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Enumeration of perfect matchings in a type of graphs with reflective symmetry*

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Dedicated to professor H.Sachs on the occasion of his 75th birthday

Abstract

Let G be a bipartite graph without $4k$-cycles. It is well known ([1,2]) that $M(G) = \sqrt{|\det A(G)|}$, where $A(G)$ is the adjacency matrix of G and $M(G)$ is the number of perfect matchings of G. In this paper we show that: if G is invariant under the reflection across some plane (or straight line) then $M(G) = |\det A(G^+)|$ under some conditions, where G^+ is a graph with half of the number of vertices of G.

1. Introduction

A perfect matching of a graph G is a set of independent edges of G covering all vertices of G. Problems involving enumeration of perfect matchings of a graph were first examined by chemists and physicists in the 1930s, for two different (and unrelated) purposes: the study of aromatic hydrocarbons and the attempt to create a theory of the liquid state.

Shortly after the advent of quantum chemistry, chemists turned their attentions to molecules like benzene composed of carbon rings with attached hydrogen atoms. For these researchers, perfect matchings of a polyhex graph corresponded to "Kekule structures", i.e., assigning single and double bonds in the associated hydrocarbon (with carbon atoms at the vertices and tacit hydrogen atoms attached to carbon atoms with only two neighboring carbon atoms). Resonant theory states that there are strong connections between combinatorial and chemical properties for such molecules (see [3]); for intance, those edges which are present in comparatively few of the perfect matchings of a graph turn out to correspond to the bonds that are of longer length, and the more perfect matchings a polyhex graph possesses the more stable is the corresponding benzenoid molecule. Since hexagonal rings are so predominant in the structure of hydrocarbons, chemists gave most of their attention to counting Kekule structures of benzenoids (see [4-8]).

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Let G be a bipartite graph. We say that G is symmetric if it is invariant under the reflection across some straight line S (say symmetry axis). In [9] M.Ciucu gave a matching factorization theorem of the number of perfect matchings of symmetric planar bipartite graph in which there are some vertices lying on the symmetric axis S but no edges intersect S. M.Ciucu's theorem expresses the number of perfect matchings of G in terms of the product of the numbers of perfect matchings of two subgraphs each one of which has nearly a half of vertices of G.

In this paper we will consider the symmetry graph G in which there is no vertices lying on the symmetric axis. Our result is not only valid for plane graphs but also valid for some graphs in $3D$. In other words, G is invariant under reflection across some symmetric plane or straight line. Our main result is to prove that the number of perfect matchings of G can be expressed by a determinant of the adjacency matrix of a graph G^+ which has a half of vertices of G.

The start point of this paper is the fact that the number of perfect matchings of some bipatite graphs can be expressed by the determinant of their adjacency matrix or by product of their eigenvalues. In order to formulate the first lemma we need to introduce the following terminology.

Let G be a bipartite graph with perfect matchings. A cycle C of G is called a nice cycle if $G - V(C)$ has at least one perfect matching, where $G - V(C)$ denotes the subgraph obtained from G by deleting all vertices of C and their incident edges. Based on the results of Dewar and Longuet-Higgins [1], Graovac et al [10] and Cvetkovic et al [11], Heping Zhang and one of the present authors gave the following lemma.

Lemma 1[12] Let G be a bipartite graph with $2n$ vertices and let A and $M(G)$ denote the adjacency matrix and the number of perfect matchings of G, respectively. Then $\det A = (-1)^n[M(G)]^2$ if and only if G has no nice cycles of length $4s$, where $s \in \{1, 2, \cdots\}$.

Now let us recall the relationship between the terms of determinant of a real matrix $A = (a_{ij})$ and the 1-factors of its associated weighted digraph. Let $A^w = (a_{ij})$ be a matrix of order n. A digraph $D^w = D(A^w)$ with n vertices labeled by the integers from 1 to n is defined as follows: If $a_{ij} \neq 0$ then there is an arc from vertex i to vertex j with associated weight a_{ij} in D^w, where $0 < i, j \leq n$. Clearly in this digraph loops are allowed and the arcs with the same head and tail are not allowed. A 1-factor of D^w is defined to be a spanning subgraph of D^w which is regular of in-degree and out-degree 1. The following results were first exploited by Konig [13] and were developed by Coates [14].

