MATCH
Communications in Mathematical and in Computer Chemistry

no. 50, February 2004
MATCDY (50) 117-132 (2004)

ISSN 0340 - 6253

BOUNDS ON THE WIENER NUMBER OF A GRAPH

H. B. WALIKAR[a], V. S. SHIGEHALLI[b], H. S. RAMANE[c]

[a]Department of Mathematics, Karnatak University's Post Graduate Centre, Post Bag No. 3, Belgaum – 590001, India, Department of Mathematics,
[b]Department of Mathematics, Basweshwar Engineering College, Bagalkot, India,
[c]Department of Master of Computer Applications, Gogte Institute of Technology, Udyambag, Belgaum – 590008, India.

(Received April 14, 2003)

ABSTRACT

The Wiener number $W(G)$ of a graph G was introduced by H. Wiener in connection with the modeling of various physico-chemical, biological and pharmacological properties of organic molecules in chemistry. The Wiener number $W(G)$ of a graph G is defined as the half of the sum of the distances between every pair of vertices of G. As such there is no exact formula to determine the value of $W(G)$, though there are some, for particular class of graphs. In this paper, we found some upper and lower bounds for $W(G)$, in terms of other graph-theoretic parameters, like radius, diameter, order, size, independence number, connectivity and chromatic number.

Corresponding Author: H. S. Ramane, E-mail: hsramane@yahoo.com

1. Introduction:

The Wiener number or the Wiener index $W = W(G)$ of a graph G was put forward in 1947 by Harald Wiener[1]. Its applications in the modeling of various physico-chemical, biological and pharmacological properties of organic molecules are outlined in several monographs[2 - 4] and reviews[5 - 9]. Wiener number is used in the study of ultrasonic sound velocities in alkanes and alcohols[10], rates of electroreduction of chlorobenzenes[11], cytostatic and antihistaminic activities of certain drugs[12], protonation constants of derivatives of salicylhydroxamic acid and their fungicidal activities[13], isomerism of fullerences[14].

Harald Wiener[1] defined the parameter $W(G)$ of a graph G when G represents the skeleton of carbon-carbon bonds between all pairs of carbon atoms of alkanes and he considered the Wiener number $W(G)$ to be the sum of the distances between the carbon-carbon bonds of alkanes and this parameter has been examined in various context namely Hosya[15] in 1971 and examined the relation between W and the distances in molecular graph. The molecular graph G, we mean the molecular structure of hydrocarbons by deleting the hydrogen atoms or simply the relationship between the carbon-carbon bonds of a molecular structure[4]. For example in the Fig. 1, gives the picture of a molecular structure of benzyl and the corresponding molecular graph.

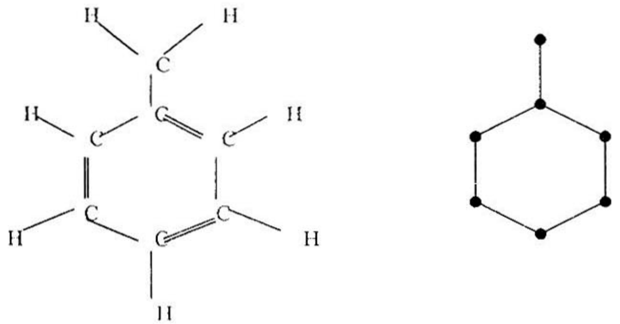

Fig. 1

Hence all molecular graphs are connected and hence, in general, throughout this paper we consider the connected graphs only while estimating its Wiener number.

As pointed out above, the Wiener number of a graph is studied in various context such as for predicting the boiling point of alkanes based on the formula: Boiling point = $\alpha W + \beta w(3) + \gamma$, where α, β, γ are empirical constants and $w(3)$ is the so-called path number, namely the number of pairs of vertices whose distance is equal to three. This type of applications has been studied in a series of papers[16 - 19] published during 1947 and 1948.

The Wiener number of a graph G is also considered independently by F. Harary[20] in sociometry, called total status of a graph denoted by $ts(G)$. If each vertex represents an individual in a society and two vertices are adjacent if there is a interpersonal relation exist between them. Hence a status of an individual or a vertex v in G denoted by $s(v)$ and is defined as the sum of the distances between all other vertices of G to v and hence in this context, the Wiener number $W(G)$ is going to be $(1/2)\sum s(v)$, where $v \in V(G)$.

