All these examples occur in a fashion of pairing when the underlying continuum of possibilities is a linear continuum, and really the pairing of choice simply refers to the direction of displacement along the linear scale. Sometimes there is a well-defined zero with respect to which the displacements are referred, as for : forward and backward raising and lowering, accelerate and decelerate, hydrophobic and hydrophilic, give and receive (though here receive is not exactly opposite to give, as receive is usually distinguished from take), and donor and acceptor. A relevant aphorism is: "*We make a living by what we get; we make a life by what we give.*" Sometimes there seems to be a natural zero, but no negatives, as with: dry and wet, faithful and unfaithful, orthodox and unorthodox, symmetric and asymmetric, theist and atheist, though agnostic is often taken to describe a 0, but it is not exactly such since agnostic is usually defined in terms of lack of "knowledge", while theism and atheism are usually defined in terms of "belief", so that one could imagine atheistic agnostics and theistic agnostics. Ignorance and enlightenment, chiral and achiral in chemistry, are other such dualities, and temperature, with cold and hot could be put in this list too, though the absolute zero of temperature is unattainable and so cold that it is out of reach.

The two directions in a general linear ordering are evidently deeply ingrained in our psyches. The alternative of partial orderings seem less natural to our psyches, and when dealing with partially ordered circumstances, a linear ordering still often seems to be imagined, with two directions recognized in our language and very thinking. Indeed, of the items that could be included in the lists of this paragraph, some may be argued to actually involve partial orders, as for the cases: of chronology (at least in relativity); of intelligence (smart and dumb); of ability (able and disabled); of modernity (ancient and modern, especially as applied to ideas, and being viewed as a deeper concept than just an identification of its chronological occurrence); of beauty (beautiful and ugly); and of disturbed and undisturbed - and perhaps even strength (weak and strong) and hardness (hard and soft) may be partially ordered. For instance, for intelligence it may be argued that there are different kinds, regarding music, or mathematics, or language, or mechanical, or yet others, and that of two people one person may be more intelligent in one area while the other is more intelligent in another area. It would be just as ill-conceived to argue which of the pairs of numbers (5,5) or (8,3) is the "larger" - if they are represented by a single number as the product of the parts (or as a geometric mean), then (5,5) (8,3) ; but if as the sum of the parts (or as an arithmetic mean), then (8,3) (5,5); and ordering according to the size of the largest part (*i.e.*, lexicographic ordering) gives the opposite result to ordering according to the size of the smallest part. It seems that the human mind exhibits a proclivity to impose a "twoness" in bidirectionality in many, even questionable, circumstances. This proclivity seems to carry over to the circumstance when one speaks about the "between" relation, which might be used again in circumstances where underlying there is likely a multi-dimensional partial order rather than a linear order.

Everyday language includes explanatory statements, which have normally the dualistic form: "q because p". Two is manifested as a great number of pairs of ideas, where one simply appends to a parent word a "negating" prefix *a-*, *ab-*, *ant(-,e-,i-)*, *di-(dis-)*, *in-*, *non-*, or *un-*. Even a marginally complete listing would be immense, so we might just mention the pairs: sexual and asexual, use and abuse, arctic and antarctic, choir and antechoir, American and anti-American (a very common dualism during cold war), freeze and antifreeze, order and disorder, own and disown, certitude and incertitude, clement and inclement, entity and non-entity, known and unknown. Some few times in English this appending process is anomalous, as with visibility and divisibility (!), and with flammable and inflammable both of which mean the same thing - perhaps in this case the idiosyncrasy of English arises in its proclivity for borrowing from other languages, such as French where "*inflammable*" means what in English is naturally meant by flammable. Some other times one of the two pairs has nearly no meaning as with denaturation and naturation, or deviation and viation.

