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The (m,k)-patch boundary code problem

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Abstract

A simple (non overlapping) region of the hexagonal tessellation of the plane is uniquely determined by its boundary. This seems also to be true for "regions" that curve around and have a simple overlap. However, Guo, Hansen and Zheng [3] constructed a pair of non isomorphic (self-overlapping) regions of the hexagonal tessellation which have the same boundary. These regions overlapped themselves several times. In this paper we prove that any region not uniquely determined by its boundary must cover some point three or more times.

Benzenoid hydrocarbons or polyhexes have been studied extensively and a rich variety of mathematical questions have arisen in the course of investigating these structures. There are several different methods for coding the boundary of a polyhex, as cyclic sequences of numbers. Independent of the coding method, it is natural to ask the following two questions. Which sequences are the boundary codes of some polyhex and, and for those that are, do they uniquely determine that polyhex? This paper is concerned with the second question. This second question was answered in the negative by Guo, Hansen and Zheng [3]. Their result then leads to the problem of deciding whether or not the boundary sequence of a given polyhex uniquely determines that polyhex. In this paper, we show that, if the natural projection of the polyhex into the hexagonal tessellation of the plane does not triple cover some point, then the polyhex is uniquely determined by its boundary sequence.

These questions about polyhexes were actually considered in a broader context in [1]. In that paper, Brinkmann, Friedrichs and Nathusius introduced the term (m,k)-patch; in this paper, we expand their definition slightly. Let m and k be integers greater than 2. By an *(m,k)-patch*, we mean a connected, plane graph Γ with a distinguished face Δ whose boundary vertices have been indexed counterclockwise around that face by the integers $1,\ldots,n$ and a sequence of non negative integers $r_1,\ldots,r_n$ such that the following conditions hold:

- all faces, other than Δ, are k-gons;
- for each vertex v, $\rho(v)+\sum_{j=1}^{h} r_{i_j} = m$, where $\rho(v)$ is the degree of v and $i_1\ldots i_h$ is the complete list of boundary indices assigned to v;
- all articulation vertices (cut vertices) lie on the boundary of Δ.

Note that, if v does not lie on the boundary of Δ, the list of boundary indices assigned to it is empty and $\rho(v)=m$. Note further that, if a boundary vertex v is assigned only one index, say i, then $r_i = m-\rho(v)$. Finally, note that a boundary vertex is assigned more than one index if and only if it is an articulation vertex. Thus, the values of the boundary sequence $r_1,\ldots,r_n$ are redundant bits of information, except at the articulation vertices of the patch.

Grünbaum and Shephard [2] showed that, for any $m,k>2$, there is a unique finite or infinite, one ended, edge-transitive plane graph with all vertex valences equal to m and all face valences equal to k. We denote this graph by $\Lambda_{(m,k)}$. When $(\frac{(m-2)(k-2)}{2}-2)<0$, $\Lambda_{(m,k)}$ is one of the Platonic graphs and, when $(\frac{(m-2)(k-2)}{2}-2)\geq 0$, $\Lambda_{(m,k)}$ is infinite. By a *drawing* of an (m,k)-patch $P=(\Gamma,\Delta,r_1,\ldots,r_n)$, we mean a graph homomorphism $\phi:\Gamma\to\Lambda_{(m,k)}$ that is a local graph isomorphism (one to one on vertex neighborhoods) and satisfies the following condition: let x_i denote the image under ϕ of the vertex on Δ indexed by i; then, for each index i, r_i is the number of edges of $\Lambda_{(m,k)}$, with end point x_i, between the edges (x_{i-1},x_i) and (x_i,x_{i+1}) in the clockwise direction. All parts of this first theorem were proved by Brinkmann, Friedrichs and Nathusius [1] for (m,k)-patches as they defined them. Their proofs easily extend to the more general patches we have defined. However, for the sake of completeness, we reprove their results here.

Theorem 1 *Let $P=(\Gamma,\Delta,r_1,\ldots,r_n)$ be an (m,k)-patch. Then*

i. *P has a drawing in $\Lambda_{(m,k)}$.*

ii. *That drawing is unique up to an automorphism of $\Lambda_{(m,k)}$.*

iii. *Furthermore, the image of Δ is uniquely determined up to an automorphism of $\Lambda_{(m,k)}$ by the boundary sequence alone.*

iv. *Finally,*

$$r_1+\ldots+r_n = (\frac{m-2}{2})n + m + (\frac{(m-2)(k-2)}{2}-2)f_k,$$

where f_k is the number of k-covalent faces of Γ.