As mentioned above, the concept of status has been introduced by F. Harary[20] and further studied by B. Zelinka[21] and R. C. Entringer, D. E. Jackson and D. E. Snyder[22]. Also it has been studied in the context of centrality of a graph, that is, the median graphs by G. Sabidussi[23] and G. S. Bloom, J. W. Kennedy and L. V. Quintas[24]. This parameter has also related to many other topological indices like Hyper Wiener index, Quassi Wiener, Kirchhoff indices, Harary indices, Szeiged indices, Cluj indices and many more. For details see[25].

Let $G = (V, E)$ be a graph on n vertices and m edges where V is vertex set and E be the edge set. Thus $|V| = n$ and $|E| = m$. A sequence of vertices $v_1, v_2, \ldots, v_k$ is said to be a path of length $k - 1$ on k vertices if v_iv_{i+1} is an edge in G for $i = 1, 2, \ldots, k - 1$. The distance between pair of vertices v_i and v_j is denoted by $d(v_i, v_j)$ is equal to the length of shortest path joining v_i and v_j in G. So the Wiener number $W(G)$ of a graph G is defined to be

$$W(G) = \sum_{i<j} d(v_i, v_j) \tag{1}$$

The Wiener number also be defined by considering so-called the distance matrix of a graph G denoted by $D(G)$ and $(i, j)^{th}$ entry in $D(G)$ is equal to $d(v_i, v_j)$. So the sum of the elements of i^{th} row of $D(G)$ is equal to

$$\sum_{j=1}^{n} d(v_i, v_j)$$

where n is the number of vertices in G and now $W(G)$ is turned out to be

$$(1/2) \sum_{i=1}^{n} \sum_{j=1}^{n} d(v_i, v_j)$$

The distance number of a vertex u of a graph G denoted by $d(u \mid G)$ and is defined as[4]

$$d(u \mid G) = \sum_{v \in V(G)} d(u, v) \tag{2}$$

Then
$$W(G) = (1/2) \sum_{u \in V(G)} d(u \mid G) \tag{3}$$

For example: Consider a graph G with vertices v_1, v_2, v_3 and v_4 as labeled in the Fig. 2.

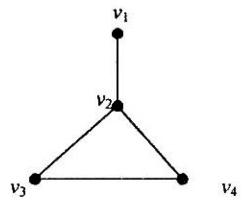

Fig. 2: G

Here $d(v_1, v_2) = 1$, $d(v_1, v_3) = 2$, $d(v_1, v_4) = 2$, $d(v_2, v_3) = 1$, $d(v_2, v_4) = 1$, $d(v_3, v_4) = 1$.

Therefore $W(G) = \sum_{i<j} d(v_i, v_j) = 1 + 2 + 2 + 1 + 1 + 1 = 8.$

Also $d(v_1 \mid G) = 1 + 2 + 2 = 5$, $d(v_2 \mid G) = 1 + 1 + 1 = 3$,

$d(v_3 \mid G) = 2 + 1 + 1 = 4$, $d(v_4 \mid G) = 2 + 1 + 1 = 4$

Therefore $W(G) = (1/2) \sum_{i=1}^{4} d(v_i \mid G) = (1/2)[5 + 3 + 4 + 4] = 8.$

The Wiener number of a graph has been calculated for particular class of graphs namely path P_n, cycle C_n, star $K_{1,n-1}$, complete bipartite graph $K_{r,s}$, complete graph K_n and many more. But as such there is no exact formula for finding a Wiener number of a general class of graphs, not even a recursive formula. But some recursive formulae can be obtained for certain class of trees and even a general class of trees. See[22, 26].

For any tree T on n vertices, the star $K_{1,n-1}$ has the lowest Wiener number and the path P_n has the largest Wiener number[22] and so we can conclude for any tree T on n vertices

$$W(K_{1,n-1}) \leq W(T) \leq W(P_n). \tag{4}$$

The question arise here what are the lower and upper bounds for the general class of graphs G. In this direction, the only known result can be found in [22] with the help of lower and upper bounds on a status $s(v)$ of a vertex v, that is, $n-1 \leq s(v) \leq ((n-1)(n-2)/2) - m$ by Entringer, Jackson and Snyder.