Two appears as "bad" in a few circumstances. Examples include: in bigamy, at least in many cultures this is "bad", in bipolar disorder (or schizophrenia), in *spina bifida* (a severe spinal defect), bipedism is "bad" in George Orwell's *Animal Farm* where the sheep are induced to bleat over and over "Four legs good, two legs bad". A famous occurrence of a double-edged tongue, as relates to the fundamental true-untrue duality, is recorded by Plutarch: *also the two-edged tongue of mighty Zeno, who, say what one would, could argue it untrue*. More commonly, two does not take a bad connotation, it often represents something deep beyond good and bad, and in some cases two takes a good connotation, often when two refers to partnering. Two is referenced in a number of circumstances involving partners. Examples include: partners in marriage, dance partners, lovers, child and parent, teacher and student (which seems to work best in the older manner when the ratio is 1 to 1) and also apprentice and master craftsman, police partners in tough neighborhoods, doubles games in sports, pilots and co-pilots, nut and bolt, male and female plugs (as for coaxial cable). Recently the term dynergy [39] has been suggested as a new word for an universal pattern creating process based on opposition, i.e., dynergy is the energy-creating process that transforms discrepancy into harmony by allowing differences to complement each other. The concept of harmony based on a dualism reminds in some way the ying-yang notion, who reached Europe and influenced middle-age European architects, as can be seen from some windows of gothic cathedrals (Fig. 5), which strongly remind the ying – yang symbol. Most sports and games entail competition between two individuals or two teams. In sports there may be an overall competition between many teams, but the direct competition is one on one, and there may be some sort of decision tree to identify an overall

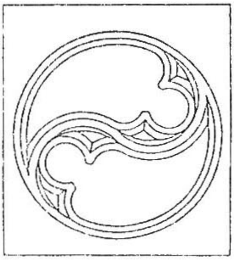

Figure 5. Window of a Gothic church.

champion. The cases are so numerous as to make any reasonable listing incomplete. There are two dice in many games of chance, and in backgammon if the dice match, then the value thrown is doubled.

Dualistic systems are successfully at play even nowadays. In modern democracies it seems that the alternating of two opposing parties is fundamental for the survival of democracy. In ideology the struggle between two opposing ideologies has till recently shaped or misshaped the politics of our world. Religions divide humanity in believers and disbelievers. This dynamic *di-vision* of the world did not always engender the growth of civilization (the crusades, the burning of heretics, the KZs, the Goulags,....can hardly be considered a dynamic factor). Moreover this *di-vision* may be perceived at the root of many philosophical systems, from Heraclitus of Ephesus, (VI BC), to the Hegelian *thesis* and *antithesis* and on to the Marxian struggle of classes, as nobility vs. bourgeoisie or bourgeoisie vs. proletarians, and which seems to hide, behind a Hegelian philosophical apparatus tied to socioeconomic considerations, the old biblical hate between believers and unbelievers. [40]

10. Last Considerations

Several of the examined pairs, if not all, might be quite reasonably be characterized as a manifestation of a fundamental duality, which is at the root of our reality, though there is some ambiguity about what is fundamental and what is not. The reflections on duality go well beyond the range of mathematics, and underline that the essential characteristic of the number two is not its magnitude and its place in the ordered aggregate of real numbers. If our universe is a 'unicum', a thing that should be proven, it seems nevertheless to have been thrown into being by a 'Shakespearean' dual choice: *to be or not to be*, and a dual choice appears again in an old Babylonian text, *how much is one God beyond the other God ?* It was too bad for the humanity that the deep truth of this text was too soon forgotten. A speech that seems to trace a line between science and religion, and centered on a duality is told by Snow, [41] it is the speech of the schoolmaster Luard to his class, *well, if you took a piece of lead, and halved it, and halved the half, and went on like that, where do you think you'd come to in the end ? Do you think it would be lead for ever ? ....If you went on long enough, you'd come to an atom of lead, an atom, do you hear, an atom, and if you split that up, you wouldn't have lead anymore. What do you think you would have ? ......pieces of positive and negative electricity. Just that. That's all you are. Just positive and negative electricity – and, of course an immortal soul.*'

A Robert Frost's poem gives a deeply human example of the good/evil dualism encoded by the two roads (fork in the road) dualism, "The Road Not Taken":

Two roads diverged in a yellow wood, / and sorry I could not travel both

I shall be telling this with a sigh

Somewhere ages and ages hence: / Two roads diverged in a wood, and I -

I took the one less traveled by, / And that has made all the difference.

