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μ-POLYNOMIAL OF A GRAPH WITH AN ARTICULATION POINT

Ivan Gutman* and Oskar E. Polansky**

* Faculty of Science, YU-34000 Kragujevac, Yugoslavia

** Max-Planck-Institut für Strahlenchemie, D-4330 Mülheim/Ruhr, F.R. Germany

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Abstract

A recurrence relation for the μ-polynomial of a graph, possessing an articulation point, is obtained. Some of its consequences are pointed out.

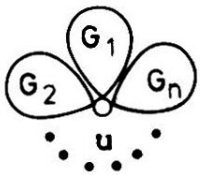
In [1] a recurrence relation for the μ-polynomial of a graph G has been determined, namely

$$\mu(G) = x\,\mu(G-u) - \sum_j \mu(G-u-v_j) - 2t \sum_k \mu(G-Z_k) \quad . \qquad (1)$$

In eq. (1) u is an arbitrary vertex of G; the first summation goes over all vertices v_j of G which are adjacent to u; the second summation goes over all cycles Z_k of G, which contain the vertex u. In formula (1) the weight t_k associated with the cycle Z_k is assumed to be the same for all cycles of G. If this is not the case, then (1) is to be replaced with its immediate generalization, viz.

$$\mu(G) = x\,\mu(G-u) - \sum_j \mu(G-u-v_j) - 2 \sum_k t_k\,\mu(G-Z_k) \quad . \qquad (2)$$

In the present work we shall be concerned with this more general form of eq. (1). Our aim is to apply formula (2) to graphs which possess an articulation point, that is to graphs having the following structure:



G

A vertex u is called an articulation point (or a cut-point) if the graph G-u has more components than the graph G.

Let $G_1, G_2, \ldots, G_n$, $n \geq 2$, be graphs with disjoint vertex sets and let u_i be an arbitrary vertex of G_i, $i = 1,2,\ldots,n$. Then the above graph G can be constructed from the graphs $G_1, G_2, \ldots, G_n$ by identifying their vertices $u_1, u_2, \ldots, u_n$. The new vertex obtained in this manner will be denoted by u.

In the following we shall often refer to the subgraph $G_i - u_i$, obtained by deleting the vertex u_i from G_i. For brevity we denote this subgraph by G_i'.

For the case $n = 2$ the relation

$$\mu(G) = \mu(G_1)\mu(G_2') + \mu(G_1')\mu(G_2) - x\,\mu(G_1')\mu(G_2') \qquad (3)$$

was given without proof in [2] and was recently proved in [3]. Using a method which differs from the approach of ref. [3] we demonstrate here that the formula

$$\mu(G) = \mu(G_1)\mu(G_2')\cdots\mu(G_n') + \mu(G_1')\mu(G_2)\cdots\mu(G_n') + \cdots$$
$$\cdots + \mu(G_1')\mu(G_2')\cdots\mu(G_n) - (n-1)x\,\mu(G_1')\mu(G_2')\cdots\mu(G_n') \qquad (4)$$

is valid for any value of $n \geq 2$. It is clear that (3) is a special case of eq. (4) for $n = 2$.

In order to derive the relation (4) we denote the vertices of G_i which are adjacent to u_i by v_{ij}, $j=1,2,\ldots,g_i$. Similarly, the cycles of G_i which contain the vertex u_i will be

denoted by Z_{ik} , $k = 1,2,\ldots,r_i$.

Bearing in mind that u is an articulation point of the graph G, we readily see that Z_{ik}, $i = 1,2,\ldots,n$, $k = 1,2,\ldots$ $\ldots,r_i$ are all the cycles of G which contain u. In other words, there is no cycle of G which would belong simultaneously to G_i and G_j, $i \neq j$. As a consequence of this we can write eq. (2) as

$$\mu(G) = x\,\mu(G-u) - \sum_{i=1}^{n} \sum_{j=1}^{g_i} \mu(G-u-v_{ij}) - $$
$$- 2 \sum_{i=1}^{n} \sum_{k=1}^{r_i} t_{ik}\,\mu(G-Z_{ik}) \qquad (5)$$

where, of course, t_{ik} is the weight associated with Z_{ik}.

