match no. 18 pp. 55-61 1985

A NOTE ON THE CIRCUIT POLYNOMIAL AND ITS RELATION TO THE μ-POLYNOMIAL

E. J. Farrell* and I. Gutman**

* Department of Mathematics, The University of the West Indies, St. Augustine, Trinidad, and

** Faculty of Science, University of Kragujevac, P.O. Box 60, 34000 Kragujevac, Yugoslavia

(received: April 1984)

Abstract: The μ-polynomial is shown to be a circuit polynomial with appropriately chosen weights. A relation is established between the two polynomials. Some deductions are then made for μ-polynomials.

1. INTRODUCTION

A *circuit* (*or cycle*) is a connected graph in which all the nodes have degree two. The number of nodes of a circuit is called its *size*. We define the circuits with one and two nodes to be a node and an edge (together with its two incident nodes), respectively. Circuits whose size is greater than two will be also called *proper circuits* (*or proper cycles*).

The graphs considered in the present paper are finite and loopless. Let G be such a graph, possessing p nodes and q edges. Let, in addition, G contain as subgraphs r distinct cycles: $Z_1, Z_2, \ldots, Z_r$.

A *circuit or cycle cover* in G is a spanning subgraph of G whose elements are circuits (cycles). Let us associate an indeterminate or weight w_α with every circuit α in G, and with every circuit cover C of G the weight

$$w(C) = \Pi_\alpha w_\alpha \ ,$$

where the product is taken over all the components of C. Then the *circuit polynomial* of G is

$$C(G;\underline{w}) = \Sigma\ w(C).$$

where $\underline{w}$ is a vector of indeterminates w_α and the summation is taken over all the circuit covers of G. This polynomial was introduced in Farrell [1] and it was shown to be a generalization of both the characteristic and matching polynomials. The basic properties of C(G;w) are given in Farrell [2].

In the present application we shall assign the weight w' to all circuits of size one, the weight w'' to all circuits of size two and the weight w_i to the proper circuit $Z_i, i=1,\ldots,r$. Then $\underline{w}$ has the form

$$\underline{w} = (\underbrace{w',w',\ldots,w'}_{p\text{ times}},\underbrace{w'',w'',\ldots,w''}_{q\text{ times}},w_1,w_2,\ldots,w_r)$$

which may be written also in the following abbreviated form

$$\underline{w} \equiv (w',w'',w_1,w_2,\ldots,w_r) .$$

The μ-polynomial $\mu(G;\, t,\, x)$ of a graph G was introduced in [3], [4] and [5]. It is defined as follows.

For a graph G with p nodes,

$$\mu(G;\underline{t},x) = \sum_{s\in S} (-1)^{c(s)}\, 2^{r(s)}\, x^{p-n(s)}\, T(s) , \tag{1}$$

where S is the set of circuit covers *without isolated nodes*, of subgraphs of G. $n(s)$, $c(s)$ and $r(s)$ are the number of nodes, components and proper cycles respectively in the cover s. $T(s) = t_{a_1}\, t_{a_2} \ldots t_{a_{r(s)}}$, where $\underline{t} = (t_1,t_2,\ldots,t_r)$ and the component t_a of $\underline{t}$ is associated with the cycle Z_a of G. Also, if s is acyclic (i.e. s has no proper cycles), then $T(s)$ is by definition equal to unity.

The μ-polynomial was introduced as a means of unifying the matching polynomial (see [6-8]) and the characteristic polynomial. It has been used in numerous investigations of the topological properties of conjugated molecules [3-5,9-12].

In the material which follows, we will establish a relation between $\mu(G;\underline{t},x)$ and $C(G;w)$. It will then follow that the μ-polynomial is a circuit polynomial with appropriately chosen weights.

Most of the previously established properties of $\mu(G;t,x)$ will then follow from the corresponding properties of $C(G;\underline{w})$. We will call a cover which has no isolated nodes, a *semi-proper circuit cover*.

2. THE RELATIONSHIPS

First of all, it should be observed that every semi-proper circuit cover, of a subgraph H of G can be extended uniquely to a circuit cover C of G, by adding the appropriate number of isolated nodes. Conversely, every circuit cover C of G can be reduced uniquely, by omitting the isolated nodes, to a semi-proper circuit cover of a subgraph of G. Hence the two kinds of covers are equinumerous.

