communications in mathematical
match
and in computer chemistry

33 (1996), 87 - 100

Phantasmagorical Fulleroids

Andreas Dress and Gunnar Brinkmann

Fakultät für Mathematik
Universität Bielefeld
D 33501 Bielefeld, Germany

Abstract

Fullerene-*like* molecules may – in principle – include other polygons than just pentagons and hexagons. In this paper, two possible geometries for such *fulleroids* are discussed, both with exactly 260 "atoms" and consisting of pentagons and heptagons, only, exhibiting icosahedral symmetry. It is shown that they are the smallest such structures (easy), how they can be found and that – up to isomorphisms – they are the only two such structures with exactly that many atoms (more difficult and based on the *Theory of Delaney Symbols*), this way presenting a complete answer to a question raised by Patrick Fowler almost a year ago.

1 Introduction

When visiting Exeter last spring, we were asked by Patrick Fowler whether a fullerene-like structure with 260 vertices consisting of pentagons and heptagons (rather than hexagons) only, and exhibiting icosahedral symmetry, could exist. More precisely, he asked whether or not there exist tilings T of the sphere (that is, embedded spherical graphs) encompassing exactly 72 pentagons and 60 heptagons, 3 of them meeting at each vertex, with a symmetry group $G = G(T)$ isomorphic to the group I of rotational (or *proper*) symmetries of the icosahedron.

Clearly, if such a tiling consists of f_5 pentagons and f_7 heptagons, the number e of its edges will be equal to $\frac{1}{2}(5f_5 + 7f_7)$ while the number v of its vertices will be equal to $\frac{1}{3}(5f_5 + 7f_7)$, so the Euler formula $f_5 + f_7 - e + v = 2$ implies

$$12 = 6f_5 + 6f_7 - 3(5f_5 + 7f_7) + 2(5f_5 + 7f_7),$$

that is,

$$12 + f_7 = f_5.$$

In addition, as no element g from the group G except the identity element can fix a heptagon, the number f_7 must be a multiple of the order of the group, that is, we must have

$$f_7 = 60 \cdot n$$

for some $n \in \mathbb{N}$. So, the numbers considered by Patrick Fowler are the minimal numbers of pentagons and heptagons for which such a structure might exist.

Fortunately, it occured to us immediately that it should be possible to find a definite answer to this question by applying to it the theory of *Delaney symbols* that had been developed in Bielefeld over the last 12 years (cf. [2],[5],[3],[4]). In a way, this theory allows to translate such questions into easily decidable algebraic-combinatoric questions similar to the way, Cartesian coordinate geometry allows to translate classical geometric questions into easily manipulable algebra. And indeed, applying our theory, we found that exactly two such structures exist, both of which are depicted in Figure 1.

In this note, we want to present a proof of this fact. We'll start with a short introduction into the theory of Delaney symbols, and we'll then detail the arguments which allow to determine the two structures depicted in Figure 1 as the only two structures – up to isomorphism – satisfying all of the above requirements.

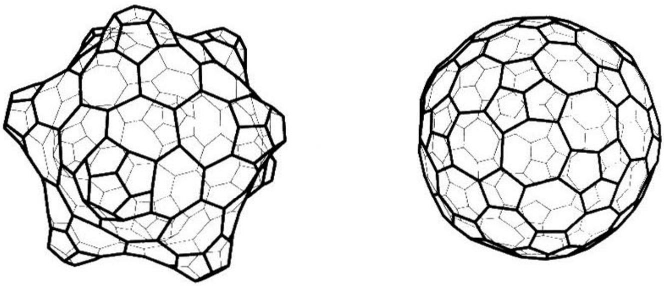

Figure 1

2 Tilings

Let M be a two-dimensional manifold, and let T be a tiling of M, that is, a partition of M into three different types of disjoint subsets, the set T_0 of *vertices* of T, the set T_1 of *edges* of T, and the set T_2 of *faces* of T.

By definition, the vertices $v_1, v_2, \ldots$ are one-point subsets, that is, subsets consisting of just one element (with which they will be identified whenever convenient); the edges $e_1, e_2, \ldots$ are subsets of M such that, for any such edge $e \in T_1$, there exists a homeomorphism $\varphi_e : (0,1) \tilde{\hookrightarrow} e$ of the open interval $(0,1)$ onto e which extends to a homeomorphism $\overline{\varphi_e} : [0,1] \to \overline{e}$ of the closed interval $[0,1]$ onto the closure $\overline{e}$ of e having the property that both, $\overline{\varphi_e}(0)$ and $\overline{\varphi_e}(1)$, are in T_0; and the faces $f_1, f_2, \ldots$ of T are open subsets of T such that, for each face $f \in T_2$, there exists a homeomorphism