Lemma 2[15] Let A^w be a matrix of order n and Ω be the set of 1-factors of D^w. Then

$$\det A^w = \sum_{h \in \Omega} (-1)^{L_h} f(h),$$

where the summation ranges over every 1-factor of D^w, and L_h is the number of even components of h and $f(h)$ is the product of the weights of arcs in h.

When A^w is a Hermite matrix the associated weighted digraph $D^w = D(A^w)$ can be considered as a weighted undirected graph $G^w = G(A^w)$ in a natural way. In fact, a 2-dicycle in D^w is considered to be an edge in G^w, two dicycles with the same vertex-set and different directions are considered as a cycle in G^w. Following Harrary [16] and Sachs [17] there is no difficulty to get a result by lemma 2. In order to do this, we need to introduce further terminology. A spanning subgraph H of graph G is called an elementary spanning subgraph of G, if each connected component of H is a loop or cycle or an edge.

Lemma 3[18] Let A^w be a Hermite matrix and Θ be the set of elementary spanning subgraphs of G^w. Then

$$\det A^w = \sum_{h\in\Theta}(-1)^{L_h}2^{L'_h}f(C_h)f(E_h)^2,$$

where the summation ranges over every elementary spanning subgraph in G^w, and L_h is the number of even components of h, f(X) is the product of the weights of edges in X, L'_h is the number of cycles in h, C_h is the set of cycles and loops in h and E_h is the set of disjoint edges in h.

2. Main result

Let G be a connected bipartite graph with a symmetry plane (or axis) S and there are not vertices lying on S (We consider S to be horizontal). Then the set of edges of G intersecting S forms an edge cut K of G. If we delete the edges of K from G, two isomorphic subgraphs (say the upper and lower half of G) are obtained (see Fig. 1).

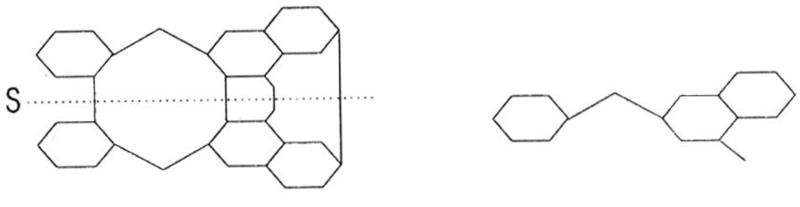

Fig. 1 A connected bipartite graph G with symmetric plane S and upper half of G.

Now we will define a graph G^+ obtained from the upper half of G and use the determinant of its adjacency matrix to calculate the number of perfect matchings of G. In fact G^+

is obtained form G by adding a loop to each vertex in the upper half of G which is an end vertex of an edge in the edge cut K. Similarly we can define a weighted graph G^- by adding a loop of weight -1 to each vertex of in lower half of G which is end vertex of an edge in the edge cut K. By this definition the graphs G^+ and G^- obtained from the graph G indicated in Fig. 1 are pictured in following Fig. 2.

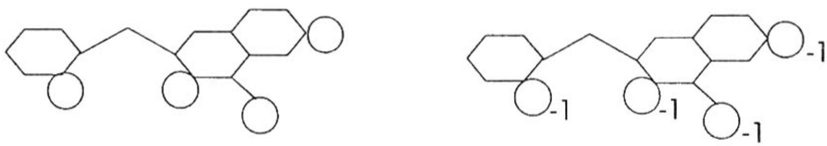

Fig. 2 The graph G^+ and weighted graph G^-.