In this paper, we derive various bounds for $W(G)$ in terms of the number of vertices, the number of edges, radius, diameter, vertex connectivity, independence number and chromatic number. These terms will be defined as and when require. For any other terms, the reader may refer books[27, 28].

2. Bounds for $W(G)$ in terms of order, size, diameter and radius:

Let $G = (V, E)$ be a graph and v be any vertex in G. The degree of a vertex v in G is the number of edges incident to it and is denoted by $deg(v)$. The eccentricity $e(v)$ of a vertex v in G is defined to be

$$e(v) = \max\{d(u, v) \mid u \in V\} \tag{5}$$

The radius and diameter of a graph G are denoted by $rad(G)$ and $diam(G)$ and defined by

$$rad(G) = \min\{e(v) \mid v \in V\} \tag{6}$$

and

$$diam(G) = \max\{e(v) \mid v \in V\} \tag{7}$$

The following theorem gives the exact value of the Wiener number $W(G)$ in terms of its order and size, when $diam(G) \leq 2$.

Theorem 1: Let G be a graph of order n and size m. Then $W(G) = n^2 - n - m$ if and only if $diam(G) \leq 2$.

Proof: suppose G be a graph of order n and size m with $diam(G) \leq 2$. Define the sets $A = \{u \in V \mid e(u) = 1\}$ and $B = \{u \in V \mid e(u) = 2\}$. Then, $|A| + |B| = n$. If $u \in A$, then $d(u \mid G) = n - 1$ and if $u \in B$, then define two sets B_1 and B_2 as $B_1 = \{v \in V \mid d(u, v) = 1\}$ and $B_2 = \{v \in V \mid d(u, v) = 2\}$.

Then,
$$\begin{aligned}
d(u \mid G) &= |B_1| + 2|B_2| \\
&= |B_1| + |B_2| + |B_2| \\
&= n - 1 + (n - 1 - |B_1|) \qquad \text{since } |B_1| + |B_2| = n - 1 \\
&= 2n - 2 - |B_1| \\
&= 2n - 2 - deg(u).
\end{aligned}$$

Therefore,
$$\begin{aligned}
W(G) &= (1/2) \sum_{u \in V} d(u \mid G) \\
&= (1/2)\Big\{ \sum_{u \in A} d(u \mid G) + (1/2) \sum_{u \in B} d(u \mid G) \Big\} \\
&= (1/2)\{(n-1)|A| + (2n - 2 - deg(u))|B|\} \\
&= (1/2)\Big\{(n-1)|A| + (2n-2)|B| - \sum_{u \in B} deg(u) \Big\} \\
&= (1/2)\Big\{(n-1)(|A| + |B|) + (n-1)|B| - \sum_{u \in B} deg(u) \Big\} \\
&= (1/2)\Big\{n(n-1) + (n-1)(n - |A|) - \sum_{u \in B} deg(u) \Big\} \\
&= (1/2)\Big\{n(n-1) + n(n-1) - (n-1)|A| - \sum_{u \in B} deg(u) \Big\} \\
&= (1/2)\Big\{2n(n-1) - \sum_{u \in A} deg(u) - \sum_{u \in B} deg(u) \Big\}. \\
&= (1/2)\Big\{2n(n-1) - \Big(\sum_{u \in A} deg(u) + \sum_{u \in B} deg(u) \Big) \Big\}. \\
&= (1/2)\Big\{2n(n-1) - \sum_{u \in V} deg(u) \Big\}.
\end{aligned}$$

$$= (1/2)\{2n(n-1) - 2m\} \qquad \text{since} \quad \sum_{u \in V} deg(u) = 2m.$$

$$= n^2 - n - m$$

which proves the one part of the proof.

On the other hand, suppose that $W(G) = n^2 - n - m$. We prove that $diam(G) \leq 2$. For, if $diam(G) = k \geq 3$ and let $u \in V$ be an arbitrary vertex in G, then define the sets A and B as $A = \{u \in V \mid e(v) = 2\}$ and $B = \{u \in V \mid e(v) \geq 3\}$, where $|A| + |B| = n$. If $u \in A$, as in the proof of the above part, we can prove that $d(u|G) = 2n - 2 - deg(u)$. If $u \in B$, define three sets B_1, B_2 and B_3 as follows:

$B_1 = \{v \in V \mid d(u, v) = 1\}$, $B_2 = \{v \in V \mid d(u, v) = 2\}$ and $B_3 = \{ v \in V \mid d(u, v) \geq 3\}$.