$$\psi_f : \{z \in \mathbb{C} \mid |z| < 1\} \tilde{\hookrightarrow} f$$

from the open disk $\{z \in \mathbb{C} \mid |z| < 1\}$ onto f which extends to a homeomorphism $\overline{\psi}_f$: $\{z \in \mathbb{C} \mid |z| \leq 1\} \tilde{\hookrightarrow} \overline{f}$ of the closed disk $\{z \in \mathbb{C} \mid |z| \leq 1\}$ onto the closure $\overline{f}$ of f, having the property that $\overline{\psi}_f(z)\,\big|_{\{z\in\mathbb{C}\,|\,|z|=1\}}$ is a union of finitely many vertices and edges (and, hence, of as many edges as vertices – their number will be called the degree of f; and f is called a k-gon if its degree is k). Moreover, for any $x \in M$, there exists a neighbourhood $U(x)$ of M such that $U(x)$ intersects only finitely many vertices, edges, and faces. As is well known, this implies that for any $x \in M$ with $x \in e$ for some $e \in T_1$, there exists a neighbourhood $U(x)$ intersecting just e and two faces $f_1, f_2 \in T_2$, the only two faces containing e in their boundary. And for any $x \in T_0$, there exists a neighbourhood $U(x)$ intersecting exactly those edges and faces which contain x in their closure which then can be labelled $e_1, e_2, \ldots, e_k$ and $f_1, f_2, \ldots, f_k$, $f_{k+1} := f_1$ respectively so that, for each $i = 1, \ldots, k$, the two faces containing e_i in their boundary are precisely f_i and f_{i+1}. The number k of faces (or edges) containing a given vertex $x \in T_0$ in their boundary will be called the degree of x. [1]

Clearly, using this terminology, we can define a tiling T of the sphere S^2 to be one of *Fowler's Phantasmagorical Fulleroids*, if all of its vertices have degree 3, all of its faces are either pentagons or heptagons, there are precisely 72 pentagons and 60 heptagons, and there exists a group G of homeomorphisms of S^2 respecting the partition T, which is isomorphic to the group I of rotational symmetries of the icosahedron.

3 The Flagspace of a Tiling

As above, we consider a tiling T of a 2-manifold M. The flagspace $\mathcal{F}(T)$ of T is defined to be the set of triples (v, e, f) consisting of one vertex $v \in T_0$, one edge $e \in T_1$, and one face $f \in T_0$ such that $(v =)\overline{v} \subseteq \overline{e} \subseteq \overline{f}$ holds. Obviously, for any vertex $v \in T_0$ of degree k, there exist $2k$ flags containing v as their "0-dimensional component", for any edge $e \in T_1$, there exist 4 flags containing e as their "1-dimensional component", and for any n-gon $f \in T_2$, there exist $2n$ flags containing f as their "2-dimensional component".

Two flags (v, e, f) and (v', e', f') are called 0-neighbours, if $v \neq v', e = e'$, and $f = f'$ holds. They are called 1-neighbours, if $v = v', e \neq e'$, and $f = f'$ holds. And they are called 2-neighbours, if $v = v', e = e'$, and $f \neq f'$ holds. From our definition of a tiling and standard two-dimensional topology, it follows easily that for any flag there exist precisely one 0-neighbour, one 1-neighbour and one 2-neighbour. Moreover, the flags containing a given vertex v of degree k form a "circular" sequence of $2k$ flags

$$(v, e_1, f_1), (v, e_2, f_1), (v, e_2, f_2), \ldots, (v, e_k, f_k), (v, e_1, f_k),$$

any two consecutive ones being alternatingly either 1- or 2-neighbours, the flags containing a given edge e form a circular sequence of 4 sequences

$$(v_1, e, f_1), (v_2, e, f_1), (v_2, e, f_2), (v_1, e, f_2),$$

any two consecutive ones being alternatingly either 0- or 2-neighbours, and the flags containing a given face f of degree k form a circular sequence

$$(v_1, e_1, f), (v_2, e_1, f), (v_2, e_2, f), \ldots, (v_k, e_k, f), (v_1, e_k, f),$$

[1] Here, for the sake of simplicity, we have introduced the special class of tilings we have called *cellular* in [4]. The experienced reader will easily verify that our arguments below also work for the more general class of tilings introduced there and that, consequently, they prove that all such – potentially more general – tilings satisfying Patrick Fowler's requirements actually are cellular.

any two consecutive ones being alternatingly either 0- or 1-neighbours.

