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A Linear Algorithm For Recognizing Planar Two-Cycle Resonant Graphs *

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Abstract

A connected graph G is said to be k-cycle resonant if, for $1 \leq t \leq k$, any t disjoint cycles in G are mutually resonant; that is there is a perfect matching M of G such that each of the t cycles is M-alternating cycle. Some necessary and sufficient conditions for a graph to be k-cycle resonant were given by Xiaofeng Guo and Fuji Zhang, and they also established some necessary and sufficient conditions for a planar graph to be 1-cycle resonant and 2-cycle resonant. A linear algorithm for deciding if a planar graph is 1-cycle resonant was established by Zhixia Xu and Xiaofeng Guo. In this paper, we establish a linear algorithm for recognizing planar 2-cycle resonant graphs.

1 Introduction.

In the topological theory of benzenoid hydrocarbons, a *hexagonal system* (or *benzenoid system*) denotes the carbon atom skeleton graph of a benzenoid hydrocarbon, that

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is a 2-connected plane graph whose every interior face is bounded by a regular hexagon. A *Kekulé structure* K of a hexagonal system H is also a perfect matching (1-factor) of H. A cycle (or circuit) C in a hexagonal system H is said to be *conjugated* or *resonant* if there is a Kekulé structure K of H such that C is a K-alternating cycle. In the conjugated circuit model [1-24], conjugated circuits with different sizes have different resonant energies. If the size of a conjugated circuit is equal to $4n+2$, then the smaller n the larger the resonance energy. So the conjugated hexagon has the largest resonance energy. On the other hand, from a purely empirical standpoint, Clar found that various electronic properties of polycyclic aromatic hydrocarbons can be predicted by appropriately defining aromatic sextets [25-35]. According to Clar's aromatic sextet theory, the Clar formula of a hexagon is a set of mutually resonant sextets with the maximum cardinal number, where sextets mean resonant hexagons and a set of mutually resonant sextets means a set of disjoint hexagons for which there is a Kekulé structure K so that all of the the disjoint hexagons are K-alternating hexagons.

For a hexagonal system H with Clar number c (the number of sextets in a Clar formula of H), Clar formula of H may be not unique, and, for $1 \leq k \leq c$, any k disjoint hexagons of H are not necessarily mutually resonant. An interesting problem is under which conditions any k disjoint hexagons of a hexagonal system H are mutually resonant? If a hexagonal system H satisfies such property, that is, for a positive integer k and $1 \leq t \leq k$, any k disjoint hexagons of H are mutually resonant, it is said to be *k-resonant* or *k-coverable*.

As a generalization of k-coverable hexagonal systems, Guo Xiaofeng and Fuji Zhang [36] introduced *k-cycle resonant* graphs. Some properties of k-cycle resonant graphs and some necessary and sufficient conditions for a graph to be k-cycle resonant were given.

Xiaofeng Guo and Fuji Zhang [37] further investigated planar k-cycle resonant graphs with $k = 1, 2$. Some new necessary and sufficient conditions for a graph to be planar 1-cycle resonant or planar 2-cycle resonant were given. Zhixia Xu and Xiaofeng Guo [38] investigated the construction and the recognition of planar 1-cycle resonant graphs, and established a linear algorithm for deciding if a planar graph is planar 1-cycle resonant. In ref. [39], the present authors have given a method for constructing any planar 2-cycle resonant graph from smaller 2-cycle resonant graphs. In the present paper, a linear algorithm for recognizing 2-cycle resonant graphs is established.

In this paper, for the basic terminologies, we refer to the books by Bondy and Murty [40] and L. Lovász and M. D. Plummer [41].

2 Preliminary Results

Definition: A connected graph G is said to be k-cycle resonant if, for $1 \leq t \leq k$, any t disjoint cycles in G are mutually resonant, that is, there is a perfect matching M of G such that each of the t cycles is an M-alternating cycle.

Theorem A. [37] A 2-connected graph G with at least k disjoint cycles is k-cycle resonant if and only if G is bipartite and, for $1 \leq t \leq k$ and any t disjoint cycles $C_1, C_2, \cdots, C_t$ in G, $G - \bigcup_{i=1}^{t} V(C_i)$ contain no odd component. ■

The above thecrem is a revision of Theorem 3.1 [36], in which the condition "2-connected" is neglected, however, the condition is implicitly used in the proof of Theorem 3.1 [36].

Theorem A'. [37] A connected graph G with at least k disjoint cycles is k-cycle resonant if and only if G is bipartite with perfect matchings and, for $1 \leq t \leq k$ and any t disjoint cycles $C_1, C_2, \cdots, C_t$ in G, $G - \bigcup_{i=1}^{t} V(C_i)$ contain no odd component. ■

A *block* of a connected graph G is either a maximal 2-connected subgraph of G or a cut edge of G.