Clearly, $|B_1| + |B_2| + |B_3| = n - 1$.

$$\begin{aligned}
\therefore d(u|G) &\geq |B_1| + 2|B_2| + 3|B_3| \\
&= |B_1| + |B_2| + |B_3| + |B_2| + 2|B_3| \\
&= n - 1 + |B_2| + |B_3| + |B_3| \\
&= (n-1) + (n - 1 - |B_1|) + |B_3| \\
&= 2n - 2 - deg(u) + |B_3| \\
&\geq 2n - 2 - deg(u) + 1 \qquad \text{since } |B_3| \geq 1 \\
&= 2n - 1 - deg(u).
\end{aligned}$$

$$\begin{aligned}
\therefore W(G) &= (1/2) \sum_{u \in V} d(u\,|G) \\
&= (1/2) \sum_{u \in A} d(u\,|G) + \sum_{u \in B} d(u\,|G) \\
&\geq (1/2)\{(2n - 2 - deg(u))|A| + (2n - 1 - deg(u))|B|\} \\
&= (1/2)\Big\{(2n-2)(|A| + |B|) - \sum_{u \in A} deg(u) + \sum_{u \in B} deg(u\} + |B|\Big\} \\
&= (1/2)\Big\{2n(n-1) - \sum_{u \in V} deg(u) + |B|\Big\} \\
&= (1/2)\{2n(n-1) - 2m + |B|\} \\
&= n(n-1) - m + (1/2)|B|
\end{aligned}$$

$\geq n(n-1) - m + 1$, as $|B| \geq 2$, since there exists at least two vertices of eccentricity greater than or equal to three, as $diam(G) \geq 3$.

$\therefore\ W(G) \geq n^2 - n - m + 1$, a contradiction to the fact that $W(G) = n^2 - n - m$, which completes the proof.

Theorem 2: For any graph G of order n, size m with $diam(G) \geq 3$,

$$W(G) \geq n^2 - n - m + 1 \tag{8}$$

holds. Further, the equality holds, if G contains exactly two vertices of eccentricity three and rest are of eccentricity two.

Proof: The proof follows from the proof of Theorem 1.

Theorem 3: Let G be a graph of order n, size m with $diam(G) = rad(G) = 3$, then

$$W(G) \leq (1/2)n(3n-5) - 2m \tag{9}$$

Proof: Suppose G be a graph with $diam(G) = rad(G) = 3$, then $e(u) = 3$ for every vertex u in G. Define the sets $A_i(u) = \{v \in V \mid d(u, v) = i\}$ for $i = 0, 1, 2, 3$.

Clearly, $\left|\bigcup_{i=0}^{3} A_i(u)\right| = n$.

$$\therefore\ |A_2(u)| + |A_3(u)| = n - 1 - deg(u) \tag{10}$$

Since $|A_0(u)| = 1$ and $|A_1(u)| = deg(u)$.

Also, $|A_2(u)| \geq 2$, for, otherwise, there is a vertex $w \in A_2(u)$ such that $e(w) \leq 2$, a contradiction. Thus,

$$\begin{aligned} d(u|G) &= \sum_{v \in V} d(u, v) \\ &= |A_1(u)| + 2|A_2(u)| + 3|A_3(u)| \\ &= deg(u) + 2(|A_2(u)| + |A_3(u)|) + |A_3(u)| \end{aligned} \tag{11}$$

But, $|A_3(u)| = n - 1 - deg(u) - |A_2(u)|$, from (10)

$\leq n - 1 - deg(u) - 2$, since $|A_2(u)| \geq 2$.

$= n - 3 - deg(u)$.

Therefore (11) becomes,

$$\begin{aligned} d(u|G) &\leq deg(u) + 2(n - 1 - deg(u)) + (n - 3 - deg(u)) \\ &= 3n - 5 - 2deg(u). \end{aligned}$$

Thus, $$W(G) = (1/2) \sum_{u \in V} d(u \mid G)$$

$$= (1/2) \sum_{u \in V} (3n - 5 - 2deg(u))$$

$$= (1/2) \left[n(3n - 5) - 2 \sum_{u \in V} deg(u)\right]$$

$$= (1/2)n(3n - 5) - 2m.$$

This completes the proof.

