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ON THE EIGENVECTORS OF A SQUARE MATRIX AND ITS TRANSPOSE .

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I. DEFINITIONS -

We consider a square matrix $M = [\, a_{ik} \,]$. Its eigenvalues are y_α , y_β , ... y_μ ... The eigenvectors associated to these eigenvalues are X_α , X_β , ... X_μ .. The components of the eigenvector X_μ are $x_{1\mu}$, $x_{2\mu}$, ... $x_{i\mu}$, ...
M' , transpose of M have the same eigenvalues. Its eigenvectors are Z_α , Z_β , ... Z_μ .... The components of Z_μ are $z_{1\mu}$, $z_{2\mu}$, ... $z_{i\mu}$ ... .
Y is the diagonal matrix, which diagonal elements are y_α , y_β , ... y_μ , ...
X is a square matrix, the columns of which are the eigenvectors X_α , X_β , ...

$$X = [\, X_\alpha \;\; X_\beta \;\; \dots \; X_\mu \; \dots \,]$$

In a similar way

$$Z = [\, Z_\alpha \; , \; Z_\beta \;\; \dots \; Z_\mu \; \dots \,]$$

Fundamental condition.

The lengths of vectors X and Z are chosen in order that their inner product should be equal 1.

$$(1) \quad X'_\mu \; Z_\mu = Z'_\mu \; X_\mu = 1$$

X'_{μ} , Z'_{μ} are transposed of respectively X_{μ} and Z_{μ}
Equation (1) could be also written as

$$(2) \quad \sum_i x_{i\mu} \; z_{i\mu} = 1$$

A numerical example is given in Table 1.

<u>T a b l e 1</u>

$$M = \begin{bmatrix} 3 & 2 & 2 & -4 \\ 2 & 3 & 2 & -1 \\ 1 & 1 & 2 & -1 \\ 2 & 2 & 2 & -1 \end{bmatrix} \qquad X = \begin{bmatrix} 1 & 1 & -2 & 0 \\ 0 & -1 & 4 & 3 \\ -1 & 0 & 1 & 1 \\ 0 & 0 & 2 & 2 \end{bmatrix}$$

$$y_{\alpha} = y_{\beta} = 1$$
$$y_{\gamma} = 2$$
$$y_{\delta} = 3$$

$$Z = \begin{bmatrix} 0 & -1 & -1 & 1 \\ 0 & -2 & -1 & 1 \\ -1 & -1 & -1 & 1 \\ +1/2 & \frac{7}{2} & 2 & -3/2 \end{bmatrix}$$

$$X'_{\gamma} \; Z_{\gamma} = (-2)(-1) + (4)(-1) + (1)(-1) + (2)(2) = 1$$

<u>II. RELATION BETWEEN M AND THE TWO SETS OF EIGENVECTORS.</u>

$$(3a) \quad M = X \, Y \, X^{-1}$$
$$(3b) \quad M' = Z \, Y \, Z^{-1}$$

By taking transpose of the left and the right side of (3b) one obtains

$$(4) \quad M = (Z^{-1})' \; Y \, Z'$$

By comparing (3a) and (4), we have
$Z' = X^{-1}$ and similarly $X' = Z^{-1}$. Therefore one obtains

$$(5a) \quad M = X \, Y \, Z'$$
$$(5b) \quad M' = Z \, Y \, X'$$

Equation (5a) could be also written as

$$(6) \quad a_{ik} = \sum_{\mu} y_{\mu} \; x_{i\mu} \; z_{k\mu}$$

Equations (5a) and (5b) could be alternatively written in the following way

$$(7a) \quad M = \sum_{\mu} y_{\mu} \; X_{\mu} \; Z'_{\mu}$$

$$(7b) \quad M' = \sum_{\mu} y_{\mu} \; Z_{\mu} \; X'_{\mu}$$

For the matrix M of Table 1. the matrices X_{μ} Z'_{μ} $\mu = \alpha, \beta, \gamma, \delta$ are presented in Table 2. The factor y_{μ} appearing in equation (7a) is indicated on the left side of the related matrix X_{μ} Z'_{μ}

T a b l e 2

1 x

$X_\alpha \backslash Z'_\alpha$	0	0	-1	1/2
1	0	0	-1	1/2
0	0	0	0	0
-1	0	0	1	-1/2
0	0	0	0	0

1 x

$X_\beta \backslash Z'_\beta$	-1	-2	-1	7/2
1	-1	-2	-1	7/2
-1	1	2	1	-7/2
0	0	0	0	0
0	0	0	0	0

2 x

$X_\gamma \backslash Z'_\gamma$	-1	-1	-1	2
-2	2	2	2	-4
4	-4	-4	-4	8
1	-1	-1	-1	2
2	-2	-2	-2	4

3 x

$X_\delta \backslash Z'_\delta$	1	1	1	-3/2
0	0	0	0	0
3	3	3	3	-9/2
1	1	1	1	-3/2
2	2	2	2	- 3

Square of matrix M. We have $M^2 = XYZ' \; XYZ'$

Since $Z' = X^{-1}$ we obtain

$$(8) \quad M^2 = XY^2 Z'$$

If we consider the matrix $M^2 = [\, b_{ik} \,]$, we obtain

$$(9) \quad b_{ik} = \sum_{\mu} y_{\mu}^{2} \; x_{i\mu} \; z_{k\mu}$$

The p-th power of matrix M. $M^p = [\, c_{ik} \,]$

We obtain in the same way $M^p = X \; Y^p \; Z'$ what can be also written as

$$c_{ik} = \sum_{\mu} y_{\mu}^{p} \; x_{i\mu} \; z_{k\mu}$$

Let us now consider a particular case of a symmetrical matrix M ($M' = M$). Let us further normalise each eigenvector to unity. One has then $X'_\mu \; X_\nu = \delta_{\mu\nu}$ and as the result one obtains $X' = X^{-1}$