It is also easy to see that M is connected if and only if there exists, for any two flags (v,e,f) and (v',e',f'), a sequence of flags $(v_0,e_0,f_0) := (v,e,f), (v_1,e_1,f_1), \ldots, (v_n,e_n,f_n) := (v',e',f')$ such that any two consecutive ones are 0-, 1- or 2-neighbours (of each other), and that M is orientable if and only if we can split the set $F(T)$ of flags into two classes such that any pair of neighbouring flags consists of exactly one flag from each of these two classes. In this case, an orientation of M can be specified by labelling the flags in one of these two classes as positively oriented flags (namely, say, those flags (v,e,f) where f is on the right side of e if e is oriented in the direction from v towards the other vertex in $\overline{e}$), while the other one may be labelled as negatively oriented.

A more intuitive approach describing the notions in this section is given at the end of the text.

4 The Flag Graph of a Tiling

To collect all these facts in a convenient way, we may introduce the *flag graph* $\Gamma(T) = (\mathcal{F}(T); \mathcal{E}_0, \mathcal{E}_1, \mathcal{E}_2)$ of T whose vertex set is the set $\mathcal{F}(T)$ of flags of T and which has three kinds of edges contained in $\mathcal{E}_0$, $\mathcal{E}_1$, and $\mathcal{E}_2$, respectively: the 0-edges consisting of all pairs of 0-neighbours, the 1-edges consisting of all pairs of 1-neighbours, and the 2-edges consisting of all pairs of 2-neighbours. The above observations can now be stated as follows:

(i) for any flag $F = (v,e,f) \in \mathcal{F}(T)$ and any $i \in \{0,1,2\}$, there exists exactly one i-edge $\{F, F'\} \in \mathcal{E}_i$ containing F;

(ii) M is connected if and only if the graph $(\mathcal{F}(T), \mathcal{E}_0 \cup \mathcal{E}_1 \cup \mathcal{E}_2)$ is connected;

(iii) M is orientable, if and only if $(\mathcal{F}(T), \mathcal{E}_0 \cup \mathcal{E}_1 \cup \mathcal{E}_2)$ is bipartite;

(iv) for any $i \in \{0,1,2\}$, the connected components of $\Gamma_i(T) := (\mathcal{F}(T), \bigcup_{j\neq i} \mathcal{E}_j)$ – the i-components of $\Gamma(T)$ – correspond in a canonical one-to-one fashion to the subsets in T_i: for a given vertex $v \in T_0$ (or edge $e \in T_1$ or face $f \in T_2$), the corresponding 0- (or 1- or 2-)component forms a cycle of flags connected alternatingly by 1- and 2- (or 0- and 2-, or 0- and 1-)edges, consisting of altogether $2k$ flags, with $k = 2$ in the case of edges $e \in T_1$ and k coinciding with the degree of v or f, respectively, in the other two cases.

In addition,we can easily reconstruct M and T from $\Gamma(T)$: Consider the standard simplex

$$\Delta := \{(x_0, x_1, x_2) \in \mathbb{R}^3 \mid x_0, x_1, x_2 \geq 0; x_0 + x_1 + x_2 = 1\}$$

and its cartesian product $\Delta \times \mathcal{F}(T)$ with $\mathcal{F}(T)$ and define two points, say $((x_0,x_1,x_2),(v,e,f))$ and $((x_0',x_1',x_2'),(v',e',f'))$, in $\Delta \times \mathcal{F}(T)$ to be equivalent if and only if $x_0 = x_0', x_1 = x_1', x_2 = x_2'$ and if – in addition – (v,e,f) and (v',e',f') are in the same connected component of $(\mathcal{F}(T), \bigcup_{x_i=0} \mathcal{E}_i)$.

So, if (x_0,x_1,x_2) is contained in the interior of Δ, none of the points $((x_0,x_1,x_2),(v,e,f))$ is equivalent to any other point. If, say, $x_0 = 0$ and $x_1 \neq 0 \neq x_2$, then $((0,x_1,x_2),(v,e,f))$ is equivalent with $((0,x_1,x_2),(v',e',f'))$ whenever (v,e,f) and (v',e',f') are connected by a 0-edge, and if, say $x_0 = x_1 = 0$ and $x_2 = 1$, then $((0,0,1),(v,e,f))$ is equivalent with $((0,0,1),(v',e',f'))$ if and only if $f = f'$, that is, if and only if (v,e,f) and (v',e',f') can be connected by a sequence of 0- and 1-edges. Now, identify any two equivalent points in $\Delta \times \mathcal{F}(T)$. It can be shown that there exists a homeomorphism between the resulting topological space and M which maps the equivalence classes of pairs of the form $((x_0,x_1,x_2),(v,e,f))$ with $x_2 \neq 0$ onto the points contained in f (surjectively if v and e are allowed to vary), while it maps those of the form $((x_0,x_1,0),(v,e,f))$ with $x_1 \neq 0$ onto the points contained in e (also surjectively if v is allowed to vary) and those of the form $((1,0,0),(v,e,f))$ onto v.