Theorem 4: Let G be a graph of order n, size m and $diam(G) = rad(G) = k \geq 3$, then

$$W(G) \geq n^2 - n - m + (n(k-2)^2)/2 \tag{12}$$

Proof: Suppose $diam(G) = rad(G) = k$. Let u be an arbitrary vertex in G and let us define the sets $A_i(u)$ as $A_i(u) = \{v \in V \mid d(u, v) = i\}$, for $i = 0, 1, 2, \ldots, k$.

Therefore $$d(u \mid G) = |A_1(u)| + 2|A_2(u)| + 3|A_3(u)| + \ldots + k|A_k(u)|$$

$$= deg(u) + 2(|A_2(u)| + |A_3(u)| + \ldots + |A_k(u)|)$$
$$+ |A_3(u)| + 2|A_4(u)| + \ldots + (k-2)|A_k(u)|)$$

$$= deg(u) + 2(n - 1 - |A_1(u)| + |A_3(u)| + 2|A_4(u)|)$$
$$+ \ldots + (k-2)|A_k(u)|)$$

$$= deg(u) + 2(n - 1 - deg(u)) + |A_3(u)| + 2|A_4(u)|$$
$$+ \ldots + (k-3)|A_{k-1}(u)| + (k-2)|A_k(u)|)$$

$$\geq 2(n-1) - deg(u) + 2 + (2)(2) + (3)(2) + \ldots + (k-3)(2) + (k-2)(1)$$

Since $|A_i(u)| \geq 2$, for $i = 2, 3, \ldots, k-1$ and $|A_k(u)| \geq 1$.

Therefore $$d(u \mid G) \geq 2(n-1) - deg(u) + 2((k-3)(k-2)/2) + k - 2$$
$$= 2(n-1) - deg(u) + (k-2)^2.$$

Therefore $$W(G) = (1/2) \sum_{u \in V} d(u \mid G)$$

$$\geq (1/2) \sum_{u \in V} [2(n-1) - deg(u) + (k-2)^2]$$

$$= (1/2)(n)(2)(n-1) - m + (1/2)n(k-2)^2$$

$$= n^2 - n - m + (1/2)n(k-2)^2$$

Hence the inequality.

Remark: The bounds in the Theorem 3 and 4 are sharp, in the sense that the upper bound in the Theorem 3 is attainable, if $G = C_6$ and the lower bound in the Theorem 4 is attainable, if $G = C_n$, for n being even.

3. Bounds for $W(G)$ in terms of the vertex connectivity, independence number and chromatic number:

The vertex connectivity $k(G)$ of a graph G is defined to be the minimum number of vertices whose removal from G results into a disconnected or a trivial graph. A graph $G = G_1 + G_2 + \ldots + G_k$ is the graph obtained by joining each vertex of G_i to all vertices of G_{i+1}, $1 \le i \le k$ (see [27]).

Theorem 5: Let G be a graph of order n, connectivity $k(G)$ and $H_1, H_2, \ldots, H_t$ be the connected components of $G - S$, where $|S| = k(G)$, then

$$W(G) \ge n(n-1)/2 + l(n - l - k(G)) \tag{13}$$

Where $l = \min\limits_{1 \le i \le t}\{|V(H_i)|\}$

Further, the inequality (13) holds if and only if $G = K_l + K_k + K_{n-l-k}$, where $k = k(G)$.

Proof: Let G be a graph with n vertices and S be any cut set of G with $|S| = k(G)$. Let $H_1, H_2, \ldots, H_t$ be the connected components of $G - S$ with $l = \min\limits_{1 \le i \le t}\{|V(H_i)|\}$

Without loss of generality, assume that $|V(H_1)| = l$, $G_1 = H_1$ and $G_2 = \bigcup\limits_{i=2}^{t} H_i$.

Then, $|V(G_1)| = l$ and $|V(G_2)| = n - k - l$. Now we have,

$$W(G) = (1/2) \sum_{u \in V} d(u|G)$$

$$= (1/2)\Big[\sum_{u \in V(G_1)} d(u|G) + \sum_{u \in S} d(u|G) + \sum_{u \in V(G_2)} d(u|G)\Big] \tag{14}$$

Now we consider the following three cases, for any arbitrary vertex u in G.

<u>Case 1</u>: Let $u \in V(G_1)$.