Therefore for a symmetrical matrix M equations (5a) an (6) are rewritten as follow

$$M = XYX' = \sum_\mu y_\mu X_\mu X'_\mu$$

$$a_{ik} = \sum_\mu y_\mu \quad x_{i\mu} \quad x_{k\mu}$$

III. RELATIONS BETWEEN EIGENVECTOR COMPONENTS AND CHARACTERISTIC POLYNOMIAL.

The characteristic equation of M is $F(My) = 0$, F is function of variable y and matrix elements a_{ik} . Variation of an element a_{ik} produces the variation of the eigenvalue y so that

$$(10) \qquad \left[\frac{\partial F}{\partial a_{ik}} + \frac{\partial F}{\partial y}\,\frac{\partial y}{\partial a_{ik}}\right]_{y=y_\mu} = 0$$

$$(11) \qquad \frac{\partial y_\mu}{\partial a_{ik}} = -\left[\frac{\frac{\partial F}{\partial a_{ik}}}{\frac{\partial F}{\partial y}}\right]_{y=y_\mu}$$

On the other hand we shall now consider the system of linear equations associated to

$$(12) \qquad Z'MX = Y$$

By equating the matrix element in the μ -th row and in the μ -th column of the both sides of equation (12) one obtains

$$(13) \qquad \sum_i \sum_k a_{ik} \quad x_{k\mu} \quad z_{i\mu} = y_\mu$$

and as the result one has

$$(14) \qquad \frac{\partial y_\mu}{\partial a_{ik}} = x_{k\mu}\, z_{i\mu}$$

Comparing (14) and (11) we have

$$(15) \qquad x_{k\mu} \quad z_{i\mu} = -\left|\frac{\frac{\partial F}{\partial a_{ik}}}{\frac{\partial F}{\partial y}}\right|_{y=y_\mu}$$

Numerical example. The characteristic equation of matrix M presented in Table 1 is $F = y^4 - 7y^3 + 17y^2 - 17y + 6$

$$\frac{\partial F}{\partial y} = 4y^3 - 21y^2 + 34y - 17$$

From Table 2 we see that for $y_\delta = 3$, $x_2 \, x_4 = -9/2$

$$\frac{\partial F}{\partial a_{4,2}} = y^2 + 5y - 6$$

If $y = y_\gamma = 3$ $\quad \frac{\partial F}{\partial a_{4,2}} = 18$, $\frac{\partial F}{\partial y} = 4$, $x_2 x_4 = -\frac{18}{4} = -\frac{9}{2}$

The relation (15) has been demonstrated earlier for the particular case of the symmetrical matrices.[1]

If y_μ is a double degenerate eigenvalue, the eigenvectors associated to y_μ are $X_{\mu'}$ and $X_{\mu''}$. The relation (14) becomes

$$(16) \quad \frac{\partial y}{\partial a_{ik}} = x_{k\mu'} z_{i\mu'} + x_{k\mu''} z_{i\mu''}$$

One has (17a) $\left[\frac{\partial F}{\partial y}\right]_{y=y_\mu} = 0$ and as the result of equation (10) one obtains

$$(17b) \quad \left[\frac{\partial F}{\partial a_{ik}}\right]_{y=y_\mu} = 0$$

The annulation of the second derivative of F and equation (17a) leads to the following relation

$$(18) \quad \left[\left(\frac{\partial y}{\partial a_{ik}}\right)^2 \frac{\partial^2 F}{\partial y^2} + 2\frac{\partial^2 F}{\partial y \, \partial a_{ik}} \frac{\partial y}{\partial a_{ik}}\right]_{y=y_\mu} = 0$$

and

$$(19) \quad \frac{\partial y_\mu}{\partial a_{ik}} = -2\left[\frac{\frac{\partial^2 F}{\partial y \, \partial a_{ik}}}{\frac{\partial^2 F}{\partial y^2}}\right]_{y=y_\mu}$$

By comparing (16) and (19), we have

$$(20) \quad x_{k\mu'} \; z_{i\mu'} + x_{k\mu''} \; z_{i\mu''} = -2\left[\frac{\frac{\partial^2 F}{\partial y \, \partial a_{ik}}}{\frac{\partial^2 F}{\partial y^2}}\right]_{y=y_\mu}$$

As a numerical example we use again the matrix M of Table 1. This matrix has a double degenerate eigenvalue

$y_\alpha = y_\beta = 1$

By using Table 2 one obtains

$$x_{1\alpha}\; z_{4\alpha} + x_{1\beta}\; z_{4\beta} = 1/2 + 7/2 = 4$$

$$\left[\frac{\partial^2 F}{\partial a_{4,1}\,\partial y}\right]_{y=1} = 8y - 16 = -8$$

$$\left[\frac{\partial^2 F}{\partial y^2}\right]_{y=1} = 12y^2 - 42y + 34 = 4$$

$$-2\left[\frac{\dfrac{\partial^2 F}{\partial a_{4,1}\,\partial y}}{\dfrac{\partial^2 F}{\partial y^2}}\right] = 4$$

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(1)
I. SAMUEL , Comptes-rendus Ac. Sc. Paris. 241, 1955,1464