5 Symmetry

Let us now assume that, in addition to M and T, we are also given a group G of homeomorphisms γ of M which respect T, that is, of homeomorphisms γ which satisfy $\gamma(v) \in T_0, \gamma(e) \in T_1$, and $\gamma(f) \in T_2$ for all $v \in T_0, e \in T_1$, and $f \in T_2$, respectively. Without loss of generality, we may also assume that G acts properly discontinuously, that is, that the only homeomorphism $\gamma \in G$ with $\gamma(v) = v, \gamma(e) = e$, and $\gamma(f) = f$ for all $v \in T_0, e \in T_1$ and $f \in T_2$ is the identity (cf. [4]). Clearly, G acts also on the flag space $\mathcal{F}(T)$ and it respects i-neighbours, so it induces automorphisms of the flag graph $\Gamma(T)$. Moreover, G acts fixed-point free on $\mathcal{F}(T)$ provided M is connected, as $\gamma(v,e,f) = (v,e,f)$ for some flag $(v,e,f) \in \mathcal{F}(T)$ implies that also the i-neighbours of (v,e,f) must remain fixed under γ, so the same must hold for their j-neighbours and so on. So any flag in the connected graph $\Gamma(T)$ must remain fixed which in turn means that any $v \in T_0$, any $e \in T_1$, and any $f \in T_2$ remains fixed under γ; so γ must be the identity.

Let us now consider the orbit space $\mathcal{D}(T,G) := G \setminus \mathcal{F}(T)$ consisting of all G-orbits of flags in $\mathcal{F}(T)$. As G respects i-neighbourhood, $\mathcal{D}(T,G)$ forms the vertex set of a graph

$$\Gamma(T,G) := (\mathcal{D}(T,G); G \setminus \mathcal{E}_0, G \setminus \mathcal{E}_1, G \setminus \mathcal{E}_2)$$

with, as above, three types of edges which we get by identifying any two 0-, 1-, or 2-edges $\{F_1, F_1'\}$ and $\{F_2, F_2'\}$ from $\Gamma(T)$ if and only if there exists some $\gamma \in G$ with $\gamma(F_1) = F_2$ and, hence, $\gamma(F_1') = F_2'$. The graph $\mathcal{D}(T,G)$ is also called the *Delaney Graph*, associated with T and G. Clearly, as above, there exists, for any flag orbit $G \cdot F$ and any $i \in \{0,1,2\}$, exactly one flag orbit $G \cdot F'$ with $\{G \cdot F, G \cdot F'\} \in G \setminus \mathcal{E}_i$.

If M is oriented and if any $\gamma \in G$ preserves this orientation, $\Gamma(T,G)$ inherits the property of $\Gamma(T)$ of being bipartite. In this case, the i-components of $\Gamma(T,G)$, that is, the connected components of

$$\Gamma_i(T,G) := (\mathcal{D}(T,G), \bigcup_{j \neq i} G \setminus \mathcal{E}_j)$$

remain cycles now corresponding to G-orbits of vertices ($i = 0$), edges ($i = 1$), or faces ($i = 2$) and consisting of $2k'$ flag-orbits $\{G \cdot F_1, G \cdot F_2, \ldots, G \cdot F_{2k'}\}$ which are alternatingly connected in a circular fashion by 1- and 2-edges ($i = 0$), or 0- and 2-edges ($i = 1$), or 0- and 1-edges ($i = 2$), while their number $2k'$ is the quotient of the number $2k$ of flags corresponding to the original vertex v, edge e, or flag f, divided by the order of the subgroup of G stabilizing v, e, or f, respectively.

Consequently, to construct a tiling T of a given manifold M with a pregiven symmetry group G, we may first discuss the way G may act on the vertices, edges, and faces of T to derive corresponding restrictions for the associated Delaney graph $\Gamma(T,G)$, we may then try to systematically construct all graphs $(\mathcal{D}; E_0, E_1, E_2)$ obeying these restrictions, and we may finally try to find all pairs (T, G) with $\mathcal{D}(T,G) \cong (\mathcal{D}; E_0, E_1, E_2)$. In general, the first task is easy, provided we have specified the conditions we require T and G to satisfy sufficiently clearly. The second task can be done by exhaustive search, sometimes even by hand (see below), in more complicated situations by computer.