Then, $d(u \mid G) = \sum_{v \in V(G)} d(u, v)$

$$= \sum_{v \in V(G_1)} d(u, v) + \sum_{v \in S} d(u, v) + \sum_{v \in V(G_2)} d(u, v)$$

$$\geq (l - 1) + k(G) + 2(n - l - k(G))$$

$$= 2n - l - k - 1.$$

Since $d(u, v) \geq 1$, if $v \in V(G_1)$, $v \in S$ and $d(u, v) \geq 2$, if $v \in V(G_2)$.

Case 2: Let $u \in S$.

Then, $d(u \mid G) = \sum_{v \in V(G)} d(u, v)$

$$= \sum_{v \in V(G_1)} d(u, v) + \sum_{v \in S} d(u, v) + \sum_{v \in V(G_2)} d(u, v)$$

$$\geq l + k - 1 + n - l - k$$

$$= n - 1.$$

Since $d(u, v) \geq 1$, if v is in either sets $V(G_1)$, S and $V(G_2)$.

Case 3: Let $u \in V(G_2)$, then we can prove that $d(u \mid G) \geq n + l - 1$ as in the Case 1.

Thus, by (14) we have

$$W(G) \geq (1/2)[(2n - l - k - 1)l + (n - 1)k + (n + l - 1)(n - l - k)]$$
$$= (n(n - 1)/2) + l(n - l - k(G)).$$

The second part of the theorem follows from the proof of the inequality itself.

A subset S of a vertex set $V(G)$ of a graph G is said to be an independent set, if no two vertices of S are adjacent in G. The independence number $\beta_0(G)$ of G is the maximum number of vertices in the independent sets in G. The following Theorem will gives the lower bound for $W(G)$ in terms of the order of G and the independence number $\beta_0(G) = \beta_0$.

Theorem 6: Let G be any connected graph of order n, then

$$W(G) \geq (1/2)[n(n - 1) + \beta_0(\beta_0 - 1)] \tag{15}$$

The equality holds if and only if $G = \overline{K}_{\beta_0} + K_{n - \beta_0}$.

Proof: Let S be the maximum independent set with $|S| = \beta_0$ and u_i be any vertex in S.

Then, $$d(u_i \mid G) = \sum_{u_j \in V(G)} d(u_i, u_j)$$

$$= \sum_{u_j \in S} d(u_i, u_j) + \sum_{u_j \in V-S} d(u_i, u_j)$$

$$\geq 2(\beta_0 - 1) + (n - \beta_0)$$

$$= n - \beta_0 - 2. \qquad (16)$$

Since $u_i \neq u_j$ and $u_i \in S$, so that there are $(\beta_0 - 1)$ vertices in S which are at a distance at least two from u_i and $d(u_i, u_j) \geq 1$, for any $u_j \in V - S$.

Next, let $u_i \in V - S$, then

$$d(u_i \mid G) = \sum_{u_j \in V(G)} d(u_i, u_j)$$

$$\geq n - 1 \qquad (17)$$

Therefore $$W(G) = (1/2) \sum_{u_i \in V(G)} d(u_i \mid G)$$

$$= (1/2) \Big[\sum_{u_i \in S} d(u_i \mid G) + \sum_{u_i \in V-S} d(u_i \mid G) \Big]$$

$$\geq (1/2)[\beta_0(n + \beta_0 - 2) + (n - \beta_0)(n - 1)] \qquad \text{from (16) and (17)}$$

$$= (1/2)[n(n-1) + \beta_0(\beta_0 - 1)].$$

Further, if $G = \bar{K}_{\beta 0} + K_{n - \beta 0}$, it is not difficult to see that $W(G) = (1/2)[n(n - 1) + \beta_0(\beta_0 - 1)]$.

Conversely, suppose $W(G) = (1/2)[n(n - 1) + \beta_0(\beta_0 - 1)]$.

We prove that $G = \bar{K}_{\beta 0} + K_{n - \beta 0}$. If possible assume that $G \neq \bar{K}_{\beta 0} + K_{n - \beta 0}$. Let S be the maximum independent set with $|S| = \beta_0$ in G. For any two vertices u and v in G, $d(u, v) = 2$ if both u and v are in S and $d(u, v) = 1$ if both u and v are in $V - S$, otherwise, it will lead to $W(G) > (1/2)[n(n - 1) + \beta_0(\beta_0 - 1)]$, a contradiction. Thus $\langle S \rangle = \bar{K}_{\beta 0}$ and $\langle V - S \rangle = K_{n - \beta 0}$.