Theorem 4 Let G be a symmetry connected bipartite graph without nice cycles of length $4s, s \in \{1, 2, 3, \cdots\}$ and there are not vertices lying on the symmetry plane (or axis) S. If the edges intersected by S (say the edge cut set K) form a matching of G and the reflection interchanges the end vertices of each edge of K, then the number of perfect matchings of G equals $|\det A(G^+)|$, where G^+ is the graph as described above, and $A(G^+)$ denotes the adjacency matrix of G^+.

Proof By a suitable labelling of vertices of G, the adjacency matrix of G has following form:

$$A(G) = \begin{bmatrix} A & R \\ R & A \end{bmatrix}.$$

By our assumption, the reflection interchanges each pair of end vertices of the edges in K. Thus R is represented to be a diagonal matrix. By lemma 1 we have

$$[M(G)]^2 = \left|\det \begin{bmatrix} A & R \\ R & A \end{bmatrix}\right| = \left|\det \begin{bmatrix} A+R & A+R \\ R & A \end{bmatrix}\right|$$

$$= \left|\det \begin{bmatrix} A+R & 0 \\ R & A-R \end{bmatrix}\right| = |\det(A+R)||\det(A-R)|.$$

Clearly $A+R$ is the adjacency matrix of G^+ and $A-R$ is the adjacency matrix of G^- as described above. Hence to prove the theorem we need only to show that

$$|\det(A+R)| = |\det(A-R)|.$$

Denoted the sets of elementary spanning subgraphs of G^+ and G^- by Θ^+ and Θ^-, there is a natural bijection between Θ^+ and Θ^- such that two corresponding elementary spanning subgraphs have the same configuration except the weights of adding loops. Now we consider the following cases.

Case 1 G^+ has even number of vertices. Since G is a bipartite graph, both of Θ^+ and Θ^- have only even cycle except loops. It is no difficulty to see that the number of loops in each elementary spanning subgraph is even. Thus any pair of corresponding elementary spanning subgraphs in Θ^+ and Θ^- have the same weights. By lemma 2 this means that

$$\det(A+R) = \det(A-R).$$

Case 2 G^+ has odd number of vertices. Since both of Θ^+ and Θ^- have only even cycle except loops. It is no difficulty to see that the number of loops in each elementary spanning subgraph is odd. Thus the weights of any pair of corresponding elementary spanning subgraphs in Θ^+ and Θ^- have the same absolute values but different signs. By lemma 2 we have

$$\det(A+R) = -\det(A-R).$$

Hence theorem 4 is proved.

Note that the above theorem can be used to count the numbers of perfect matchings for polytopes as well as non-plane graphs. We give the following example of non-plane graph.

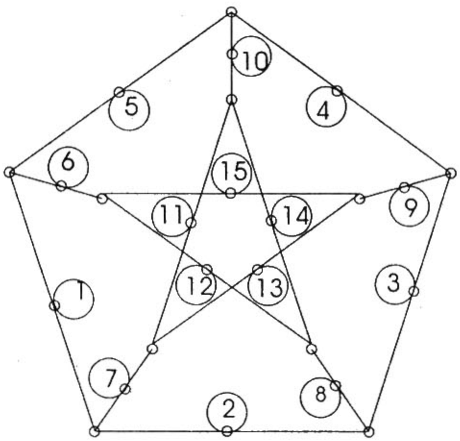

Fig. 3 The graph G^+.