The third task, finally, depends a bit on the topology of M. While for general M, complicated questions relating to topological aspects of combinatorial group theory may have to be studied, the case is relatively easy in case M is simply connected: in this case, the Delaney graph $\mathcal{D}(T,G)$ together with the – generally pregiven – information on the degrees of the vertices and faces of the tiling we want to construct suffice to imply – up to isomorphism – the uniqueness of the triple (M,T,G) in question and even to guarantee its existence provided the Delaney graph fulfills all our requirements (including a number of easily specified compatibility requirements, cf. [4]). While the uniqueness result is relatively easy to prove – it is based on a well-known formula describing the fundamental group of a CW-complex in terms of its 2-skeleton –, the existence result is more difficult to establish in full generality, but can easily be circumvented in specific cases (again, see below for a simple and illustrative example) by constructing T and G explicitly.

6 Fowler's Phantasmagorical Fulleroids

Let us now apply these considerations to the above mentioned fulleroid problem, that is, let us try to find the Delaney graphs of all spherical tilings T with proper icosahedral symmetry group, consisting of 60 heptagons and 72 pentagons with exactly three of those meeting at each vertex.

Obviously, the 60 heptagons give rise to $60 \cdot 14$ flags, while the 72 pentagons give rise to $72 \cdot 10$ flags. So, the flag space $\mathcal{F}(T)$ of any such tiling consists of exactly $60 \cdot 14 + 72 \cdot 10 = 60 \cdot (14+12) = 60 \cdot 26$ flags. As the symmetry group I acts fixed-point free on those flags, the associated Delaney graph $\mathcal{D}(T,I)$ must contain exactly 26 vertices. Moreover, as the symmetry group I must also act fixed-point free as well as transitively on the set of heptagons of T, that is on the corresponding set of $\{0,1\}$-components of $\mathcal{F}(T)$, the Delaney graph $\mathcal{D}(T)$ must contain exactly one $\{0,1\}$-component of order 14, representing the heptagons. Similarly, the five-fold symmetries in I must stabilize pentagons, the three-fold symmetries must stabilize vertices, and the two-fold symmetries must stabilize edges. In all three cases, all stabilized elements must be symmetry-equivalent to each other as the fixed points of the corresponding subgroups of order 5, 3, or 2, respectively, of I are wellknown to be conjugate to each other. So, in the Delaney graph, we must have exactly one $\{0,1\}$-component, one $\{0,2\}$-component, and one $\{1,2\}$-component of order 2, while all the remaining $\{0,2\}$-components must be of order 4, the remaining $\{1,2\}$-components must be of order 6 and the 10 vertices not contained in the $\{0,1\}$-components of order 2 and 14 must form just one more $\{0,1\}$-component representing those 60 pentagons which have no internal symmetry. And, finally, as we deal with a proper symmetry group, acting on an orientable 2-manifold, the Delaney graph must be bipartite.

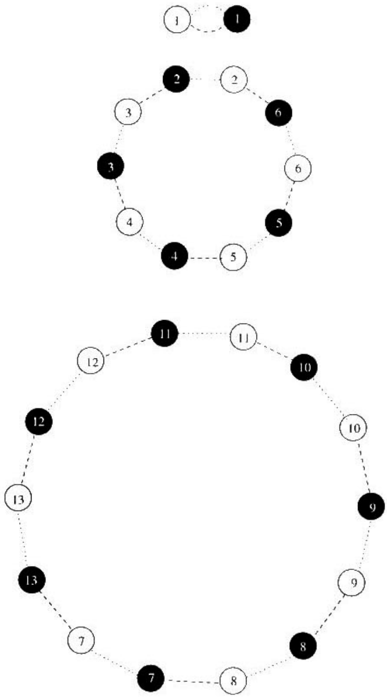

Figure 2a

Figure 2b

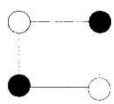

Figure 2c

Figure 2d

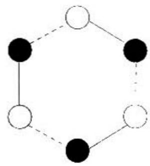

Figure 2e

In other words, we have to find all connected bipartite graphs

$$(V; E_0, E_1, E_2) = (V_0 \dot{\cup} V_1; E_0, E_1, E_2)$$

consisting of a set V of 26 vertices, split into two disjoint subsets V_0 and V_1, each of cardinality 13, and three edge sets $E_0, E_1, E_2 \subseteq V_0 \times V_1$ such that the restriction $(V; E_0, E_1)$ looks like the graph depicted in Fig. (2a), while

(Assertion 0-2) the restriction $(V; E_0, E_2)$ consists of one component isomorphic to the graph depicted in Fig. (2b) and 6 components isomorphic to the graph depicted in Fig. (2c)

and

(Assertion 1-2) the graph $(V; E_1, E_2)$ consists of one component isomorphic to the graph depicted in Fig. (2d) and 4 components isomorphic to the graph depicted in Fig. (2e).