Further, if $u \in S$ and $v \in V - S$, we claim that $d(u, v) = 1$, for, otherwise

$$\sum_{u \in S} d(u \mid G) > n - \beta_0.$$

and thereby $W(G) > (1/2)[n(n - 1) + \beta_0(\beta_0 - 1)]$ holds, a contradiction.

Thus $G = \bar{K}_{\beta 0} + K_{n - \beta 0}$ holds.

Next Theorem deals with the lower bound on $W(G)$ in terms of the number of vertices and the chromatic number of G.

A coloring of a graph G is an assignment of colors to its vertices in such a way that no two adjacent vertices have the same color. The minimum number of colors in any coloring of G, is called the chromatic number of G and is denoted by $\chi(G)$. The set of all vertices with any one color in a coloring of G is independent and is called the color class of G. A graph $G = K_{n1,\, n2,\, \dots,\, nk}$ is said to be complete bipartite graph if its vertex $V(G)$ can be partitioned into disjoint sets $V_1, V_2, \dots, V_k$ where $|V_i| = n_i$, such that no two vertices in any V_i, $1 \le i \le k$ are adjacent and each vertex of V_i is adjacent to all vertices of V_j, $1 \le i, j \le k$ (for details, see [27]).

Theorem 7: Let G be any connected graph of order n with chromatic number $\chi(G) = t$, then

$$W(G) \ge (1/2)[n(t+1) - 2t] \tag{18}$$

Further the equality holds if and only if $G = K_{n1,\, n2,\, \dots,\, nt}$.

Proof: Suppose $\chi(G) = t$, then the vertex set $V(G)$ of G can be partitioned into t color classes $C_1, C_2, \dots, C_t$ such that no two vertices in any C_i are adjacent and let $|C_i| = n_i$, for $i = 1, 2, \dots, t$.

Thus $\quad n = \sum_{i=1}^{t} n_i.$

Let $u \in C_i$, for $i = 1, 2, \dots, t$.

Then,
$$\begin{aligned} d(u \mid G) &= \sum_{v \in V(G)} d(u, v) \\ &= \sum_{v \in C_i} d(u, v) + \sum_{v \in V - C_i} d(u, v) \\ &\ge 2(n_i - 1) + (n - n_i) \\ &= n - n_i - 2. \end{aligned}$$

Since $d(u, v) \ge 2$, if $v \in C_i$ and $d(u, v) \ge 1$, if $v \in V - C_i$.

Therefore, $\quad W(G) = (1/2) \sum_{u \in V(G)} d(u \mid G)$

$$= (1/2)\Big[\sum_{i=1}^{t} \sum_{u_i \in C_i} d(u|G)\Big]$$

$$\geq (1/2) \sum_{i=1}^{t} (n + n_i - 2)$$

$$= (1/2)\Big[\, nt + \sum_{i=1}^{t} n_i - 2t \,\Big]$$

$$= (1/2)[nt + n - 2t]$$

$$= (1/2)[n(t + 1) - 2t].$$

Further, if G if $G = K_{n1,\, n2,\, \ldots,\, nt}$, then, it is not difficult to see that $W(G) = (1/2)[n(t + 1) - 2t]$. On the other hand, if $W(G) = (1/2)[n(t + 1) - 2t]$ and $\chi(G) = t$, then the vertex set $V(G)$ can be partitioned into the color classes $C_1, C_2, \ldots, C_t$ such that $|C_i| = n_i$, for $i = 1, 2, \ldots, t$. Now, we claim that any two vertices u and v belonging to two different color classes are adjacent. For, if $u \in C_i$ and $v \in C_j$, for $i \neq j$ are not adjacent,

then $\sum_{v \in V - C_i} d(u, v) > n - n_i$, which in turn implies that

$d(u|G) > n + n_i - 2$ and thereby it will lead to $W(G) > (1/2)[n(t + 1) - 2t]$, a contradiction. Again, if both u and v belongs to the same color class then $d(u, v) = 2$, otherwise, it leads to the same contradiction. Hence $G = K_{n1,\, n2,\, \ldots,\, nt}$ holds.

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