Theorem B. [37] Let G be a k-cycle resonant graph, then

(i) for a 2-connected block G' of G with the maximum number k^* of disjoint cycles, if $k^* < k$, G' is k^*-cycle resonant, otherwise G' is k-cycle resonant;

(ii) the forest induced by all the vertices of G not in any 2-connected block of G has a unique perfect matching. ■

From the theorem B, we know that a non-2-connected k-cycle resonant graph can be constructed from some disjoint 2-connected k-cycle (or k^*-cycle if $k^* < k$, where k^* is the maximum number of disjoint cycles) resonant graphs and a forest with perfect matching by adding some edges between the 2-connected graphs and the forest so that the resultant graph is connected and the added edges are cut edges. Hence we need only to consider 2-connected k-cycle resonant graphs.

Let G be a connected graph, and H a subgraph of G. A vertex in H is said to be an *attachment vertex* of H if it is incident with an edge in $E(G) \setminus E(H)$. A *bridge* B of H in G is either an edge in $E(G) \setminus E(H)$ with two end vertices being in H, or a subgraph of G induced by all the edges in a connected component B' of $G - V(H)$ together with all the edges with an end vertex in B' and the other in H. The vertices in $V(B) \cap V(H)$ are also attachment vertices of B to H. A bridge with k attachment vertices is called a *k-bridge*.

The attachment vertices of a k-bridge B of a cycle C in G divide C into k edge-disjoint paths, called the *segments* of B. Two bridges of C avoid one another if all the attachment vertices of one bridge lie in a single segment of the other bridge, otherwise they overlap.

For a bipartite graph, we always color its vertices black and white so that adjacent vertices have different colors.

Additional necessary and sufficient conditions for a graph to be planar 1-cycle resonant and 2-cycle resonant was given by Xiaofeng Guo and Fuji Zhang [37]:

Theorem C. [37] A 2-connected graph G is planar 1-cycle resonant if and only if G is bipartite and, for any cycle C in G, any bridge of C has exactly two attachment vertices which have different colors. ■

Theorem D. [37] A 2-connected graph G is planar 1-cycle resonant if and only if G is bipartite and, for any cycle C in G, any two bridges of C avoid one another and, for any 2-connected subgraph B of G with exactly two attachment vertices, the attachment vertices of B have different colors. ■

On the basis of these necessary and sufficient conditions, Zhixia Xu and Xiaofeng Guo [38] have given a modification of that as follows.

A *plane graph* is an embedding of a planar graph. We call the boundary of the exterior face of a 2-connected plane graph G the *outer cycle* of G.

Theorem E. [38] Let G be a 2-connected plane bipartite graph, then G is 1-cycle resonant if and only if any bridge of the outer cycle C of G has exactly two different colored attachment vertices and for any maximal 2-connected subgraph H of any bridge B of C, the following conditions are satisfied:

(1) H is 1-cycle resonant;

(2) H has exactly two different-colored attachment vertices u and v;

(3) u and v avoid any bridge of the outer cycle of H. ■

According to Theorem E, Zhixia Xu and Xiaofeng Guo [38] provided a linear algorithm for deciding a plane graph to be 1-cycle resonant.

Algorithm F. [38] Let G be a 2-connected plane bipartite graph,

1. Color all the vertices of G so that adjacent vertices get different colors.

2. Let C_0 be the outer cycle of G, check the bridges of C_0 to see whether any bridge of C_0 has exactly two different-colored attachment vertices. If not so, go to step 11.

3. If any bridge of C_0 is a path, go to step 10.

4. Denote by $G_1, G_2, \cdots, G_r$ all the maximal 2-connected subgraphs of the bridges of C_0. Set $i = 1$.

5. Check: (1) whether two attachment vertices of G_i are different-colored (Since G is 2-connected and any bridge of C_0 has exactly two attachment vertices, G_i also has exactly two attachment vertices); if yes, let them be u and v; (2) for the outer cycle C_i of G_i, whether u and v avoid any bridge of C_i in G_i; (3) whether any bridge of C_i has exactly two different-colored attachment vertices. If any one of the three conditions is not satisfied, go to step 11;

6. If $i \neq r$, set $i = i + 1$ and return to step 5.

7. If any bridge of the outer cycles of $G_1, G_2, \cdots, G_r$ is a path, go to step 10.

8. Let $H_1, H_2, \cdots, H_s$ be all the maximal 2-connected subgraphs of the bridges of the outer cycles of $G_1, G_2, \cdots, G_r$.

9. Set $G_i = H_i$, $i = 1, 2, \cdots, s$, and set $r = s$, $i = 1$, go to step 5.

10. Stop, G is 1-cycle resonant.

11. Stop, G is not 1-cycle resonant. ■

Theorem G. [38] Algorithm F is linear with respect to the number of vertices. ■

In what follows we give some terminologies and notions relative to planar 2-cycle resonant graphs.

For a 2-connected subgraph B in G with exactly two attachment vertices, we call $G[E(G)\text{-}E(B)]$ the *complement* of B in G, denoted by $\overline{B}$.