We will now argue that – up to isomorphism – there are exactly 2 such graphs Γ_1 and Γ_2 satisfying all of our requirements, viz. those depicted in Fig. (3a/b). To this end, we label the vertices as suggested in Fig. (2a). Next, we observe that – by connectedness – the 2-edge meeting vertex ① must connect this vertex with one from the other two $\{0, 1\}$ components. So, without loss of generality, we may assume that it either meets vertex ❷ or vertex ❼, that is, we have established the 2-edge labelled $\boxed{a}$ in Fig. (3a/b). Using (Assertion 0-2), it follows that another 2-edge, labelled $\boxed{b}$ in Fig. (3a/b), must connect vertex ❶ with vertex ② or ⑦, respectively.

Next, using (Assertion 1-2), it follows that a 2-edge, labelled $\boxed{c}$ in Fig. (3,a/b), must connect the vertices ③ and ❻ (or ⑧ and ⓭, respectively) which in turn, using (Assertion 0-2) again, implies the existence of still another 2-edge, labelled $\boxed{d}$, connecting ❸ and ⑥ (or ❽ and ⑬, respectively).

Now, using (Assertion 1-2) again, we observe that the 2-edges meeting ④ and ❺ (or ⑨ and ⓬, respectively) must connect these two vertices with a pair of vertices connected by a 1-edge. In the first case (Fig. 3a), this cannot be the pair ❹,⑤, as this would contradict connectedness, so – using again the freedom of labelling – we may assume that it is the pair ❼,⑧, leading to the 2-edges $\boxed{e}$ and $\boxed{f}$. Similarly, in the second case (Fig. 3b), it can be neither of the pairs ❾ and ⑩, ❿ and ⑪, or ⓫ and