Example 5 Suppose that G_1 is the Petersen graph, $V(G_1)$ and $E(G_1)$ are the vertex-set and edge-set of G_1, and let $S(G_1)$ denote the 1-subdivision graph of G_1, that is obtained from G_1 by replacing every edge of G_1 by a path of length 2. Since G_1 has 15 edges, there are 15 subdividing vertices in $V(S(G_1))$. We denote the set of subdividing vertices by $\{i|1 \le i \le 15\}$. Take two copies of $S(G_1)$, denoted by $S(G_1^{'})$ and $S(G_1^{''})$, respectively. Let G be a graph such that $V(G) = V(G_1^{'}) \cup V(G_1^{''})$, $E(G) = E(G_1^{'}) \cup E(G_1^{''}) \cup K$, where $K = \{(i, i^{'})|1 \le i \le 15\}$, and $\{i|1 \le i \le 15\}(\{i^{'}|1 \le i^{'} \le 15\})$ is the set of subdividing vertices in $V(S(G_1^{'}))(V(S(G_1^{''})))$. It is obvious that G is a symmetry connected non-plane bipartite graph without nice cycles of length $4s$, $s \in \{1, 2, \cdots\}$. Hence G satisfies the conditions of theorem 4. By theorem 4 the number of perfect matchings of G equals $|\det A(G^+)|$, where G^+ is pictured in Fig. 3. By using computer software–Matlab, we can get easily that

$$|\det A(G^+)| = 6144.$$

Hence the number of perfect matchings of G is 6144.

Corollary 6 Let G be a symmetric plane bipartite graph without nice cycles of length $4s, s \in \{1, 2, \cdots\}$, and there are not vertices lying on the symmetry axis S. If the reflection interchanges the end vertices of each edge intersected by S, then we have

$$M(G) = |\det A(G^+)|.$$

Proof We prove that all edges intersected by S form a matching of G. First we consider the leftmost edge $(u, u^{'})$ in the edge cut set K (edges intersected by S). Since the reflection intersecting S interchanges u and $u^{'}$. If there is another edge $(u, w^{'})$ incident with u and intersects S, then there is another edge $(u^{'}, w)$ incident with $u^{'}$ and intersects S. One can see that $(u, w^{'})$ and $(u^{'}, w)$ are crossing, a contradiction. Now we delete $(u, u^{'})$ from G and consider the leftmost edge of $G - (u, u^{'})$. Repeating the previous discussion, we conclude that all the edges crossing by S form a matching of G. Thus the conclusion follows from theorem 4.

Remark 7 Let G be a plane bipartite graph with a perfect matching M. There is an linear algorithm to determine whether or not G has a nice cycle of length $4s, s \in \{1, 2, \cdots\}$ (see [12] Algorithm 13).

In section 1 we established the relation between the weighted graph G and the Hermite matrix $A(G)$ (say adjacent matrix of weighted graph G). If we change the weights of all the edges and loops to be 1, we obtain the underlying graph of G. We say a weighted graph is symmetry if its underlying graph is symmetry and the weights are constant on the orbit of reflection. G^+ has edge e^+ and the corresponding edge of G^- is denoted by e^-. Define weight $w^+ := w(e^+)$ of edge e^+ of G^+ and analogous we define $w^- := w(e^-)$ of G^-. Clearly we have $|\det A^+(w^-)| = |\det A^-(w^-)|$, where $A^+ := A(G^+), A^- := A(G^-)$ or $|\det A^+(w^+)| = |detA^-(w^+)|$. Where we stand $w^\pm$ for all weights of edges of graph

$G^{\pm}$. The weight of one perfect matching M is defined to the product of weights of edges contained in M. We denote the sum of the weights of all perfect matchings of G by $M(G)$. As pointed out in [12], lemma 1 can be extended to the weighted graphs. Furthermore, theorem 4 can be extended as follows.

Theorem 8 Let G be a symmetric connected bipartite weighted graph without nice cycles of length $4s, s \in \{1, 2, \cdots\}$. If there are not vertices lying on the symmetry plane (axis) and all the edges intersecting symmetry plane (axis) form a matching K of G, then

$$M(G) = |\det A(G^{+})|,$$

where G^{+} is obtained from G by adding a loop to each vertex in the upper half of G which is an end vertex of one edge e in K and the weight of the adding loop equals the weight of edge e.

Corollary 6 can also be extended in a similar way.

Remark 9 In [19] Sachs extended lemma 1 to bipartite graphs with multiple edges which can also be considered as weighted graph. Thus theorem 8 is also valid for bipartite graph with multiple edges.

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