A path P in a graph G is said to be a *chain* if all internal vertices of P are of degree 2 in G and the degree of any end vertex of P is not equal to 2 in G. The set of internal vertices of a chain P in G is denoted by $V_I(P)$.

A vertex u of a graph G is said to be *cycle-related* to another vertex v of G, denoted by $u \Rightarrow v$, if u is contained in a 2-connected block of G and any cycle containing u must also contain v. If v is also cycle-related to u, then u and v are *mutually cycle-related*, denoted by $u \Leftrightarrow v$.

Property H. [37] If a vertex u of a connected graph G is cycle-related to another vertex v of G, then u and v belong to a same 2-connected block B in G and all the edges in $B - v$ incident with u are cut edges of $G - v$. ■

Theorem I. [37] A 2-connected graph G is planar 2-cycle resonant if and only if G is planar 1-cycle resonant, and

(i) for a chain P of even length and end vertices v_1 and v_2, $G - V_I(P)$ has exactly two blocks each of which is 2-connected, and v_1 and v_2 are cycle-related to the common vertex w of the two blocks.

(ii) for a chain P of odd length and end vertices v_1 and v_2 such that $G - V_I(P)$ is not 2-connected, either (a) $G - V_I(P)$ has exactly three blocks, each of which is a 2-connected, and each of v_1 and v_2 is cycle-related to the other attachment vertex of the block containing it, and the attachment vertices of the third block are mutually cycle-related in the third block, or (b) any two 2-connected blocks of $G - V_I(P)$ are disjoint,

(iii) for a 2-connected subgraph B_1 of G with exactly two attachment vertices, if $\overline{B_1}$ is not 2-connected and every block of $\overline{B_1}$ is 2-connected, then $\overline{B_1}$ has exactly three blocks, say B_2, B_3, B_4, and the attachment vertices of each of B_1, B_2, B_3, B_4 are mutually cycle-related in the block. ■

Let G be a graph, u and v are two distinct vertices in G. Let P^* be a path with end vertices u and v, and $V(G) \cap V(P^*) = \{u, v\}$. Let $(G + P^*)_{(u,v)}$ denote the graph $G \cup P^*$.

Let G be a planar 1-cycle resonant graph. If G has no disjoint cycles, we call it *simple* planar 1-cycle resonant graph. Obviously, planar 1-cycle resonant graphs with cyclomatic number $\nu(G) = 1, 2$ are simple. Generally, if G consists of at least two chains of odd length with the same end vertices, then G is simple planar 1-cycle resonant. We call this kind of graphs *parallel-odd-chain*.

Theorem J. [39] A 2-connected planar graph G with cyclomatic number $\nu(G) \geq 2$ is simple 1-cycle resonant if and only if G is bipartite, and in the vertices with degree greater than 2 there is a vertex v such that $G - v$ is a tree and the color of v is different from all the other vertices with degrees greater than 2. ■

From Theorem J, we know that if G is 2-connected simple planar 1-cycle resonant, then there is a vertex v the color of which different from all the other vertices with degrees greater than 2, such that $G - v$ is a tree, and v is in all cycles of G. We call the vertex v *cycle-common vertex* of G. So, a simple planar 1-cycle resonant graph has a cycle-common vertex. Since the cycle-common vertex of a simple planar 1-cycle resonant graph is colored differently from all the other vertices with degrees greater than 2, then we have the following corollary.

Corollary 1. A simple planar 1-cycle resonant graph G with cyclomatic number $\nu(G) \geq 2$ has at most two cycle-common vertices. ■

Having the aid of Theorem I, we can get four structure models of a 2-connected planar 2-cycle resonant graph G as follows:

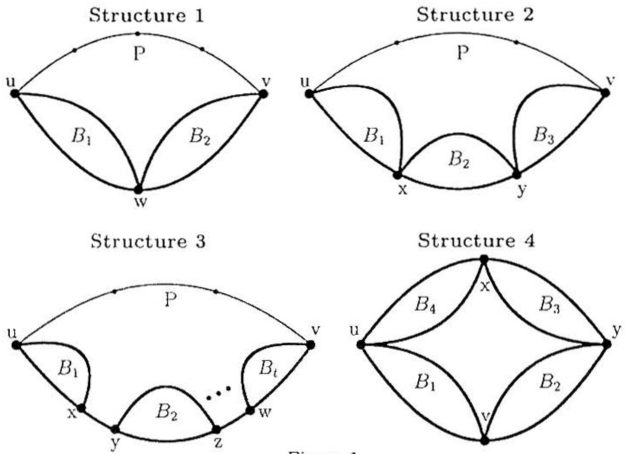

Figure 1.

Structure 1: If there is a chain P of even length and end vertices u and v, then $G - V_I(P)$ has exactly two 2-connected blocks B_1 and B_2 with a common vertex w, and $u \Rightarrow w$ in B_1 and $v \Rightarrow w$ in B_2, as shown in Figure 1.