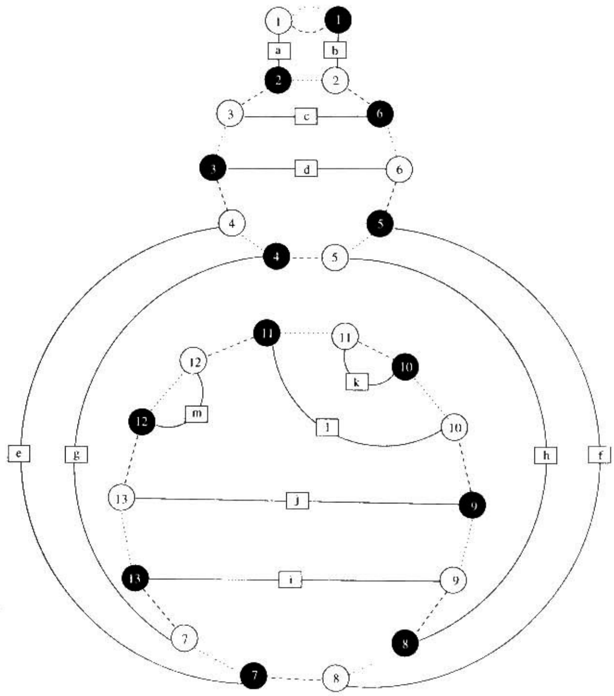

Figure 3a

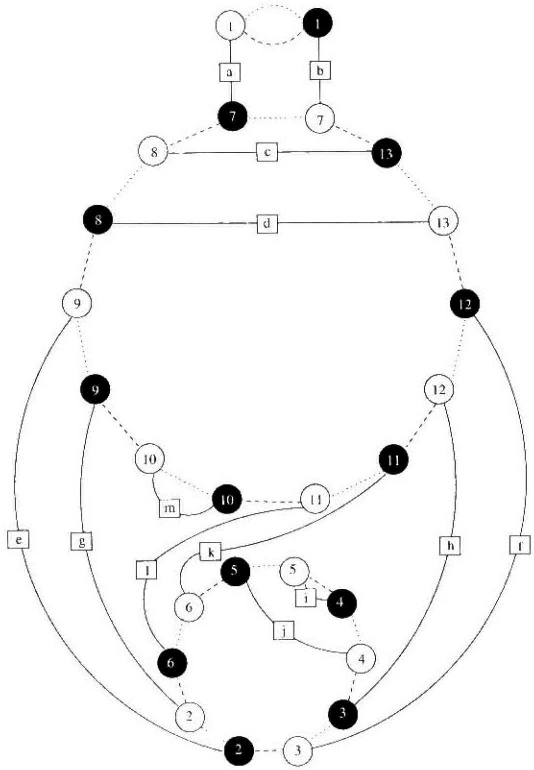

Figure 3b

⑫: Connecting ⑨ and ⓬ with the first (or the last) pair would – by (Assertion 0-2) lead a further 2-edge connecting ❿ and ⑫ (or ❾ and ⑪, respectively) which would enforce the two vertices ⑪ and ⓫ (or ⑩ and ❿) to be connected by 2-edges with a pair of vertices connected by a 0-edge in view of (Assertion 0-2) and, simultaneously, connected by a 1-edge in view of (Assertion 1-2), which is impossible. Connecting instead ⑨ and ⓬ with ❿ and ⑪, (Assertion 0-2) would force us to connect ❾ with ⑩ and ⑫ with ⓫ by 2-edges, contradicting connectedness as well as the fact that there is only one $\{1,2\}$-component consisting of 2 vertices, only. So, – using again the freedom of labelling – we may assume without loss of generality that the two 2-edges emanating at ④ and ❺ in Fig. 3a connect these two vertices with the pair ❼,⑧, while the two 2-edges emanating at ⑨ and ⓬ in Fig. 3b connect these two vertices with the pair ❷,③, leading in both cases to the 2-edges $\boxed{e}$ and $\boxed{f}$. Using (Assertion 0-2) once more, we get two more 2-edges, labelled $\boxed{g}$ and $\boxed{h}$.

In the situation depicted in Fig. 3a, this in turn forces us to connect ⑨ and ⓭ by a 2-edge labelled $\boxed{i}$ and ❾ and ⑬ by a 2-edge labelled $\boxed{j}$, using first (Assertion 1-2) and then (Assertion 0-2). As we still have to place a $\{1,2\}$-component of order 2 somewhere into Fig. 3a and as the remaining two pairs of vertices connected by a 1-edge are symmetrically placed with respect to all the 0- and 1-edges and the 2-edges labelled $\boxed{a}$ to $\boxed{j}$, we may – without loss of generality – now connect the vertices ❿ and ⑪ by the next 2-edge labelled $\boxed{k}$ which then enforces us to place the remaining 2-edges $\boxed{l}$ and $\boxed{m}$ as depicted in Fig. 3a, once again using (Assertion 0-2) and (Assertion 1-2) consecutively. The resulting structure obviously fulfills all of our requirements.

In the situation depicted in Fig. 3b, we now consider where to place the 2-edge contained in the unique $\{1,2\}$-component of order 2. We cannot connect ❿ and ⑪ by a 2-edge as this would force us to connect also the vertices ⑩ and ⓫ by a 2-edge, producing a $\{1,2\}$-component encompassing at least 8 vertices. So, by symmetry, we are left with the possibility of connecting ❹ and ⑤ by the next 2-edge labelled $\boxed{i}$. Now, using (Assertion 0-2) and (Assertion 1-2) alternatingly, we find ourselves compelled to place the remaining 2-edges $\boxed{j}$, $\boxed{k}$, $\boxed{l}$, and $\boxed{m}$ as depicted in Fig. 3b leading to the unique second alternative for a graph satisfying all of our requirements.

In other words, we have established

Theorem 1 *There exist – up to isomorphisms – exactly* 2 *graphs* Γ_1 *and* Γ_2 *which are potential Delaney graphs of spherical tilings exhibiting proper icosahedral symmetry, and consisting of* 72 *pentagons and* 60 *heptagons with exactly* 3 *of them meeting at each vertex, namely those depicted in Fig. 3a/b.*

It follows from the easier parts of the theory explained above that, consequently, there are at most two such spherical tilings, and it follows from the more difficult parts of that theory that exactly two such spherical tilings must exist. Fortunately, we do not need the more difficult part as the spherical tilings constructed by means of that theory (and the computer programs based on it, cf. [1]) can be inspected in Fig. 1, giving ample evidence of their existence. So, altogether we can now state

Theorem 2 *There are exactly two distinct potential fulleroid isomers with icosahedral symmetry comprising 260 carbon atoms which exclusively form pentagonal and heptagonal cycles.*

7 Flags and the Barycentric Subdivision

There is a simple geometric interpretation of the space $\mathcal{F}(T)$ of flags of T (cf. Fig. 4): Recall that, for any tiling T, one can construct another tiling T', called a barycentric subdivision of T, by

(i) choosing one point $p_e \in e$ for each edge $e \in T_1$, cutting each edge e into two edges e_1 and e_2 this way, and one point $p_f \in f$ for each $f \in T_2$,