PROOF: We proceed by induction on the number of edges in Γ. The conclusions are easily checked for the trivial patch consisting of a single edge.

Assume then that Γ has at least two edges. Suppose first that Γ is a tree and that the vertex with index i is a pendant vertex. Let Γ' denote the graph obtained by deleting this vertex and its attaching edge. eliminate the indices i and $i+1$ and then reduce each boundary vertex index by 2 from $i+2$ on. Define $r'_j = r_j$, for $j = 1, \ldots, i-2$; define $r'_{i-1} = r_{i-1} + r_{i+1} + 1$ and define $r'_j = r_{j+2}$, for $j = i, \ldots, n-2$. One easily sees that $P' = (\Gamma', \Delta', r'_1, \ldots, r'_{n-2})$ is also an (m,k)-patch. By the induction hypothesis, the four conclusions hold for P'. And, since the pendant vertex can be reattached in just one way and be consistent with the boundary sequence $r_1, \ldots, r_n$, the first three conclusions hold for P as well. Since neither Γ nor Γ' have any k-faces, we have:

$(\frac{m-2}{2})(n-2) + m = r'_1 + \ldots + r'_{(n-2)} = r_1 + \ldots + r_n + 1 - r_i = r_1 + \ldots + r_n + 1 - (m-1)$.
Hence $r_1 + \ldots + r_n = (\frac{m-2}{2})(n-2) + m + (m-2) = (\frac{m-2}{2})(n) + m$, as required.

Now suppose that Γ is not a tree. Then there is an edge in the boundary of Δ that also bounds a k-face. Select and fix such an edge and k-face. Let i and $i+1$ be the indices of the selected edge. Let Γ' denote the graph obtained by deleting this edge. Δ' is now the union of Δ and the selected k-face. We must insert indices for the vertices around the inside of the deleted k-face and adjust the boundary sequence: define $r'_j = r_j$, for $j = 1, \ldots, i-1$; $r'_i = r_i + 1$; $r'_j = 0$, for $j = i+1, \ldots, i+k-2$; $r'_{i+k-1} = r_{i+1} + 1$; $r'_j = r_{j-k+2}$, for $j = i+k, \ldots, n+k-2$. One easily sees that $P' = (\Gamma', \Delta', r'_1, \ldots, r'_{n+k-2})$ is an (m,k)-patch.

Again, by the induction hypothesis, the four conclusions hold for P'. Consider a drawing of P' and note that, since $r'_j = 0$, for $j = i+1, \ldots, i+k-2$, the images in $\Lambda_{(m,k)}$ of the vertices indexed $i, \ldots, i+k-1$ are the vertices of a face. Hence the edge joining the images of the vertices indexed i and $i+k-1$ may be added to get the required drawing of P and it easily follows that the first three conclusions hold for P. Since one k-covalent face has been eliminated, we have $f'_k = f_k - 1$. Thus

$$r'_1 + \ldots + r'_{n+k-2} = (\tfrac{m-2}{2})(n+k-2) + m + (\tfrac{(m-2)(k-2)}{2} - 2)(f_k - 1).$$

We also have

$$r'_1 + \ldots + r'_{n+k-2} = r_1 + \ldots + r_n + 2.$$

Hence:

$$\begin{aligned} r_1 + \ldots + r_n &= (\tfrac{m-2}{2})(n+k-2) + m + (\tfrac{(m-2)(k-2)}{2} - 2)(f_k - 1) - 2 \\ &= (\tfrac{m-2}{2})n + m + (\tfrac{(m-2)(k-2)}{2} - 2)f_k, \end{aligned}$$

as required.□

We will have need of the following corollary later.

Corollary 1 *If* $P = (\Gamma, \Delta, r_1, \ldots, r_n)$ *and* $P' = (\Gamma', \Delta', r'_1, \ldots, r'_n)$ *are* (m,k)-*patches with* $r'_i = r_i$ *for* $i = 1, \ldots, (n-1)$, *then* $r'_n = r_n$.