Structure 2: For a chain P of odd length and end vertices u and v such that $G - V_I(P)$ is not 2-connected, $G - V_I(P)$ has exactly three blocks B_1, B_2 and B_3, each of which is 2-connected, and each of u and v is cycle-related to the other attachment vertex of the block containing it, and the attachment vertices of the third block are mutually cycle-related in the third block, that is, $u \Rightarrow x$, $v \Rightarrow y$ and $x \Leftrightarrow y$, as shown in Figure 1.

Structure 3: For a chain P of odd length and end vertices u and v such that $G - V_I(P)$ is not 2-connected, any two 2-connected blocks of $G - V_I(P)$ are disjoint. In addition, any chain P_i in $G - V_I(P)$ induced by non-2-connected blocks of $G - V_I(P)$ is of odd length, as shown in Figure 1, where B_i is a 2-connected block, for $i = 1, 2, \cdots, t$, $(t \geq 2)$.

Structure 4: For a 2-connected subgraph B_1 of G with exactly two attachment vertices, if $\overline{B_1}$ is not 2-connected and every block of $\overline{B_1}$ is 2-connected, then $\overline{B_1}$ has exactly three blocks, say B_2, B_3, B_4, and the attachment vertices of each of B_1, B_2, B_3, B_4 are mutually cycle-related in the block, as shown in Figure 1.

From these four structure models, we can see that a 2-connected planar 2-cycle resonant graph G consists of two kind of subgraphs: chains and 2-connected subgraphs with exactly two attachment vertices, based on some rules. We call the two kind of subgraphs *structural-chain* and *structural-brick* of G respectively.

Theorem K. [39] A 2-connected graph G with cyclomatic number $\nu(G) \geq 3$ is planar 2-cycle resonant if and only if there is a 2-connected subgraph B_1 in G with exactly two attachment vertices such that G is one of the Structures 1, 2, 3, 4 (shown in Fig. 1), and for every 2-connected structural-brick B_i of G with exactly two attachment vertices u_i and v_i, $(B_i+P^*)_{(u_i,v_i)}$ is 2-cycle resonant or simple 1-cycle resonant, $i = 1, 2, \cdots, t$, $(t \geq 2)$. Moreover, if G is of the Structure 1, then there is at least one 2-connected structural-brick B of G with exactly two attachment vertices u and v such that $(B + P^*)_{(u,v)}$ is planar 2-cycle resonant.

3 Properties of Planar 2-Cycle Resonant Graphs

In this section, we give some properties relative to the algorithm for recognizing 2-cycle resonant graphs.

By Theorem I [37], we have the following property.

Property 1. Let G be a 2-connected planar 2-cycle resonant graph, B a 2-connected subgraph of G with exactly two attachment vertices. If $\overline{B}$ is neither a path nor 2-connected, then G is one of the Structures 1, 2, 3, 4, in which B is a structural-brick.■

Property 2. Let G be a plane 2-cycle resonant graph, C_0 the outer cycle of G. Let u and v be two attachment vertices of a bridge of C_0, SB a union of a segment $\widehat{uv}$ of C_0 separated by u and v and all bridges of C_0 attaching to the segment $\widehat{uv}$. Then $SB \cup C_0$ is either simple 1-cycle resonant or 2-cycle resonant. ■

Proof: Since G is 1-cycle resonant, then every pair of bridges of C_0 avoid each other, and SB is a 2-connected subgraph of G with exactly two attachment vertices u and v. Obviously, $\overline{SB}$ is connected, but not 2-connected. If $\overline{SB}$ is a path, then $G = SB \cup C_0$, the result holds. If $\overline{SB}$ is not a path, by Property 1 and Theorem K, $(SB + P^*)_{(u,v)}$ is either simple 1-cycle resonant or 2-cycle resonant, where P^* is a path with odd length, and $V(SB) \cap V(P^*) = \{u, v\}$. We can take $C_0 - V_I(\widehat{uv}) = P^*$, then $SB \cup C_0 = (SB + P^*)_{(u,v)}$, which is either simple 1-cycle resonant or 2-cycle resonant. ■

Property 3. A 2-connected planar graph G with cyclomatic number $\nu(G) \geq 2$ is simple 1-cycle resonant if and only if either G is a parallel-odd-chain, or there is a 2-connected subgraph with exactly two attachment vertices such that G is of the Structure 1, and for every 2-connected structural-brick B of G with two attachment vertices u and v, both B and $(B + P^*)_{(u,v)}$ are also simple planar 1-cycle resonant.

Proof : Necessity. Let G be a 2-connected planar simple 1-cycle resonant graph with cyclomatic number $\nu(G) \geq 2$. By Corollary 1, G has at most two cycle-common vertices.

In what follows, we show that if G has two cycle-common vertices then G is a parallel-odd-chain, and if G has exactly one cycle-common vertex then G is of the Structure 1.

Let w_1 and w_2 be two cycle-common vertices of G, then w_1 and w_2 have different colors. For every vertex $u \in V(G) \setminus \{w_1, w_2\}$, we have that $deg_G(u) = 2$, since the color of cycle-common vertex is different from all the other vertices with degree greater than 2. Hence G is a parallel-odd-chain.