Haiku is one of the shortest forms of poetry ever composed. It was born and developed in Japan during the XVIth century. Originally it was a poem that could consist of 36, 50 or 100 verses, and the first three verses were called *hokku*. It was to be these three verses that would become independent and develop into haikus in the 'hands' of many Japanese poets during and after the XVIth century. [42] Such haikus consist of two scenes, as is evident from the haiku by Matsuo Bashoo (1644-1694), here the first verse is a scene and the other two verses are the other and unique scene: *A day of quiet gladness / Mount Fuji is veiled / In misty rain.*

A recent theoretical discovery that makes the distinction between the whole and its parts entirely relative emphasizes the importance of duality even in quite new scientific domains. This duality is emerging in the quantum-relativistic string theory, where elementary (the parts) and compounded particles (the whole) appear to be interchangeable. [43] Thus, a set of quarks may give rise to solitons that are monopoles; a set of monopoles may give rise to solitons that are quarks. Thus, at the scale of the Planck constant, reality is intimately entwined with duality, and the option if a particle is irreducible or is itself made up of more fundamental particles becomes a matter of method. String theory still awaits experimental validation.

It is told that Socrates (468-399 BC) answered to Euripides' (480-406 BC) question concerning what did he think of Heraclitus, the obscure, with an intriguing remark: *what I have understood was excellent, and I think that it was excellent also what I haven't understood.*

11. Notes and References

† On sabbatical leave, present address: Prof. L. Pogliani, Unidad de Investigacion en Diseno de Farmacos y Conectividad Molecular, Facultdad de Farmacia, Departamento de Quimica-Fisica, 46100 Burjassot (Valencia), Spain.

[a] Democritus was a good mathematician, who determined the volumes of the pyramid and of the cone. He may have imagined an ideal mathematical line to consist of discrete atomic points.

[b] So long as one combines the rationals only through addition, subtraction, multiplication,

division of finite repetitions of such.

[c] Greek mathematicians did not speak of the area of one figure, but of the ratio of two surfaces, a definition, which could not, because of the problem of incommensurability, be made precise before a satisfactory concept of number had been developed in the XIXth century.

[1] D. Tirosh, & R. Stavy, *Int.J.Sci.Educ.* 18 (1996) 653-668.

[2] C.B. Boyer & U.A. Merzbach, *A History of Mathematics* (Wiley, New York, 1989).

[3] D.E. Smith, *History of Mathematics* (Dover, New York, 1953, Vol. 1).

[4] L. Pogliani, M. Randić & N. Trinajstić, *Int.J. Math Educ.Sci.Technol.* 29 (1998) 729-744.

[5] L. Pogliani, M. Randić & N. Trinajstić, *Kem.Ind.* (Zagreb) 47 (1998) 387-396.

[6] L. Pogliani, M. Randić & N. Trinajstić, *Int.J.Math.Educ.Sci.Technol.* 31 (2000) 811-824.

[7] G. Ifrah, *From One to Zero* (Penguin Books, New York, 1987).

[8] K. Menninger, *Number Words and Number Symbols* (Dover, New York, 1969).

[9] D. Wells, *The Penguin Dictionary of Curious and Interesting Numbers* (Penguin Books, London, 1997).

[10] L. Felici, (ed.) *Enciclopedia dei Simboli,* Garzanti, Milano, Italian version of 1989, *Knaurs Lexikon der Symbole* (Droemersche Verlagsanstalt Th. Knaur Nachf, München, 1991).

[11] http://www.circleofouroboros.com/

[12] P.J. Davis & R. Hersch, *The Mathematical Experience* (Penguin Books, New York, 1990).