Let s be a semi-proper circuit cover of a subgraph H of G with n(s) nodes. Then s can be extended to a cover C of G, by adding p-n(s) isolated nodes, where p is the number of nodes in G. Let us assign weights to the components of C as follows. Each node will be assigned the weight x (i.e. $w' = x$), each edge the weight -1 (i.e. $w'' = -1$) and the proper cycle Z_{a_k}, $k=1,\dots,r(s)$, the weight $-2t_{a_k}$. Then the weight of C will be

$$w(C) = x^{p-n(s)} (-1)^e (-2t_{a_1})(-2t_{a_2}) \dots (-2t_{a_{r(s)}}) ,$$

where e is the number of edges in C and r(s) is the number of proper cycles in C. Clearly, if s has c(s) components, then

$$e + r(s) = c(s) \quad \text{and therefore} \quad e = c(s)-r(s) .$$

Hence

$$\begin{aligned} w(C) &= x^{p-n(s)} (-1)^{c(s)-r(s)} (-2)^{r(s)} t_{a_1} \cdot t_{a_2} \cdots t_{a_{r(s)}} \\ &= x^{p-n(s)} (-1)^{c(s)} 2^{r(s)} T(s), \end{aligned}$$

where $T(s) = \prod_{i=1}^{r(s)} t_{a_i}$.

Therefore, by summing over all circuit covers in G, we get

$$\Sigma w(C) = \sum_{s \in S} x^{p-n(s)} (-1)^{c(s)} 2^{r(s)} T(s)$$

$$= \mu(G;\underline{t},x).$$

Our discussion leads to the following theorem,

Theorem 1

$$\mu(G;\underline{t},x) = C(G; (x,-1,2t_1,2t_2,\ldots,2t_r))$$

This theorem shows that the μ-polynomial is an appropriately weighted circuit polynomial.

3. SOME DEDUCTIONS

We will denote the characteristic polynomial of G by $\phi(G;x)$. The matching polynomial of G in which each node is given the weight x and each edge, the weight -1, will be denoted by $\alpha(G;x)$. $\alpha(G;x)$ is also called the acyclic polynomial of G (see [13] and [14]) .

The following lemmas were established in [1] and [14], respectively.

Lemma 1

$$\phi(G;x) = C(G;(x,-1,-2,-2\ldots,-2)).$$

Lemma 2

$$\alpha(G;x) = C(G;(x,-1,0,0,\ldots,0)).$$

These two lemmas, together with Theorem 1, yield the following previously known ([5]) result.

Theorem 2

(i) $\phi(G;x) = \mu(G;\underline{1},x)$, where $\underline{1}$ is the vector obtained form $\underline{t}$ by putting each component equal to 1 i.e. $\underline{1} = (1,1,1, \ldots)$.

(ii) $\alpha(G;x) = \mu(G;\underline{0},x)$,

where $\underline{0}$ is the vector $(0,0,0, \ldots)$.

Many further properties of $(G;\underline{t},x)$, which have been established in [5], can be easily deduced from results already given in [2].

REFERENCES

[1] E. J. Farrell, J. Comb. Theory B 26, 111 (1979).

[2] E. J. Farrell, Discrete Math. 25, 121 (1979).

[3] I. Gutman, Chem.Phys. Letters 66, 595 (1979).

[4] I. Gutman, Z. Naturforsch. 35a, 458 (1980).

[5] I. Gutman and O. E. Polansky, Theoret. Chem. Acta 60, 203 (1981).

[6] E. J. Farrell, J. Comb. Theory B 27, 75 (1979).

[7] I. Gutman, Match 6, 75 (1979).

[8] C. D. Godsil and I. Gutman, J. Graph Theory 5. 137 (1981).

[9] O. E. Polansky and M. Zander, J. Mol. Struct. 84, 361 (1981).

[10] O. E. Polansky and A. Graovac, Match 13, 151 (1982).

[11] A. Graovac, O. E. Polansky and N. N. Tyutyulkov, Croat. Chem. Acta 56, 325 (1983).

[12] A. Graovac, I. Gutman and O. E. Polansky, Monatsh. Chem. 115, 1 (1984).

[13] I. Gutman, Publ. Inst. Math. (Beograd) 22, 63 (1977).

[14] E. J. Farrell, Ars Combinatoria 9, 221 (1980).