Let w be only one cycle-common vertex of G, then all vertices in $V(G) \setminus \{w\}$ with the color of w's have degree 2 in G; and, in $V(G) \setminus \{w\}$, there are at least two vertices with degrees greater than 2 in G, which have different color from w's. Since $G - w$ is a tree, between any pair of vertices in $V(G) \setminus \{w\}$ with degree greater than 2 in G, there exists a path P' with even length not containing w. Then the path P' contains a chain P of even length with two end vertices u_1 and u_2. Let T be the tree of $G - w$, then $T - V_I(P)$ has two component trees T_1 and T_2. Let $u_i \in V(T_i)$, $i = 1, 2$. Since w is adjacent with all pendant vertices of the tree T in G, and any pendant vertex of T_i is also a pendant vertex of T, so $G[V(T_i) \cup \{w\}] = B_i$ is a 2-connected subgraphs of G with two attachment vertices u_i, w, and u_i and w have different colors, for $i = 1, 2$. Hence G is of the Structure 1. If B_i is a cycle, then both B_i and $(B_i + P_i^*)_{(u_i,w)}$ are simple planar 1-cycle resonant. If B_i is not a cycle, then T_i is the tree of $B_i - w$. For any vertex v with degree greater than 2 in B_i, we have that $deg_G(v) > 2$, then the color of w is different from all the other vertices with degree greater than 2 in B_i. By Theorem J, we have got that B_i is simple planar 1-cycle resonant, and $(B_i + P_i^*)_{(u_i,w)}$ is also simple planar 1-cycle resonant (since u_i and w have different colors).

Sufficiency. If G is a parallel-odd-chain, obviously it is simple 1-cycle resonant. Now let G be of the Structure 1, P the even length chain with two end vertices u_1 and u_2 in the Structure 1, and B_i a 2-connected structural-brick with two attachment vertices u_i, w, $i = 1, 2$, where w is the common vertex of B_1 and B_2. $(B_i + P_i^*)_{(u_i,w)}$ is simple planar 1-cycle resonant, $i = 1, 2$. By Theorem J, we show that G is simple planar 1-cycle resonant. For $i = 1, 2$, in $(B_i + P_i^*)_{(u,w)}$, u_i and w have degrees greater than 2 and are differently colored, as B_i is 2-connected and P_i^* is a chain of odd length. Since $(B_i + P_i^*)_{(u_i,w)}$ is simple planar 1-cycle resonant, by Theorem J, then one of u_i and w must be a cycle-common vertex. Further, we can claim that w is a cycle-common vertex of $(B_i + P_i^*)_{(u_i,w)}$. Otherwise, in $(B_i + P_i^*)_{(u_i,w)}$, there is a cycle containing u_i but w, this is in contradiction with $u_i \Rightarrow w$ in $(B_i + P_i^*)_{(u_i,w)}$. Let T_i be the tree of $(B_i + P_i^*)_{(u_i,w)} - w$, $i = 1, 2$, then $(T_1 - V_I(P_1^*)) \cup (T_2 - V_I(P_2^*)) \cup P$ is a tree of $G - w$. For any vertex v with degree greater than 2 in G, either $deg_{(B_1+P_1^*)_{(u_1,w)}}(v) > 2$ or $deg_{(B_2+P_2^*)_{(u_2,w)}}(v) > 2$. Then the color of w

is different from all the other vertices with degree greater than 2 in G. Since G is of the Structure 1, then G is bipartite and planar. So, by Theorem J, we have that G is simple planar 1-cycle resonant. ■

From Property H, we have the following properties, which is usefully for checking wether a vertex of a 2-connected graph G is cycle-related to another vertex of G.

Property 4. Let G be a 2-connected planar bipartite graph. Then

(1) $u \Rightarrow v$ in G if and only if $u \Rightarrow v$ in $(G + P^{*})_{(u,v)}$

(2) Let B be a 2-connected subgraph of G with exactly two attachment u and v. If $u \Rightarrow v$ in G, then $u \Rightarrow v$ in B. ■

Property 5. Let G be a 2-connected plane bipartite graph, C_0 the outer cycle of G, and $\{u, v\} \subseteq V(C_0)$. Let every bridge B of C_0 has exactly two attachment vertices, and B is not 2-connected. Then $u \Rightarrow v$ in G if and only if, for any bridge B of C_0 with an attachment vertex u, v is another one attachment vertex of B, and u is incident with only one edge in B. ■

For the sake of designing algorithm, we shall characterize a class of planar 2-cycle resonant graphs: G is a planar 2-cycle resonant graph, C is a cycle of G, and all the bridges of C attach to same pair of attachment vertices. At first, we will investigate the structure of a bridge of a cycle in planar 2-cycle resonant graph G. Let C be a cycle of G and B a bridge with attachment vertices u and v of C. By property 1, if B is not a path then $\overline{B}$ is a structural-brick in one of the Structures 1, 2, 3, 4. From Theorem K, we can determine the structure of B as following five types.