The deletion of the vertex u from G causes its decomposition into disconnected parts $G_1',G_2',\ldots,G_n'$. Consequently,

$$\mu(G-u) = \mu(G_1')\mu(G_2')\cdots\mu(G_n') = \prod_{\ell=1}^{n} \mu(G_\ell') \quad . \qquad (6)$$

For the same reason,

$$\mu(G-u-v_{ij}) = \mu(G_1')\cdots\mu(G_{i-1}')\mu(G_i-u_i-v_{ij})\mu(G_{i+1}')\cdots\mu(G_n') \qquad (7)$$

and

$$\mu(G-Z_{ik}) = \mu(G_1')\cdots\mu(G_{i-1}')\mu(G_i-Z_{ik})\mu(G_{i+1}')\cdots\mu(G_n') \quad . \qquad (8)$$

Substituting (6)-(8) back into (5) and performing appropriate algebraic manipulations, we arrive at

$$\mu(G) = \prod_{\ell=1}^{n} \mu(G_\ell')\{x - \sum_{i=1}^{n} \mu(G_i')^{-1}[\sum_{j=1}^{g_i} \mu(G_i-u_i-v_{ij}) +$$
$$+ 2 \sum_{k=1}^{r_i} t_{ik}\,\mu(G_i-Z_{ik})]\} \quad . \qquad (9)$$

Applying eq. (2) to the vertex u_i of G_i we straightforwardly obtain

$$\mu(G_i) = x\,\mu(G_i') - \sum_{j=1}^{g_i} \mu(G-u_i-v_{ij}) - 2\sum_{k=1}^{r_i} t_{ik}\,\mu(G_i-Z_{ik}) \quad . \tag{10}$$

From (10) we see that the expression in the square brackets in (9) is equal to $x\,\mu(G_i') - \mu(G_i)$. Therefore,

$$\mu(G) = \prod_{\ell=1}^{n} \mu(G_\ell')\{x + \sum_{i=1}^{n} [\mu(G_i)/\mu(G_i') - x]\} =$$

$$= \prod_{\ell=1}^{n} \mu(G_\ell')\{\sum_{i=1}^{n} \mu(G_i)/\mu(G_i') - (n-1)x\} \tag{11}$$

from which (4) follows immediately.

Eq. (11) can be transformed also into another form, viz.

$$\mu(G)/\mu(G-u) - x = \sum_{i=1}^{n} [\mu(G_i)/\mu(G_i') - x] \tag{12}$$

where, in accordance with eq. (6), the product of $\mu(G_i')$'s is replaced by $\mu(G-u)$.

Introducing an auxiliary function

$$\Omega(H, v) =: \mu(H)/\mu(H-v) - x \tag{13}$$

where H is a graph and v one of its vertices, we may write eq. (12) as

$$\Omega(G, u) = \sum_{i=1}^{n} \Omega(G_i, u_i) \quad . \tag{14}$$

This form of eq. (4) reveals a certain additivity property of the μ-polynomial of graphs containing articulation points.

We wish to point out another consequence of eq. (4). Let among the graphs $G_1, G_2, \ldots, G_n$ there are m ($m \leq n$) mutually isomorphic ones. Without loss of generality we may assume that these are $G_1, G_2, \ldots, G_m$. We further suppose that the vertices $u_1, u_2, \ldots, u_m$ have been chosen so that the subgraphs $G_1', G_2', \ldots, G_m'$ are also mutually isomorphic. Then $\mu(G)$ contains m-1 times $\mu(G_1')$ as a factor. In particular, if $m = n$, then

$$\mu(G) = [\mu(G_1')]^{n-1}[n\ \mu(G_1) - (n-1)x\ \mu(G_1')] \quad . \tag{15}$$

Needless to say, all relations obtained so far hold for both the characteristic and the matching polynomial. One has simply to set $t_k = 1$ for all Z_k (characteristic polynomial) or $t_k = 0$ for all Z_k (matching polynomial). In particular, the additivity property (14) is maintained if, as a special case of (13), one chooses

$$\Omega(H, v) = \phi(H)/\phi(H-v) - x \tag{13 a}$$

or

$$\Omega(H, v) = \alpha(H)/\alpha(H-v) - x \tag{13 b}$$

where ϕ and α stand for the characteristic and the matching polynomial, respectively.

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