PROOF: Consider drawings of both P and P'. Since $r'_1 = r_1$, we may assume that the images of the vertices indexed n, 1 and 2 coincide. But then, by Part (iii) of the theorem the images of the vertices indexed i coincide, for all i. It follows that $r'_n = r_n$.□

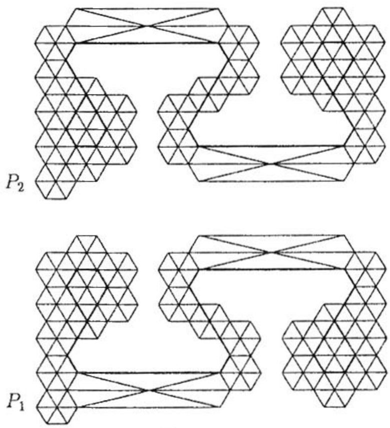

Figure 1:

A simple (non overlapping) region of $\Lambda_{(m,k)}$ is uniquely determined by its boundary. Thus, if the drawing of an (m, k)-patch is one to one, then the patch is uniquely determined by its boundary sequence. This seems also to be true if the drawing of the patch curves around and has a simple overlap. It is natural to ask if every (m, k)-patch is uniquely determined by its boundary sequence. That the answer to this question is "no" was demonstrated by Guo, Hansen and Zheng [3] with their construction of two non-isomorphic (3,6)-patches having the same boundary sequence. In Figure 1, we have drawn a triangulated version (two (6,3)-patches) of their example. One feature of this example is that, when drawn in $\Lambda_{(m,k)}$, the image has a double overlap; that is, some point of $\Lambda_{(m,k)}$ is covered three times. The main result of this paper is that this is true of all such examples.

A sequence $r_1, \ldots, r_n$ is an (m, k)-*boundary sequence* if there is an (m, k)-patch with it as boundary sequence. An (m, k)-boundary sequence $r_1, \ldots, r_n$ is said to be *ambiguous* if there exist two (m, k)-patches, with that boundary sequence, that admit no isomorphism preserving the boundary sequence. Note, if there exist two (m, k)-patches with the same boundary sequence that admit no isomorphism preserving boundary sequence, then, by adding one more k-face, we can construct two (m, k)-patches with the same boundary sequence which admit no isomorphism whatsoever. The Guo-Hansen- Zheng example was constructed this way. If the lower left hand hexagons are removed from both of the patches in Figure 1 and the resulting patches have their boundary vertices indexed clockwise starting from the lower left, then the two patches are isomorphic but there is no isomorphism that preserves the boundary sequence. Once the hexagon is replaced, the patches are no longer isomorphic.

Theorem 2 *Let $P = (\Gamma, \Delta, r_1, \ldots, r_n)$ be an (m,k)-patch with $(\frac{(m-2)(k-2)}{2} - 2) \geq 0$ and with an ambiguous boundary sequence. Let ϕ be any drawing of P. Then there exist three distinct vertices of Γ that are mapped by ϕ onto the same vertex in $\Lambda_{(m,k)}$.*

PROOF: Let $P_i = (\Gamma_i, \Delta_1, r_1, \ldots, r_n)$, for $i = 1,2$ be two (m,k)-patches with the same boundary sequence and assume that they admit no isomorphism which preserves the boundary indices. Let ϕ_i, $i = 1,2$ be drawings of these two patches. We wish to show that each of these drawings triple covers some point in its image. Proceeding by induction on the number of edges in the patch, we note that result holds vacuously for all patches with a small number of edges.

Suppose that Γ has a pendant vertex. Note that the vertex with index i is a pendant vertex if and only if $r_i = m - 1$. Thus any pendant vertex in one of the patches corresponds to a pendant vertex in the other. Removing these pendant vertices simultaneously results in a smaller pair of patches with identical boundary sequences. Furthermore, any boundary index preserving isomorphism between the smaller patches extends to a boundary index preserving isomorphism between the original patches. Thus, by the induction hypothesis, each of the induced drawings of these smaller patches cover some point of $\Lambda_{(m,k)}$ three times. Hence the original drawings cover some point of $\Lambda_{(m,k)}$ three times.