Type 1. B has exactly one chain P with even length induced by cut edges of B, one end vertex of which is u or v, say v. $B - (V_I(P) \cup \{v\})$ is the only 2-connected block B' of B, and $u \in V(B')$. Let w be the common vertex of P and $\overline{P}$ in B, then $w \Rightarrow u$ in B'.

Type 2. B has exactly one chain P with odd length induced by cut edges of B. If one of end vertices of P is u or v, say v, then $B - (V_I(P) \cup \{v\})$ has exactly two 2-connected blocks B_1 and B_2 with common vertex w. Let x be the end vertex of P distinct from v and $x \in V(B_1)$, then $u \in V(B_2)$, and $x \Rightarrow w$ in B_1, $u \Leftrightarrow w$ in B_2.

Type 3. B has exactly one chain P with odd length induced by cut edges of B. If both end vertices x and y of P are not u and v, then $B - V_I(P)$ has exactly two 2-connected blocks B_1 and B_2. Let $\{x, u\}$ and $\{y, v\}$ be attachment vertices sets of B_1 and B_2 respectively, then $x \Rightarrow u$ in B_1 and $y \Rightarrow v$ in B_2.

Type 4. B has at least two chains induced by cut edges of B. Then any two 2-connected blocks of B are disjoint, and the attachment vertices u and v of bridge B are not attachment vertex of 2-connected blocks of B. Moreover, any chain induced by cut edges of B is odd length.

Type 5. B has no chain induced by cut edges of B. Then B has exactly three 2-connected blocks, and the attachment vertices of each of them are mutually cycle-related in the block.

Moreover, for any 2-connected block B' with two attachment vertices x and y in bridges of the Types 1, 2, 3, 4, 5, $(B' + P^{\bullet})_{(x,y)}$ is either planar simple 1-cycle resonant or planar 2-cycle resonant.

Let G be a planar 2-connected graph and C a cycle of G, and all the bridges of C attach to same pair of attachment vertices u and v. If G is planar 2-cycle resonant, by Theorem K, G can be classified three kinds of graph as following:

Class 1. Any bridge of C is either a path or one of the Types 1, 2, 4, and one of u or v is incident with only one edge in every bridge of C. Moreover, if all bridges of C are of the Type 1 or paths, there is at least one 2-connected block B' with two attachment vertices u' and v' in bridges of C, such that $(B' + P^{\bullet})_{(u',v')}$ is planar 2-cycle resonant.

Class 2. There is exactly one bridge of the Type 3 on C, and any other bridge of C is either a path or a bridge of the Type 4.

Class 3. There is exactly one bridge of Type 5 on C, and any other bridge of C is either a path or a bridge of the Type 4.

In the light of above statements and Theorem K, we have the next theorem.

Theorem 6. Let G be a 2-connected graph and C a cycle of G, all the bridges of C attach to same pair of attachment vertices. Then G is planar 2-cycle resonant if and only if G is one of the Classes 1, 2, 3. Moreover, if G is of the Class 1 and all bridges of C are of the Type 1 or paths, there is at least one 2-connected block B' with two attachment vertices u' and v' in bridges of C, such that $(B' + P^{\bullet})_{(u',v')}$ is planar 2-cycle resonant. ■

Combining Theorem 6 and Property 5, we have following corollary.

Corollary 7. Let G be a planar 2-cycle resonant graph and C a cycle of G. If all bridges of C attach to same pair of attachment vertices u and v, then

(1) $u \Rightarrow v$ in G if and only if G is of the Class 1 and u is incident with only one edge in every bridge of C.

(2) $u \Leftrightarrow v$ in G if and only if G is the Class 1 and any bridge of C is the Type 4 or a path. ■

4 A Linear Algorithm for Recognizing Planar 2-Cycle Resonant Graphs

Let G be a 2-connected planar graph. Zhixia Xu and Xiaofeng Guo [38] have given the linear Algorithm F for deciding if G is 1-cycle resonant. We now establish a linear algorithm for recognizing planar 2-cycle resonant graphs. At first, we give the basic outline of a method how to determine wether or not G is planar 2-cycle resonant.

Basic outline: Let SB_0 be a 2-connected subgraph of G with exactly two attachment vertices u_0 and v_0 such that $\overline{SB_0}$ is neither a path nor 2-connected. Let SB_i be all the maximal 2-connected subgraphs of $\overline{SB_0}$ with two attachment vertices u_i and v_i, $1 \leq i \leq k$. Check wether or not G is one of the Structures 1, 2, 3, 4, in which SB_i are the structural-bricks, and wether or not u_i and v_i satisfy the cycle-related conditions required by the structure models, for $0 \leq i \leq k$. If not, then G is not 2-cycle resonant. If yes, then G can be decomposed into smaller 2-connected subgraphs SB_0, SB_1, SB_2, $\cdots$, SB_k. For $(SB_i + P_i^*)_{(u_i,v_i)}$, $0 \leq i \leq k$, we repeat to check them with the method above. If all the smaller 2-connected subgraphs obtained in certain decomposition course are parallel-odd-chains, then stop checking, and G is either 2-cycle resonant or Simple 1-cycle resonant. If the structure model in every decomposition is the Structure 1, by Property 3, we know that G is simple planar 1-cycle resonant. If there is a structure model in the decomposition process, that is not the Structure 1, by Theorem K, we have that G is planar 2-cycle resonant.