[13] E.T. Bell, *The Magic of Numbers* (Dover, New York, 1974).

[14] C. S. Ogilvy & J.J. Anderson, *Excursion in Number Theory* (Dover, NY, 1988, p. 154)

[15] D.J. Klein & W.A. Seitz, *Chemical Applications of Topology and Graph Theory* (R.B. King (ed.), Elsevier, Amsterdam, 1983).

[16] N. Trinajstić, *Chemical Graph Theory* (CRC Press, Boca Raton, 1983).

[17] A.T. Balaban, (ed.) *Chemical Applications of Graph Theory* (Academic Press, London, 1986).

[18] I. Gutman & O.E. Polansky, *Mathematical Concepts in Organic Chemistry* (Springer-Verlag, Berlin, 1986).

[19] L. Pogliani, *MATCH Commun.Math.Comput.Sci* 49 (2003) 141-152.

[20] G. Gamow, *One, Two, Three.......Infinity* (Dover, New York, 1974).

[21] J. Hurley, *Am.J.Phys.* 54 (1986) 25-28.

[22] V. Ambegoakar, & A.A. Clerk, *Am.J.Phys* 67 (1999) 1068-1073.

[23] H. Genz, *Wie die Zeit in die Welt kam* (Rowohlt, Hamburg, 1999, ch. 2).

[24] E. Fermi, *Termodinamica* (Boringhieri, Torino, 1956, chs. 2, and 3).

[25] M.W. Zemansky & R.H. Dittman, *Heat and Thermodynamics* (McGraw-Hill, Singapore, 1981, chs. 6, and 19).

[26] E.H. Wichman, *Quantum Physics* (Berkeley Physics Course, McGraw-Hill, New York, 1967).

[27] B. Jaffe, *Crucibles: the Story of Chemistry*,(Dover, New York, 1976).

[28] J.R. Partington, *A Short Story of Chemistry* (Dover, New York, 1989).

[29] M.J. Nye, *A History of Modern Physics 1800-1954. Vol IV: The Question of the Atom,* (Tomash Publ.Co., Los Angeles, 1986, chs. 1, and 2).

[30] A.L. Lehninger, *Biochemistry* (Worth Pub.Co., New York, 1977, ch. 11).

[31] L. Pauling & E.B. Wilson Jr., *Introduction to Quantum Mechanics* (Dover, New York, 1963).

[32] H.-W. Schütt, *Auf der Suche nach dem Stein der Weisen* (Beck, C.H. Verlag, München, 2000, p. 572).

[33] J. Van Brakel, *Philosophy of Chemistry* (Leuven Univ. Press, Leuven, Belgium, 2000).

[34] P.W. Atkins, *Physical Chemistry* (Oxford Univ. Press, Oxford, 1990, ch. 4).

[35] D. Halliday, R. Resnick & J. Walker, *Fundamentals of Physics Extended* (Wiley, New York, 1997, ch. 45).

[36] V. Illingworth, (ed.) *The Penguin Dictionary of Physics* (Penguin Books, London, 1991).

[37] J.D. Barrow, *Theories of Everything* (Vintage, London, 1992).

[38] H.B. Callen, *Thermodynamics and Introduction to Thermostatistics* (Wiley, New York, 1985, chs. 5, and 6).

[39] G. Doczi, *The Power of Limits* (Shambhala, Boton, 1999, ch. 1).

[40] J. Benda, *The Treason of the Intellectuals* (W.W. Norton & Co., London, 1956), translated from French by R. Aldington (*La Trahison des Clercs*, first published in 1928).

[41] C.P. Snow, *The Search* (Charles Scribner's Sons, New York, 1958, p. 10).

[42] R. De la Fuente & Y. Kawamoto, *Haijin, Antologia del Haiku* (Hiperión, Madrid, 1992).

[43] M. Pauri, *Science & Education* 12 (2003) 467-477.