Next suppose that the boundary edge between the vertices indexed i and $i+1$ bounds a k-face in both patches. Then removing that edge and reindexing (as described above) results in a smaller pair of patches with identical boundary sequences. Again, any boundary index preserving isomorphism between the smaller patches extends to a boundary index preserving isomorphism between the original patches and, invoking the induction hypothesis, we conclude that each of the original patches covers some point of $\Lambda_{(m,k)}$ three times.

Now suppose that there are two boundary indices that, in each patch, index the same articulation vertex. Without loss of generality we assume these indices are 1 and i. It follows that a boundary vertex of Γ indexed by $j \in \{2, \ldots, i-1\}$ and a boundary vertex with index in $\{i+1, \ldots, n\}$ must lie in different lobes of Γ. Thus, the subgraph of Γ obtained by deleting all lobes containing a boundary vertex indexed by $\{i+1, \ldots, n\}$ (but retaining the articulation vertex), yields a smaller patch with its distinguished face bounded by the portion of boundary of Δ with vertices indexed by $\{1, \ldots, i-1\}$ and with $\hat{r}_1, r_2 \ldots, r_{i-1}$ as boundary sequence where $\hat{r}_1$ represents the appropriate adjustment at the articulation vertex. Clearly then, each patch consists of two smaller patches joined at the specified articulation vertex.

Consider the smaller patches with boundary indexed by $1, \ldots, i-1$. Clearly their boundary sequences match for the vertices indexed $2, \ldots, i-1$; that they also agree at 1 follows from Corollary 1. The same will be true for the other pair of smaller patches. Boundary index preserving isomorphisms between corresponding smaller patches combine to give a boundary index preserving isomorphism between the original patches. Hence, at least one of the pairs of smaller patches are non isomorphic and have a common ambiguous boundary sequence. Each of these and, thus, each of the original patches covers some point of $\Lambda_{(m,k)}$ at least three times.

Thus, we may assume that our patches have no pendant vertices, no articulation vertices with corresponding indices and that the boundaries have no corresponding edges that bound k-faces in both patches. We have reduced the pair of patches in Figure 1 by simultaneously throwing out corresponding boundary edges that bound k-faces in both patches and corresponding pendant vertices and attaching edges. The resulting reduced patches are indicated by the heavy edges in Figure 1 and are redrawn in Figure 2. We say that an edge of the boundary of a patch is a *face edge* if it bounds some k-face otherwise we say that it is a *path edge*. A key feature of these reduced patches is that, if e is a face edge in one of the patches, then the corresponding edge in the other patch must be a path edge.

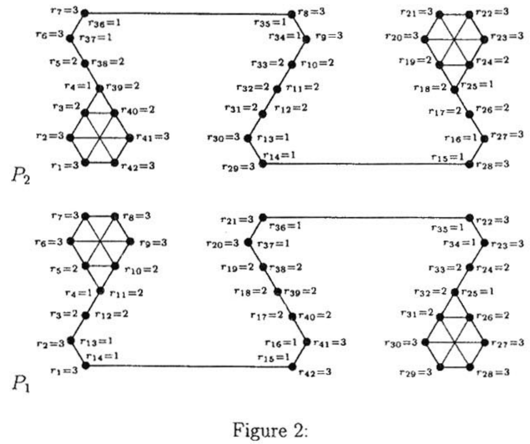

Figure 2:

The lobes (maximal 2-connected subgraphs) of these reduced patches are either single edges or small 2-connected patches. Since these reduced patches have no pendant vertices, some of the small 2-connected subpatches must be "pendant", that is, be attached to the rest of the graph by a single articulation vertex.

We now proceed to prove that any drawing of P_1 triple covers some point; by symmetry the result will then hold for P_2 as well. Suppose that some triple of indices, h, i and j correspond to a single articulation vertex in P_2. Since no two of h, i and j can correspond to the same articulation vertex in P_1, they must correspond to three distinct vertices in P_1. But any drawing of P_1 must map these three vertices onto the same vertex of $\Lambda_{(m,k)}$, the image of the corresponding single vertex of P_2. We assume then that P_2 admits no articulation vertex common to three or more lobes.