Based on the Theorem K, Property 3, and Theorem 6, we can establish a linear algorithm for recognizing planar 2-cycle resonant graphs.

Let C_0 be the outer cycle of a 2-connected plane bipartite graph G, and let every bridge of C_0 has exactly two attachment vertices that have different colors. Let P be a chain on C_0 with two end vertices u and v. If u and v are also two attachment vertices of a bridge, we call the chain P *bridge-chain* of outer cycle C_0 of G.

Let G be a 2-connected graph, and $u, v \in V(G)$. If u and v need not satisfy the cycle-related condition of $u \Rightarrow v$ or $v \Rightarrow u$, it is said that u and v are *cycle-related free* in G. For the convenience of designing the algorithm, we introduce a index $cr(G_{u,v})$ of illustrating cycle-related relation that u and v ought to satisfy in G, where u and v are two vertices appointed in $V(G)$, let $cr(G_{u,v}) = 1,\ 2,\ 3,\ 4$ mean by $u \Rightarrow v$, $v \Rightarrow u$, $u \Leftrightarrow v$, u and v are cycle-related free in G respectively.

Algorithm 8. Let G be a 2-connected plane bipartite graph with cyclomatic number $\nu(G) \geq 3$.

1. Embed G in the plane, and color all the vertices of G so that adjacent vertices get different colors. Set $G \in \mathcal{G}$, $\mathcal{G}_1 = \emptyset$, $\mathcal{G}_2 = \emptyset$, $cr(G_{u,v}) = 0$ and $2\text{-}cr = 0$.

2. Take out a graph G from $\mathcal{G}$. Let C_0 be the outer cycle of G. Find all the bridges $B_1, B_2, \cdots, B_r$ of the outer cycle C_0 of G and check whether any bridge of C_0 has exactly two different-colored attachment vertices. If not, go to step 12.

3. Find all the maximal 2-connected subgraphs $G_1, G_2, \cdots, G_t$ of the bridges of C_0 and check whether the attachment vertices u_i and v_i of G_i have different colors and are not both on $V(C_0)$. If not, go to step 12.

4. If $cr(G_{u,v}) \neq 0$, then:

(1) Check wether u and v avoid every bridge of outer cycle C_0 in G. If not, go to step 12.

(2) Check wether u and v satisfy the cycle-related condition corresponding to $cr(G_{u,v})$. If not, go to step 12.

(3) Set $G = (G + P^*)_{(u,v)}$, and let the outer cycle C_0 of G is the outer cycle of $(G + P^*)_{(u,v)}$, (that is, P^* is a bridge of C_0).

5. Let P be a bridge-chain of the outer cycle C_0 with end vertices x_0 and y_0 in G, and $B_1, B_2, \cdots, B_s$ all the bridges of C_0 with the two attachment vertices x_0 and y_0. Let $SB_0 = P \cup (\bigcup_{i=1}^{s} B_i)$ (then SB_0 is a 2-connected subgraph of G with exactly two attachment vertices x_0 and y_0).

6. If $\overline{SB_0}$ is not a path, then let SB_i be all the maximal 2-connected subgraphs of $\overline{SB_0}$ with two attachment vertices x_i and y_i, $1 \leq i \leq m$.

(1) Check wether G is one of the Structures 1, 2, 3, 4, in which SB_i are the structural-bricks, for $0 \leq i \leq m$. If not, go to step 12.

(2) For every one of the structural-bricks SB_i, $0 \leq i \leq m$, check wether the two attachment vertices x_i and y_i satisfy the cycle-related condition demanded by the structure model. If not, then go to step 12.

(3) If G is not the Structure 1, then set $2\text{-}cr = 1$.

(4) Set $SB_0 \cup C_0 \in \mathcal{G}_1$. Set $SB_i \cup C_0 \in \mathcal{G}_2$, for $1 \leq i \leq m$. (Then C_0 is the outer cycle of $SB_i \cup C_0$, and all bridges of outer cycle of $SB_i \cup C_0$ are the bridges of outer cycle C_0 of G, which attach to the segment of C_0 contained in SB_i. We take the segment of C_0 not contained in SB_i for a bridge-chain on C_0 in $SB_i \cup C_0$). Take out a graph G from

$\mathcal{G}_2$, return to step 5.