(ii) choosing homeomorphic imbeddings

$$\varphi_{(v,f)} : [0,1] \hookrightarrow \overline{f}$$

and

$$\varphi_{(e,f)} : [0,1] \hookrightarrow \overline{f}$$

for each pair (v,f) and (e,f) with $v \in \overline{f}$ and $e \subseteq \overline{f}$ so that

$$\varphi_{(v,f)}(0) = v, \; \varphi_{(e,f)}(0) = p_e,$$

and

$$\varphi_{(v,f)}(1) = \varphi_{(e,f)}(1) = p_f$$

in such a way that, restricted to $(0,1)$, all the images

$$k_{(e,f)} := \varphi_{(e,f)}((0,1))$$

and

$$k_{(v,f)} := \varphi_{(v,f)}((0,1))$$

are disjoint, and then

(iii) putting

$$T_0' := T_0 \cup \{p_e \mid e \in T_1\} \cup \{p_f \mid f \in T_2\},$$
$$T_1' := \{k_{(e,f)} \mid e \in T_1, f \in T_2, e \subseteq \overline{f}\} \cup \{k_{(v,f)} \mid v \in T_0, f \in T_2, v \in \overline{f}\} \cup \{e_1, e_2 \mid e \in T_1\},$$

and defining T_2' as the connected components of the complement of $T_0' \cup \bigcup_{k \in T_1'} k$.

It is easily seen that

- each face f' of T_2' is a (topological) triangle, i.e. a 3-gon, its boundary containing exactly one vertex $v = v(f')$ from T_0, one vertex of the form p_e for some edge $e = e(f') \in T_1$ and one vertex of the form p_f for some face $f = f(f') \in T_2$.

- f' is completely determined by the triple $v(f'), e(f')$, and $f(f')$

- and given a triple (v, e, f), there exists a face $f' \in T_2'$ in the barycentric subdivision T' of T with $v = v(f'), e = e(f')$, and $f = f(f')$ if and only if the triple (v, e, f) forms a flag. In other words, the flags of T correspond in a one-to-one fashion to the (triangular) faces of the barycentric subdivision T' of T. It is also easy to see, that two flags are 0- (1-, or 2-)neighbours of each other if and only if the closures of the corresponding faces f_1', f_2' of T' share the edge of T' connecting the vertices $p_{e(f_1')} = p_{e(f_2')}$ and $p_{f(f_1')} = p_{f(f_2')}$ (or $v(f_1') = v(f_2')$ and $p_{f(f_1')} = p_{f(f_2')}$, or $v(f_1') = v(f_2')$ and $p_{e(f_1')} = p_{e(f_2')}$, respectively).

This simple geometric interpretation of $\mathcal{F}(T)$ is rather useful if one wants to visualize the constructions that are being performed in reference to $\mathcal{F}(T)$ and the unicity results stated in terms of $\mathcal{F}(T)$ later on.

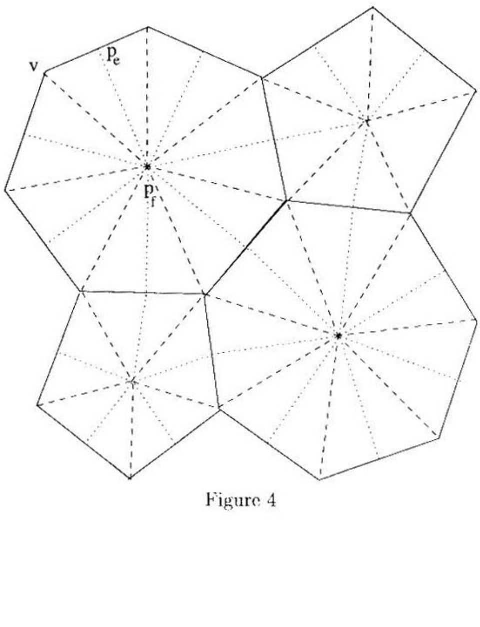

Figure 4

References

[1] O. Delgado Friedrichs, A. Dress, and D. Huson. Tilings and symbols — a report on the uses of symbolic calculation in tiling theory. In *Computer algebra in science and engineering*, pages 273–286. World Scientific, London, 1995.

[2] A.W.M. Dress. Regular polytopes and equivariant tessellations from a combinatorial point of view. In *Algebraic Topology*, pages 56–72. SLN 1172, Göttingen, 1984.

[3] A.W.M. Dress. Presentations of discrete groups, acting on simply connected manifolds. *Adv. in Math.*, 63:196–212, 1987.

[4] A.W.M. Dress and D.H. Huson. On tilings of the plane. *Geometriae Dedicata*, 24:295–310, 1987.

[5] A.W.M. Dress and R. Scharlau. Zur Klassifikation äquivarianter Pflasterungen. *Mitteilungen aus dem Math. Seminar Giessen*, 164:83–136, 1984.