Let 1 and i denote the two indices of an articulation vertex u of P_1 attaching one of its pendant patches. We have indicated such a pair in Figure 3; and we will continue to track the steps of the proof in that figure. Without loss of generality, we may assume that the edges of the boundary segment of Δ_1 with indices $1, \ldots, i$ are the boundary edges of this pendant patch. Since they are face edges in P_1, the corresponding edges in P_2 must be path edges. In view of the previous paragraph, $1, \ldots, i$ are the indices on one side of an elementary path in P_2 such that the vertices with indices $2, \ldots, i-1$ have degree 2. If $j+1$ is the index from the other side of the vertex indexed i, then the indices $j+1, \ldots, j+i$ are the indices on the other side of that path. Because of Part (iii) of Theorem 1, the drawings of the boundaries of both P_1 and P_2 match up and vertices having indices 1 and i in either patch, must be mapped onto the same vertex in $\Lambda_{(m,k)}$.

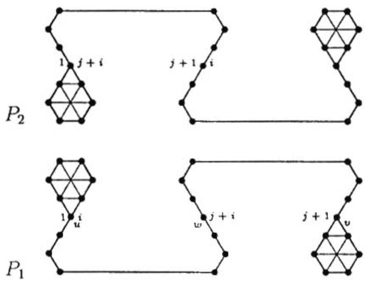

Figure 3:

Let v denote the vertex of P_1 with index $j+1$ and let w denote the vertex of P_1 with index $i+j$. By the argument just employed u, v and w must all be mapped onto the same vertex in $\Lambda_{(m,k)}$. We must show that they are indeed distinct vertices of Γ_1. If $u = v$, then there would be a pair of articulation vertices with the same indices i and j in both patches; a possibility already eliminated. Hence $u \neq v$ and by the same argument $u \neq w$. All that remains is to show that $v \neq w$. We conclude the proof by showing that the assumption, $v = w$, leads to a contradiction.

Considering the pendant patch of P_1 attached at u, we have, by Part (iv) of

Theorem 1, that

$$r_2 + \ldots + r_{i-1} + r_u = \binom{m-2}{2}(i-1) + m + cf_u,$$

where r_u is the boundary index of u for this small patch, $c = \left(\frac{(m-2)(k-2)}{2} - 2\right)$ and f_u is the number of faces in this small patch. Since $cf_u \geq 0$ and $r_u < m$,

$$r_2 + \ldots + r_{i-1} > \binom{m-2}{2}(i-1).$$

Now $v = w$ only if the edges of P_1 with vertex indices $j+1, \ldots, j+i$ enclose a small patch too. In that case, the same arguments give:

$$r_{j+2} + \ldots + r_{j+i-1} > \binom{m-2}{2}(i-1).$$

Adding these two inequalities gives

$$r_2 + \ldots + r_{i-1} + r_{j+2} + \ldots + r_{j+i-1} > (m-2)(i-1).$$

However, as we have already noted, the vertices along this path in P_2 all have valence 2. Thus, for $h = 2, \ldots, i-1$, we have $2 + r_h + r_{j+i+1-h} = m$ giving the contradictory equality:

$$r_2 + \ldots + r_{i-1} + r_{j+2} + \ldots + r_{j+i-1} = (m-2)(i-2).\square$$

As we noted above, the excluded cases, $(\frac{(m-2)(k-2)}{2} - 2) < 0$, correspond to patches that embed in one of the five Platonic maps. One of the key observations in the above proof is that if a simple circuit encloses a finite area in one patch it cannot enclose a finite area on "the other side" in the other patch. This of course is not true when $\Lambda_{(m,k)}$ is finite. Brinkmann, Friedrichs and Nathusius [1] showed that there are no ambiguous $(3,3)$, $(3,4)$ or $(4,3)$-patches. One can easily construct ambiguous $(3,5)$ and $(5,3)$-patches similar to the patches constructed by Guo, Hansen and Zheng [3]. Such patches do cover some point three times. But, it remains open as to whether or not ambiguous patches exist that wrap around the dodecahedron or icosahedron in a special way covering no point more than twice. However, if such special patches exist, they must be rather small: by the pigeonhole principle, a $(3,5)$-patch with more than 24 pentagons or a $(5,3)$-patch with more than 40 triangles must cover some point three or more times whether it is ambiguous or not.

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