7. If $\overline{SB_0}$ is a path, then let $SB_0 \cup C_0 \in \mathcal{G}_1$. If $\mathcal{G}_2 \neq \emptyset$, then take out a graph from $\mathcal{G}_2$ and return to step 5. If $\mathcal{G}_2 = \emptyset$ and $\mathcal{G} \neq \emptyset$, then return to step 2.

8. Take out a graph G from $\mathcal{G}_1$ (the outer cycle of G is C_0, and all bridges of C_0 of G attach to same pair of attachment vertices x_0 and y_0). Check wether G is one of the Classes 1, 2, 3, or a parallel-odd-chain. If not, go to step 12.

9. If G is one of the Classes 1, 2, 3, then

(1) If there is a bridge of C_0 which is not the Type 1, then set $2\text{-}cr = 1$.

(2) For every maximal 2-connected subgraphs G_i with two attachment vertices u_i and v_i of the bridges of C_0 in G, set $G = G_i$, $u = u_i$, $v = v_i$, and $G \in \mathcal{G}$. Determine the value of $cr(G_{u,v})$ according to the cycle-related relation of u and v demanded by the type of the bridge in which G is contained.

10. If $\mathcal{G}_1 \neq \emptyset$, then return to step 8. If $\mathcal{G}_1 = \emptyset$ and $\mathcal{G} \neq \emptyset$, then return to step 2.

11. Stop. If $2\text{-}cr = 1$, then G is planar 2-cycle resonant. If $2\text{-}cr = 0$ then G is simple planar 1-cycle resonant.

12. Stop. G is neither simple planar 1-cycle resonant nor 2-cycle resonant.

Theorem 9. Algorithm 8 is linear with respect to the number p of vertices of G.

Proof: According to the basic outline of determining wether or not G is 2-cycle resonant, we designed the Algorithm 8. By Theorem K, Property 3 and Theorem 6, Algorithm 8 is valid. Next, we investigate the complexity of the algorithm.

Let G be a 2-connected planar bipartite graph with p vertices and q edges. Clearly, it takes $O(P)$ operations to embed G in the plane and to color the vertices of G in step 1.

The operations in steps 2, 3, and 4(1) are also the main body of Algorithm F, from Theorem G, we know that the total operation time of these operations is $O(P)$.

In steps 4(2) and 6(2), we have to determine cycle-related relation of two appointed vertices u and v. Before step 4, all bridges of outer cycle of G have already been determined. So, if we have to check wether $u \Rightarrow v$, by Property 5, we need only to check, for every bridge B with the attachment vertex u, wether v is another attachment vertex of B, and wether u is incident with only one edge in B. In the course of decomposing G in Algorithm 8, every structural-brick in a structure model with two attachment vertices u and v contains all bridges attaching to u and v, and u and v will require checking cycle-related relation demanded by structure model. There are at most three structural-bricks with same pair of attachment vertices u and v in the course of decomposing G. So for every bridge B of outer cycle of G, the boundary of B is checked at most three times in the operation of checking cycle-related relation of two appointed vertices.

In step 6(1), to determine wether G is one of the Structures 1, 2, 3, 4, we need only to find the boundary of the interior face f that is enclosed with the boundaries of SB_0 and $\overline{SB_0}$. The boundary of the interior face f consists of the boundaries of some bridges of C_0 and some segments of C_0. In the symmetric difference of C_0 and the boundary of interior face f, every cycle with exactly two attachment vertices is the boundary of a structural-brick SB_i. In step 8, to determine wether G is one of the Classes 1, 2, 3, or a parallel-odd-chain, we need only run along the boundary of every bridge of outer cycle of G. So, in the operations of determining the structure model and the class of G, for every bridge B of outer cycle of G, the boundary of B is checked at most two times.

Hence summing over all iterations, each edge of G is checked in steps 4, 6, 8 at most ten times.

Moreover, there is a kind of graphs $(G_i+P^*)_{(u_i,v_i)}$ in the operation course of Algorithm 8, where G_i is a 2-connected block of a bridge B of outer cycle of G with two attachment vertices u_i and v_i; P^* is a path with odd length, and $E(P^*) \not\subseteq E(G_i)$. The number of all 2-connected blocks in bridges of outer cycle of G is not greater than half of the number of edges that are contained in boundaries of all bridges of outer cycle of G, then the number of P^* appearing in the operation course of Algorithm 8 is at most $q/2$ times. P^* is a bridge of outer cycle of G_i, and P^* is checked at most seven times in all iterations. We can let $|E(P^*)| = 1$, that is, P^* is a additional edge. So the total operation time of checking these additional edge is $O(q) = O(p)$.

To sum up the above, we have got that Algorithm 8 is linear with respect to the number p of vertices of G. ■

If G is not 2-connected, and any two maximal 2-connected subgraphs of G have no common vertex, we can check every maximal 2-connected subgraphs of G by Algorithm 6. If every maximal 2-connected subgraph of G is either planar simple 1-cycle resonant or planar 2-cycle resonant, and the forest induced by all the vertices of G not in any maximal 2-connected subgraph of G has a perfect matching, then G is 2-cycle resonant.

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